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Stochastic resonance in multi-threshold systems

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Abstract

We discuss the dynamical behaviour of multi-threshold systems in the presence of noise and periodic inputs. Here, the *stochastic resonance* phenomenon displays some peculiarities such as a clear dependence on the noise statistics and the presence of a multi-peaked characteristic curve, which are not observed in simple bistable systems. This phenomenon is described without reference to any frequency matching condition as a special case of the well-known *dithering* effect.

1. Introduction

Dynamical models based on threshold systems are common in many fields of scientific research. A few examples are: digital communication (e.g. analog to digital conversion), neurobiology (e.g. neuron firing), natural events (e.g. avalanches), laser systems (e.g. laser threshold). Recently threshold systems acted on by noisy signals received considerable attention [1-6]. These systems are highly non-linear and much of the work presented in the literature focused on the study of the input-output response characteristic in the presence of a small periodic signal embedded in a large noise background. It was found that when the noise intensity matches a proper value, the periodic component in the system output reaches a maximum. This is usually considered the fingerprint of the phenomenon of stochastic resonance (SR) [2] [7,9] (for a recent review on SR see Ref. [8]).

Even if the use of the word *resonance* for this phenomenon has been questioned since the very beginning, it has been recently demonstrated [10] that, for a diffusion process in a double well system, the meaning of *resonance* as the matching of two characteristic frequencies (or physical time scales) is indeed appropriate for such a phenomenon if the residence time of the two states (in the bistable case) is taken into account as the order parameter. The resonant condition can be obtained either by changing the noise intensity or by changing the input signal frequency. As we show below, this frequency matching condition, instead, does not apply to the threshold systems that we consider here. In this case, in fact, the output signal enhancement typical of the SR phenomenon can be obtained for non-periodic signals as well.

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[2] For a more comprehensive list of references see the Web site on SR at http://www.pg.infn.it/SR/.

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In the following we propose a description of this effect in terms of noise enhanced threshold crossings, which is similar to the description of a well-known technique employed in digital signal processing and commonly called *dithering*. The dithering technique was proposed to increase the performances of analog-to-digital converters. In digital signal processing, an analog signal is sampled at discrete times and converted into a sequence of numbers. Since the register length is finite, the conversion procedure, called signal *quantization*, results in distortion and loss of signal detail. In order to avoid distortion and recover signal detail, it has become common practice, since the sixties, to add a small amount of noise (called dither signal) to the analog signal before quantization. A detailed description of this technique is beyond the purpose of this work. The interested reader can refer to Ref. [11] and references therein. Here, we limit ourselves to observing that analog to digital conversion is a member of a wide class of phenomena which can be interpreted by using a dynamical system in the presence of thresholds, as a model.

2. Two-threshold system

We start considering the output y of a two-threshold system S. Let x be the input signal,

$$\begin{aligned} y(t) &= -1, \quad &&\text{for } x < -\tfrac{1}{2}, \\ &= 0, &&\text{for } -\tfrac{1}{2} < x < \tfrac{1}{2}, \\ &= 1, &&\text{for } x > \tfrac{1}{2}. \end{aligned} \tag{1}$$

In this system, there are two symmetrical thresholds centered around zero. As the input signal we use $x(t) = A\sin(\omega_0 t) + \xi(t)$. Here $\xi(t)$ represents white additive noise [3] whose statistics is described below. The periodic component [4] of the input signal has an amplitude A which is smaller than the threshold value so that, in the absence of noise, the system output y is always equal to zero. The addition of noise induces random jumps above the upper threshold and below the lower threshold causing y to switch from 0 to 1 or -1.

We are interested in monitoring the amplitude of the periodic component in the system output. For this reason, we consider the $y(t)$ time series and compute the corresponding power spectral density $S_y(\omega)$. The statistical weight of the periodic component in the output signal can be monitored by measuring the height of the narrow peak in $S_y(\omega)$ at $\omega = \omega_0$. To eliminate the effect of random jumps, we subtract the continuous background $N(\omega_0)$ and define $P_y = \sqrt{S_y(\omega_0) - N(\omega_0)}$.

In Fig. 1 we show the noise intensity dependence of P_y for two different noise statistics: Gaussian and uniform; data points have been obtained by digital simulation of system (1). Here σ represents the noise standard deviation. The uniform probability density function (PDF) is defined as

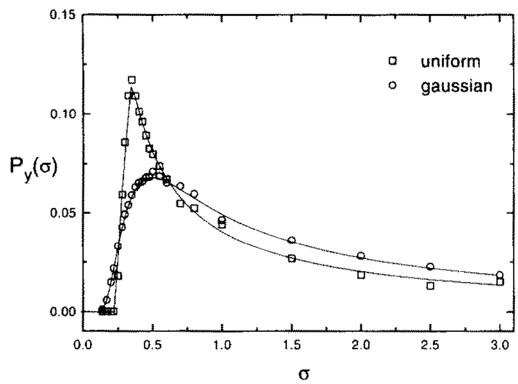
$$f_u(\xi) = \frac{1}{L}, \qquad \text{for } -\tfrac{1}{2}L \leqslant \xi_u \leqslant \tfrac{1}{2}L \tag{2}$$

and zero elsewhere. Here, $\sigma = L/\sqrt{12}$.

As can be seen, $P_y(\sigma)$ shows the typical SR profile: a sharp increase up to a maximum value (*resonant condition*) and a slow decrease. Apart from these features, common to both cases, some differences are apparent so that the system S is quite sensitive to the noise statistics. The uniform noise curve is generally sharper and more pronounced, and the maxima occur at slightly different noise values. The differences between the two curves are more evident at small noise values and decrease with increasing σ and A.

[3] Here and in the following we consider noise sources with a power spectral density flat up to a cutoff frequency $\bar{\omega}$. The standard deviation σ is used as a measure of the noise intensity.

[4] With $\omega_0 \ll \bar{\omega}$.



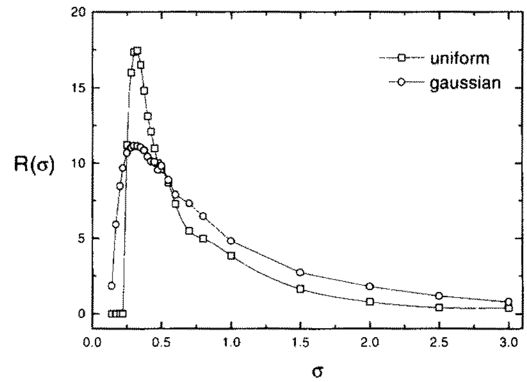


Fig. 1. P_y versus σ for Gaussian noise (circles) and uniform noise (squares). Input amplitude $A = 0.2$ a.u.; theoretical predictions A_y^1 are shown with solid lines, $k = 0.7$. The data presented in this paper have been obtained by means of digital simulation. Statistical errors are within 10%.

Fig. 2. SNR, $R(\sigma)$, versus σ for Gaussian noise (circles) and uniform noise (squares). The SNR is in dB, other parameters as in Fig. 1.

In the description of the SR phenomenon it is common practice to monitor the signal-to-noise ratio (SNR) defined as the ratio between the energy stored in the spectral peak at frequency $\nu_0 = \omega_0/2\pi$ and the noise background under the peak. It is worth remembering that such a definition, although quite popular in the SR community, differs from other definitions of the same quantity already used in data analysis. Most importantly, such a definition, when applied to two mathematically pathological quantities like a harmonic function and a white noise process, has to be taken with some care. For example, it is clear that the SNR, here, becomes dependent on the spectral bin amplitude. In Fig. 2, we show the SNR, $R(\sigma)$, for the two cases of Fig. 1. As above, the two curves show the typical SR behaviour with a clear dependence on the noise statistics: uniform noise performs much better compared to the Gaussian one. As for the periodic component amplitude, differences between the two curves are more evident at small noise values and decrease with increasing σ and A. Maxima positions approximately coincide and are not sensitive to the forcing amplitude A.

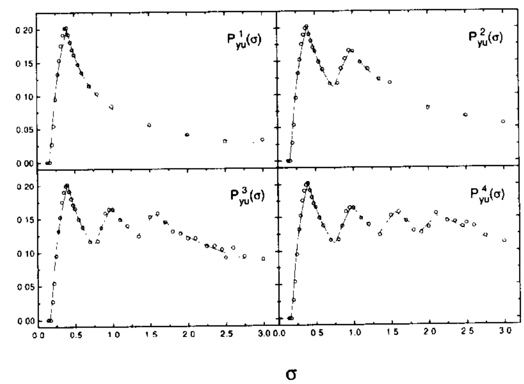
3. Multi-threshold system

The signal quantization, in the usual analog-to-digital conversion procedure, can be considered as a common example of a multi-threshold system at work. The continuous (analog) input signal is compared with a number of different thresholds and the system output is selected as the digital level which is closer to the sampled input (uniform quantization). Here we consider the quantization procedure as a generalization of the system S already introduced in Section 2. Such a procedure can be represented as follows,

$$\begin{aligned} y(t) &= -n, && \text{for } \tfrac{1}{2}(2n+1)b < x < -\tfrac{1}{2}(2n-1)b, \\ &= 0, && \text{for } -\tfrac{1}{2}b < x < \tfrac{1}{2}b, \\ &= n && \text{for } \tfrac{1}{2}(2n-1)b < x < \tfrac{1}{2}(2n+1)b. \end{aligned} \tag{3}$$

For a given n, we have $N_\ell = 2n + 1$ levels and $2n$ thresholds. b is the quantization step. For $n = 1$ and $b = 1$ we find the case of Section 2.

In Figs. 3 and 4, we show $P_y^n(\sigma)$ for the Gaussian and uniform noise cases, respectively. In both cases, we experimentally (by means of digital simulation) explore $n = 1, 2, 3, 4$. The addition of new thresholds introduces



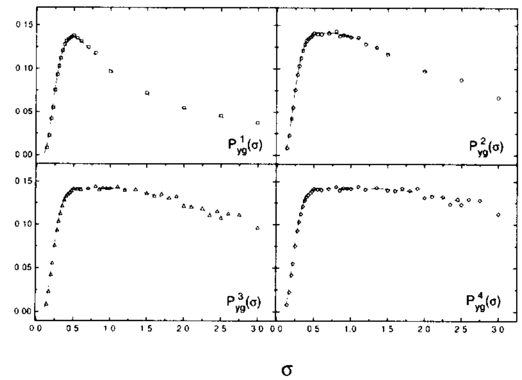
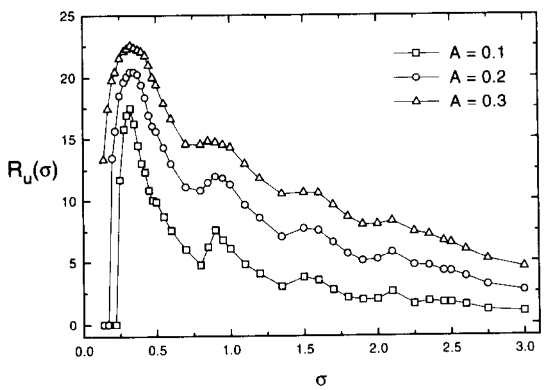


Fig. 3. P^n_{yu} versus σ with uniform noise for $n = 1$ (upper left), $n = 2$ (upper right), $n = 3$ (lower left) and $n = 4$ (lower right). $A = 0.2$. Theoretical predictions A^n_{yu} are shown by solid lines, $k = 0.7$ as in Fig. 1.

Fig. 4. P^n_{yg} versus σ with Gaussian noise for $n = 1$ (upper left), $n = 2$ (upper right), $n = 3$ (lower left) and $n = 4$ (lower right). $A = 0.2$. Theoretical predictions A^n_{yg} are shown by solid lines, $k = 0.7$ as in Fig. 1.



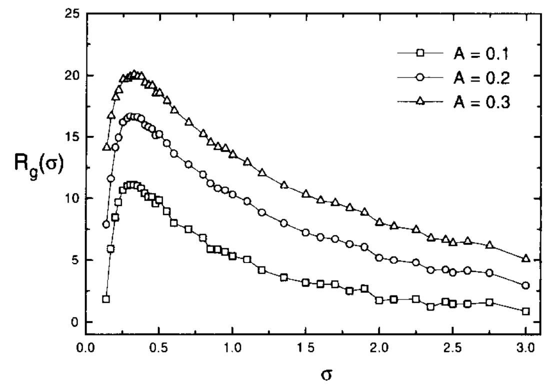


Fig. 5. SNR, R_u, versus σ with uniform noise for $n = 4$, for three different periodic signal amplitudes: $A = 0.1$ (squares), $A = 0.2$ (circles) and $A = 0.3$ (triangles). SNR is in dB.

Fig. 6. SNR, R_g, versus σ with Gaussian noise for $n = 4$, for three different periodic signal amplitudes: $A = 0.1$ (squares), $A = 0.2$ (circles) and $A = 0.3$ (triangles). SNR is in dB.

new features in the SR-like curves: (a) The SR-like behaviour is apparent also in the $n > 1$ case, both for the uniform and Gaussian noise. (b) As soon as the increasing noise explores the presence of new thresholds, two distinct qualitative behaviours occur: In the uniform noise case (Fig. 3), $P^n_y(\sigma)$ shows n distinct maxima, in correspondence with $L = L_{max}$ ($L_{max} = \frac{1}{2}(2m - 1)b + A$ for $m = 1, \ldots, n$), decreasing in amplitude. When $n \to \infty$, for large L, the curve oscillates around the asymptotic value $P^\infty_y(\infty)$ (dotted line in Fig 3). (c) In the Gaussian case (Fig. 4), for finite n, $P^n_y(\sigma)$ shows a single maximum, in analogy with the $n = 1$ case. The bell shaped curve tends to become wider and smoother as n increases. When $n \to \infty$, for large σ, the curve is characterized by a horizontal asymptote with value $P^\infty_y(\infty)$ (dotted line in Fig. 4), the same as in Fig. 3.

In Figs. 5 and 6, we show the SNR for the case with $n = 4$, for three different values of the forcing amplitude A ($A = 0.1, 0.2, 0.3$), using both the uniform and Gaussian cases, respectively. Also, for this quantity, in the uniform noise case, we find the characteristic multi-peaked shape already seen for the amplitude. As previously stated, the positions of the maxima are quite insensitive to the forcing amplitude value.

4. Theoretical model

A theoretical description of the system (3) output, for every n, can be obtained on the basis of simple considerations of the properties of the stochastic forcing term PDFs and without reference to any synchronization phenomenon, as in the usual SR description. It can be shown that such a statistical approach can be applied also to the description of the *dithering* effect. Here, we reproduce only the most relevant results. For a detailed treatment see Ref. [11].

In order to have a large periodic component in the system output, the action of the additive noise should produce upward jumps (and inhibit downward jumps) in coincidence with the half period in which $x(t) > 0$. The same requirement holds for the opposite situation of the downward jumps in the half period when $x(t) < 0$. When the added noise is white, the frequency of the periodic term does not play any role and the optimal condition for a large periodic output, stated above, can be expressed in terms of the differences of the statistical distribution function $F(\sigma)$ between two values. Ignoring the details of the actual shape of the periodic forcing, we fix these values at A and $-A$, i.e. equal to the amplitude of the periodic forcing. We obtain

$$A_y^n(\sigma) = \sum_{m=1}^{n} (F\{[\tfrac{1}{2}(2m-1)b + A]/\sigma\} - F\{[\tfrac{1}{2}(2m-1)b - A]/\sigma\}). \tag{4}$$

$A_y^n(\sigma)$ is a measure of the probability that y jumps from the lower state to the upper state when $x > 0$ *and* does not when $x < 0$. Whence, for the Gaussian case

$$A_{yg}^n(\sigma) = k\sum_{m=1}^{n} (\Phi\{[\tfrac{1}{2}((2m-1)b + A]/\sigma\} - \Phi\{[\tfrac{1}{2}(2m-1)b - A]/\sigma\}) \tag{5}$$

and for the uniform case

$$A_{yu}^n(L) = B_{yu}^n(L) + C_{yu}^n(L),$$

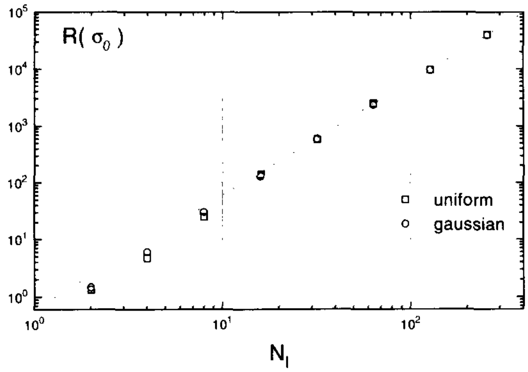
with

$$\begin{aligned} B_{yu}^n(L) &= k\sum_{m=1}^{n} 0, \qquad \text{for } \tfrac{1}{2}L < \tfrac{1}{2}b - A, \\ &= k\sum_{m=1}^{n} \left(\frac{1}{L}[\tfrac{1}{2}L - \tfrac{1}{2}(2m-1)b + A] + \frac{2A}{L}(m-1)\right), \\ &\qquad \text{for } \tfrac{1}{2}(2m-1)b - A \leqslant \tfrac{1}{2}L \leqslant \tfrac{1}{2}(2m-1)b + A, \\ &= k\sum_{m=1}^{n} \frac{2A}{L}m, \qquad \text{for } \tfrac{1}{2}(2m-1)b + A \leqslant \tfrac{1}{2}L \leqslant \tfrac{1}{2}(2m+1)b - A, \end{aligned}$$

and

$$C_{yu}^n(L) = \frac{2A}{L}n, \qquad \text{for } \tfrac{1}{2}L > \tfrac{1}{2}(2m+1)b - A.$$

Following the reasoning developed above, we can interpret $A_y^n(\sigma)$ as proportional to the amplitude of the periodic component of the output $y(t)$, at the input frequency ν_0, i.e. $A_y^n(\sigma) = kP_y^n(\sigma)$, where k is a constant



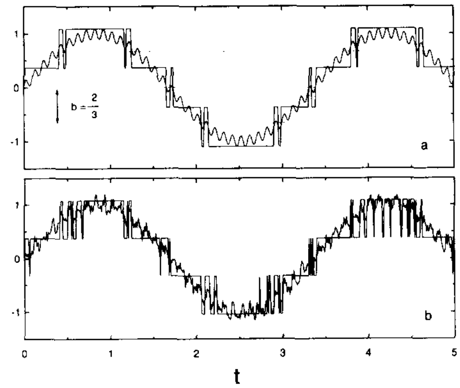


Fig. 7. SNR, $R(\sigma_0)$, versus N_ℓ with Gaussian (circles) and uniform (squares) noise. The dotted line is proportional to N_ℓ^2. Parameter values: $\sigma_0 = 1$, $A = 0.1$ a.u.

Fig. 8. Time series: input signal $z(t)$ and output signal $y(t)$, without dither noise (upper) and with dither noise (lower), for the case with $N_\ell = 4$. Parameter values: $A_1 = 1$, $A = 0.1$ a.u.

related to the spectral bin amplitude. The agreement between theoretical predictions (continuous curve) and experimental data (points) is remarkable (see Figs. 1, 3 and 4). The horizontal asymptote, $A_y^\infty(\infty) = A$, follows immediately for both signal distributions (see Figs. 3 and 4).

5. SNR versus number of levels

The dependence of the output signal of a multi-threshold system on the number of thresholds can be more easily addressed within the scheme elaborated in digital data analysis, under the name of *quantization effects*: the quantity $\eta = y - x$ called *quantization error* is introduced to take into account the changes in the output signal due to the coarseness of the input amplitude quantization. η is usually treated as an additive noise whose statistical properties depend upon the input signal x. It has been shown that the minimum loss of statistical data from the input x occurs when the quantization error can be made independent of x (noise whitening). Searching for a technique to realize such an independence condition, it was proposed to add an external signal (dither) before quantization. In fact, it can be shown [12] that, when the input signal is a *complicated* signal, i.e. a signal "which fluctuates rapidly in a somewhat unpredictable manner" [13] the quantization error can be treated as a white additive noise of amplitude

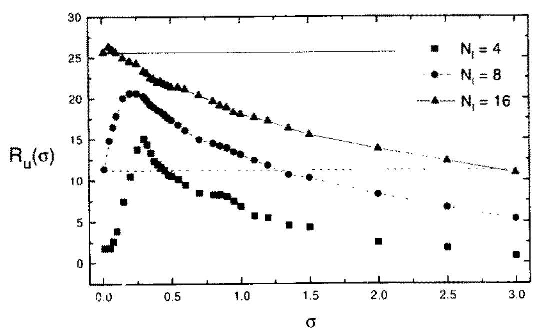
$$-\tfrac{1}{2}b \leqslant \eta(t) \leqslant \tfrac{1}{2}b, \tag{6}$$

with uniform distribution and standard deviation $\sigma_\eta = b/\sqrt{12}$. It is evident that adding to the input signal a noisy dither can represent an effective way to making the input signal more "complicated".

Another way to decrease the effect of the quantization noise is to increase the number of quantization levels. If the input dynamic range is kept fixed, the number of output levels N_ℓ scales as $N_\ell \propto 1/b$ [5]. In Fig. 7, we show the SNR for system (3) for a fixed input noise value σ_0, when the number of levels N_ℓ is changed. Both the Gaussian and the uniform noise cases follow the same N_ℓ^2 law. Such a behaviour can be easily explained considering the quantization noise as an external noise which superimposes on the input signal with variance

$$\sigma_\eta^2 = \tfrac{1}{12}b^2 \propto N_\ell^{-2}. \tag{7}$$

[5] $N_\ell = 2^{\mathrm{nb}}$ where nb is the number of bits.



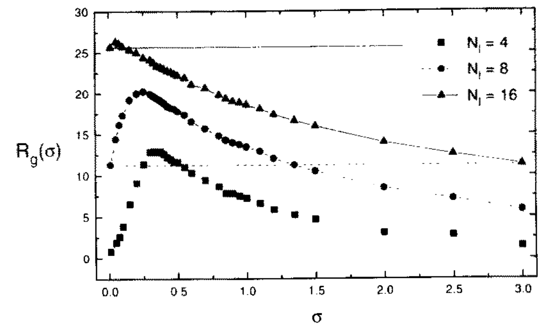


Fig. 9. SNR, R_u, versus σ with uniform noise for $n = 4$, for three different numbers of levels: $N_\ell = 4$ (squares), $N_\ell = 8$ (circles) and $N_\ell = 16$ (triangles). SNR is in dB, $A = 0.2$ a.u.

Fig. 10. SNR, R_g, versus σ with Gaussian noise for $n = 4$, for three different numbers of levels: $N_\ell = 4$ (squares), $N_\ell = 8$ (circles) and $N_\ell = 16$ (triangles). SNR is in dB, $A = 0.2$ a.u.

The output SNR thus depends on the quantization noise as $(\sigma_\eta^2)^{-1} \propto N_\ell^2$. Such a N_ℓ^2 law is more strictly fulfilled (see Fig. 7) for large N_ℓ, where the assumptions made on the quantization noise are fulfilled, i.e. "the signal is sufficiently complex and the quantization steps sufficiently small so that the amplitude of the signal is likely to traverse many quantization steps when going from sample to sample" [13].

As an example let us consider the following signal,

$$z(t) = A_1 \sin(\omega_1 t) + A \sin(\omega_0 t), \tag{8}$$

with $A \ll A_1$. For this signal the assumption of "complexity" is not tenable any more and deviation from the N_ℓ^2 law is observed. Once again, to whiten the quantization noise and increase the SNR, we can increase the number of quantization levels (number of bits) or add a dither signal, or both. In Fig. 8, we present the effect of the quantization procedure on $z(t)$, with and without the dither noise, for the case with $N_\ell = 4$. The addition of noise significantly increases the number of jumps corresponding to the threshold values making the presence of the small periodic signal more evident. In Figs. 9 and 10, the SNR versus dither noise values, for the uniform and Gaussian case, respectively, for three different numbers of levels, is reported together with the SNR in the absence of noise.

The most significant features can be listed as follows. (i) The SNR versus σ shows the usual SR profile, as expected. (ii) The increment, measured over the SNR in the absence of noise, is remarkable for the low N_ℓ case and decreases when N_ℓ increases. In the low N_ℓ regime the decrement of the quantization noise is mainly due to the action of the dither noise which helps the signal to span the quantization step. Once the N_ℓ is sufficiently large to allow the small periodic component in the input signal to explore the quantization step without the help of the noise, the effect of the dither is significantly reduced (see e.g. the case with $N_\ell = 16$). Further increasing the dither noise simply causes the decrement of the SNR as expected in a linear system. (iii) The differences between the uniform and Gaussian cases are apparent only in the small N_ℓ limit, when the role of noise is effective [11] and tend to disappear when N_ℓ is large.

6. Stochastic resonance and dithering

For the threshold systems that we consider here, the frequency matching condition typical of the SR phenomenon does not apply when we have only one characteristic frequency (periodic forcing). As expected, the change of the input frequency does not produce any effect on $P_y(\sigma)$ [2,3]. For this reason the output signal enhancement typical of the SR phenomenon can be obtained, here, for non-periodic signals as well.

It seems reasonable to conclude that the SR in the threshold systems considered here, far from being a *resonant* phenomenon, can be more correctly interpreted as a special case of the dithering effect [4,6,11] consisting of a threshold crossing process aided by noise.

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