Tekh. Fiz. **35**, 585 (1965) [Sov. Phys.-Tech. Phys. **10**, 461 (1965)].
[15]E. S. Solov'ev, R. N. Il'in, V. A. Oparin, I. T. Serenkov, and N. V. Federenko, Zh. Eksp. Teor. Fiz. Pis'ma Red. 10, 300 (1969) [JETP Lett. 10, 190 (1969)].
[16]F. R. Innes and O. Oldenberg, J. Chem. Phys. **37**, 2427 (1962).
[17]A. B. Prag and K. C. Clark, J. Chem. Phys. **39**, 799 (1963).
[18]R. A. Young, R. L. Sharpless, and R. Stringham, J. Chem. Phys. **40**, 117 (1964).
[19]K. M. Evenson and D. S. Burch, J. Chem. Phys. **45**, 2450 (1966).
[20]O. Oldenberg, Physical Sciences Research Papers Report No. 323, Office of Aerospace Research, United States Air Force, 1967 (unpublished).
[21]A. N. Wright and C. A Winkler, *Active Nitrogen* (Academic, New York, 1968), Chap. 3.
[22]H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One and Two Electron Atoms* (Springer, Berlin, 1957).
[23]M. Matsuzawa, J. Chem. Phys. **55**, 2685 (1971).
[24]R. S. Van Dyck, C. E. Johnson, and H. A. Shugart, Phys. Rev. A **4**, 1327 (1971).
[25]For both apparatuses used in the present experiment there are one or more (depending upon electron-gun voltages) unidentified very fast peaks in the TOF spectra. These peaks are probably due to neutral atoms and molecules produced by charge exchange of fast ions in the excitation region. See also C. E. Johnson, Phys. Rev. A **5**, 2688 (1972).
[26]C. E. Johnson, Phys. Rev. A **7**, 872 (1973).
[27]D. S. Bailey, J. R. Hiskes, and A. C. Riviere, Nucl. Fusion **5**, 41 (1965).
[28]E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge U. P., Cambridge, England, 1935), Chap. XV.
[29]Reference 28, p. 375.
[30]H. G. Kuhn, *Atomic Spectra*, 2nd ed. (Academic, New York, 1969).
[31]C. A. Nicolaides, Phys. Rev. A **6**, 2078 (1972).

PHYSICAL REVIEW A VOLUME 8, NUMBER 2 AUGUST 1973

Charged-Particle Scattering in the Presence of a Strong Electromagnetic Wave

Norman M. Kroll*
Department of Physics, University of California, San Diego, La Jolla, California 92037

Kenneth M. Watson†
Department of Physics, University of California, Berkeley, California 94720
and Lawrence Berkeley Laboratory, Berkeley, California 94720
(Received 21 March 1973)

A simple and convenient approximation is obtained for the multiphoton energy-transfer processes which accompany the scattering of a charged particle by a scattering potential in the presence of a strong external electromagnetic field. It is expressed in terms of the differential elastic-scattering cross section combined with known functions, and is valid when the scattering potential is weak or when the wave frequency is small. A detailed form of the classical limit is obtained.

I. INTRODUCTION AND SUMMARY

When a charged particle, to be referred to henceforth as an electron, scatters in the presence of an electromagnetic wave, it may exchange energy with the electromagnetic field. Because, on the average, energy-absorbing encounters dominate energy-emitting encounters, the process is of central importance in the study of plasma heating by electromagnetic waves and in the study of gas breakdown. It has been extensively discussed both classically[1] and quantum mechanically,[2] the latter in the context of inverse bremsstrahlung and stimulated bremsstrahlung of single photons. When the electromagnetic field is strong, however (or when the frequency is low enough), many photons can be emitted or absorbed in a single scattering process. It is the purpose of this note to analyze these multiphoton processes and to relate them to the classical description.

The electromagnetic field will be approximated by a classical spatially homogeneous electric field throughout. The distinction between the classical and quantum treatment resides in the description of the asymptotic states of the electron before and after the collision. In the classical description, the electron follows a classical orbit, with oscillating velocity. Scattering processes are characterized by an instantaneous incident velocity determined by the phase (henceforth referred to as the scattering phase α) of the electric field at the scattering instant. While the scattering itself is assumed to be instantaneous and elastic, change in time-averaged electron energy occurs as the result of the change in electron direction. For a given scattering angle (referred to the time-averaged directions) the energy change is determined by the scattering phase α. In the quantum case, initial and final electron states are described by solutions of the Schrödinger equation. Energy changes are determined by the number of photons emitted or absorbed. A correspondence between the photon number and scattering phase is established. It proves to be useful in the discussion of the classical limit and in the establishment of a relation between multiphoton processes and elastic scattering cross sections.

The classical theory is summarized in Sec. II. Section III is devoted to the development of a convenient formalism for the quantum case. Section IV consists of the application of the formalism when the elastic scattering process is adequately described by the Born approximation. Section V is devoted to a low-frequency approximation for those cases in which the Born approximation to the elastic scattering is inadequate. The classical and strong-field limits are the same for both of our approximations. Concluding comments are made in Sec. VI.

Our principal result may be summarized by the following formula:

$$\frac{d\sigma_\nu(\vec{q}(\nu),\vec{q}_0)}{d\Omega}=\frac{q(\nu)}{q_0}J_\nu^2(x)\frac{d\sigma_{el}(\epsilon,\vec{Q})}{d\Omega}, \tag{1.1}$$

where $d\sigma_\nu(\vec{q}(\nu),\vec{q}_0)/d\Omega$ is the differential cross section for scattering from (time-averaged) initial momentum $\vec{q}_0$ to final momentum $\vec{q}(\nu)$ with the emission ($\nu>0$) or absorption ($\nu<0$) of ν photons of angular frequency ω, so that

$$q^2(\nu)/2m=q_0^2/2m-\nu\hbar\omega; \tag{1.2}$$

$d\sigma_{el}/d\Omega$ is the differential elastic scattering cross section for scattering in the absence of the electromagnetic field, evaluated at momentum transfer

$$\vec{Q}\equiv\vec{q}(\nu)-\vec{q}_0 \tag{1.3}$$

and energy

$$\epsilon=\frac{q_0^2}{2m}+\nu\hbar\omega\frac{\hat{a}\cdot\vec{q}_0}{\hat{a}\cdot\vec{Q}}+(\nu\hbar\omega)^2\frac{m}{2(\hat{a}\cdot\vec{Q})^2}. \tag{1.4}$$

The electric field direction is denoted by $\hat{a}$. $J_\nu(x)$ denotes the Bessel function of order ν.

The magnitude of multiphoton cross sections is determined by the parameter x, which for a vector potential of amplitude a is given by

$$x=-e\vec{a}\cdot\vec{Q}/mc\hbar\omega. \tag{1.5}$$

In terms of incident radiation flux P in $\mathrm{MW/cm^2}$ and wavelength λ in μm,

$$x=-0.352\lambda^2P^{1/2}\,\hat{a}\cdot\vec{Q}/mc. \tag{1.6}$$

Equation (1.1) holds whenever the Born approximation provides an adequate description of the elastic process, in which case the elastic cross section depends upon the momentum transfer only. When the Born approximation is inadequate, Eq. (1.1) holds only at low frequencies, and Eq. (1.4) is reliable only for

$$\left|\frac{\nu\hbar\omega}{\hat{a}\cdot\vec{Q}}\frac{mc}{e}\right|=\left|\frac{\nu}{x}\right|<1.$$

This latter condition is equivalent to the requirement that $\nu\hbar\omega$, the energy transfer, be no larger than that which is allowed classically and avoids the apparant singularity of Eq. (1.4) at $\hat{a}\cdot\vec{Q}=0$. This condition need not be imposed when the Born approximation is adequate, but it is to be noted that the Bessel function becomes small when it is violated.

II. SUMMARY OF CLASSICAL THEORY

The instantaneous kinetic momentum $\vec{p}=m\vec{v}$ of an electron in the spatially uniform vector potential $\vec{A}$ is given by

$$\vec{p}=\vec{q}-(e/c)\vec{A}, \tag{2.1}$$

where $\vec{q}$ is a constant vector which represents the time-averaged value of $\vec{p}$ and $\vec{A}$ may be conveniently taken to have the form $\vec{a}\cos\omega t$. An electron with time-averaged momentum $\vec{q}_0$ will arrive at a scattering center at some instant t and hence with incident momentum $\vec{p}_0$ determined by (2.1). Scattering is assumed to take place instantaneously (i.e., in a time short compared to $1/\omega$) and elastically from an incident $\vec{p}_0$ to a final $\vec{p}$. Thus, $\cos\omega t$ has the same value immediately after the collision as it had immediately before, and the time-averaged final moment $\vec{q}$ is determined by (2.1). We have, therefore,

$$\vec{q}-\vec{q}_0=\vec{p}-\vec{p}_0=\vec{Q} \tag{2.2}$$

and, since $p^2=p_0^2$,

$$q^2-q_0^2=2(e/c)\vec{a}\cdot\vec{Q}\cos\omega t. \tag{2.3}$$

In a typical scattering process $\vec{q}$ and $\vec{q}_0$ are to be regarded as the observables. Equation (2.3) then determines the value of $\cos\omega t$ (let $\omega t=\alpha$ $\equiv$ "scattering phase" henceforth) at which the scattering took place, and Eq. (2.1) then determines the values $\vec{p}$ and $\vec{p}_0$ (and hence the energy) involved in the interaction with the scatterer.

The scattering rate per electron per unit scatterer density into solid angle $d\Omega_p$, in the phase interval $d\alpha$ is given by

$$dR=\frac{p_0}{m}\frac{d\sigma(p_0,p)}{d\Omega}d\Omega_p\frac{d\alpha}{\pi}. \tag{2.4}$$

(Since α appears only via $\cos\alpha$ we may take $0<\alpha<\pi$.) To define a cross section we use the time-averaged electron-flux at incident momentum $\vec{q}_0$. Thus,

$$d\sigma=(m/q_0)\,dR. \tag{2.5}$$

For comparison with the quantum theory it is convenient to express $d\Omega_p\, d\alpha$ in terms of $d\Omega_q\, d(q^2)$.

$$d\Omega_p\, d\alpha = \frac{2d_3p}{p}\, d\alpha\, \delta(p^2 - p_0^2)$$

$$= \frac{2d_3q\, d\alpha}{p}\, \delta[q^2 - q_0^2 - 2(e/c)\vec{a}\cdot\vec{Q}\cos\alpha]$$

$$= \frac{q}{p}\, d(q)^2\, d\Omega_q\, \frac{1}{|2(e/c)\vec{a}\cdot\vec{Q}\sin\alpha|}\,, \qquad (2.6)$$

yielding, finally,

$$d\sigma = \frac{q}{q_0}\,\frac{d\sigma(\vec{p},\vec{p}_0)}{d\Omega}\,\frac{d\Omega_q\, dq^2}{|2\pi(e/c)\vec{a}\cdot\vec{Q}\sin\alpha|}\,. \qquad (2.7)$$

III. FORMALISM FOR INDUCED MULTIPHOTON PROCESSES IN ELECTRON SCATTERING

The Schrödinger equation for an electron moving in a vector potential $\vec{A}$ and scattering potential V is

$$\frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla} - \frac{e}{c}\vec{A}\right)^2\Psi + V\Psi = i\hbar\dot{\Psi}\,. \qquad (3.1)$$

In strong fields, stimulated processes dominate spontaneous processes by an enormous factor. Hence it is appropriate to treat $\vec{A}$ as a c number. In addition we take (as before) $\vec{A}$ to be spatially independent. Consequently, the A^2 term can be eliminated from Eq. (3.1). Let

$$\Psi = \exp\left(-\frac{i}{\hbar}\int^t \frac{e^2}{2mc^2}A^2\, dt'\right)\varphi(t)\,. \qquad (3.2)$$

Then

$$\left(\frac{-\hbar^2}{2m}\nabla^2 + \frac{ie\hbar}{mc}\vec{A}\cdot\vec{\nabla} + V\right)\phi = i\hbar\dot{\varphi}\,. \qquad (3.3)$$

For $V=0$, Eq. (3.3) has the plane-wave solutions (we use X in place of φ for $V=0$ solutions)

$$X_{\vec{k}} = e^{i\vec{k}\cdot\vec{r}}\exp\left[-\frac{i\hbar}{2m}\int^t\left(k^2 - \frac{2e}{c\hbar}\vec{k}\cdot A\right)dt'\right] \qquad (3.4)$$

corresponding to the classical solutions with time-averaged momentum $\vec{q} = \hbar\vec{k}$. One may construct the retarded Green's function, $G(\vec{r}-\vec{r}', t, t')$, defined by the differential equation

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + \frac{ie\hbar}{mc}\vec{A}\cdot\vec{\nabla} - i\hbar\frac{\partial}{\partial t}\right)G = \delta(\vec{r}-\vec{r}')\delta(t-t') \qquad (3.5)$$

and the requirement that G vanish for negative $(t-t')$. For positive $(t-t')$ one finds

$$G = \frac{i}{\hbar(2\pi)^3}\int\exp\left[-\frac{i\hbar}{2m}\int_{t'}^{t}\left(k^2 - \frac{2e}{\hbar c}\vec{k}\cdot\vec{A}\right)dt''\right] \times e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}\, d\vec{k}\,. \qquad (3.6)$$

The solution $\varphi_{\vec{k}_0}$ to Eq. (3.3) corresponding to an incoming plane wave of momentum $\vec{q}_0 = \hbar\vec{k}_0$ plus outgoing waves satisfies the integral equation

$$\varphi_{\vec{k}_0} = X_{\vec{k}_0} - \int d\vec{r}'\int_{-\infty}^{t} dt'\, GV(\vec{r}')\varphi_{\vec{k}_0}(\vec{r}', t')\,. \qquad (3.7)$$

To determine a scattering cross section, we need the asymptotic form of $\varphi_{\vec{k}}$ at large r. It is convenient at this point to specialize to $\vec{A} = \vec{a}\cos\omega t$. Then

$$G = \frac{i}{\hbar(2\pi)^3}\int d\vec{k}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} \times\exp\left[\frac{-i\hbar}{2m}\left(k^2 t - \frac{2e}{\hbar c\omega}\vec{k}\cdot\vec{a}\sin\omega t\right)\right] \times\exp\left[\frac{i\hbar}{2m}\left(k^2 t' - \frac{2e}{\hbar c\omega}\vec{k}\cdot\vec{a}\sin\omega t'\right)\right]. \qquad (3.8)$$

Making use of the fact that $\Phi_{\vec{k}_0,\vec{k}}$ defined by

$$\Phi_{\vec{k}_0,\vec{k}}(r', t') \equiv \exp\left[\frac{i\hbar}{2m}\left(k_0^2 t' - \frac{2e}{\hbar c\omega}\vec{k}\cdot\vec{a}\sin\omega t'\right)\right] \times\varphi_{\vec{k}_0}(r', t') \qquad (3.9)$$

is periodic in t' with period $2\pi/\omega$, we expand it in a Fourier series as follows:

$$\Phi_{\vec{k}_0,\vec{k}} = \sum_{\nu=-\infty}^{\infty} e^{i\nu\omega t'}\Phi_{\vec{k}_0,\vec{k},\nu}(\vec{r}')\,. \qquad (3.10)$$

The second term of Eq. (3.7) may be written $\sum_\nu s_{\vec{k}_0,\nu}(\vec{r}, t)$, where

$$S_{\vec{k}_0,\nu}(\vec{r}, t) = \frac{-i}{\hbar(2\pi)^3}\int d\vec{k}\, d\vec{r}'\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} \times\exp\left[\frac{-i\hbar}{2m}\left(k^2 t - \frac{2e}{\hbar c\omega}\vec{k}\cdot\vec{a}\sin\omega t\right)\right] \times\int_0^t dt'\exp\left[i\left(\frac{\hbar}{2m}(k^2 - k_0^2) + \nu\omega\right)t'\right] \times V(\vec{r}')\Phi_{\vec{k}_0,\vec{k},\nu}(\vec{r}')\,. \qquad (3.11)$$

Using the familiar methods of scattering one finds that at large r, in the direction $\vec{k}(\nu)$,

$$S_{k_0,\nu}(r, t) = -\frac{m}{2\pi\hbar^2}\frac{e^{ik(\nu)r}}{r} \times\exp\left[-\frac{i\hbar}{2m}\left(k^2(\nu)t - \frac{2e}{\omega m\hbar}\vec{k}(\nu)\cdot\vec{a}\sin\omega t\right)\right] \times\int d\vec{r}'\, e^{-i\vec{k}(\nu)\cdot\vec{r}'}V(\vec{r}')\Phi_{k_0,k(\nu),\nu}(\vec{r}')\,, \qquad (3.12)$$

where

$$\frac{1}{2m}[\hbar k(\nu)]^2 = \frac{1}{2m}(\hbar k_0)^2 - \nu\hbar\omega\,. \qquad (3.13)$$

Equation (3.13) tells us that $S_{\vec{k}_0,\nu}$ is associated with the stimulated bremsstrahlung of ν photons for ν positive and the inverse bremsstrahlung of $|\nu|$ photons for ν negative.

Making use of (3.9), (3.10), and (3.13) we see that the final factor of Eq. (3.12) may be reexpressed in accordance with

$$\int d\vec{r}'\, e^{-i\vec{k}(\nu)\cdot\vec{r}'} V(\vec{r})\Phi_{\vec{k}_0,\vec{k}(\nu),\nu}(\vec{r}')$$
$$=\frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt\,[X_{\vec{k}(\nu)}, V\varphi_{\vec{k}_0}] \equiv \langle X_{\vec{k}(\nu)}, V\varphi_{\vec{k}_0}\rangle\,. \quad (3.14)$$

In Eq. (3.14) the term in square brackets is an inner product in the usual sense of integration over $\vec{r}$ and the term in angular brackets is defined as the indicated time average.

The probability current associated with the incident wave $X_{\vec{k}_0}$ is $(\vec{q}_0/m) - e\vec{A}/mc$, which is just the classical instantaneous velocity. For defining an incident flux we take the time-averaged value. The probability current associated with the spherical wave solutions in $S_{\vec{k}_0,\nu}$ is, at large r,

$$\frac{1}{r^2}\left(\frac{\vec{q}(\nu)}{m} - \frac{e\vec{A}}{mc}\right).$$

Again we take the time average to define a cross section. The total outgoing wave probability current contains cross terms between terms of different ν. These cross terms do not vanish on time averaging but do vanish when an average is taken over a macroscopically small range of r. They should, therefore, be omitted in determining the cross sections. Finally, we obtain

$$\frac{d\sigma(\vec{q}(\nu),\vec{q}_0)}{d\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 \frac{q(\nu)}{q_0}\,|\langle X_{\vec{k}(\nu)}, V\varphi_{\vec{k}_0}\rangle|^2\,. \quad (3.15)$$

Equation (3.15) is a straightforward generalization of the well-known expressions for scattering in a static potential.[3]

IV. WEAK-POTENTIAL APPROXIMATION

When the scattering potential V is weak, one can approximate $\varphi_{\vec{k}_0}$ by $X_{\vec{k}_0}$ in (3.15). Then using (3.4), (1.5), and (3.13) we have

$$[X_{\vec{k}(\nu)}, VX_{\vec{k}_0}] = \exp(ix\sin\omega t - i\nu\omega t) \times [e^{-i\vec{k}(\nu)\cdot\vec{r}}, Ve^{i\vec{k}_0\cdot\vec{r}}] \quad (4.1)$$

and, from (3.14), we have

$$\langle X_{\vec{k}(\nu)}, VX_{\vec{k}_0}\rangle = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \exp(ix\sin\omega t - i\nu\omega t)\,dt \times [e^{-i\vec{k}(\nu)\cdot\vec{r}}, Ve^{i\vec{k}_0\cdot\vec{r}}]$$
$$= J_\nu(x)\,(e^{-i\vec{k}(\nu)\cdot\vec{r}}, Ve^{i\vec{k}_0\cdot\vec{r}})\,, \quad (4.2)$$

which yields

$$\frac{d\sigma(\vec{q}(\nu),\vec{q}_0)}{d\Omega} = \frac{q(\nu)}{q_0} J_\nu^2(x)\frac{d\sigma_B(\vec{Q})}{d\Omega}. \quad (4.3)$$

The subscript B refers to first Born approximation for the elastic scattering cross sections. It is apparent from Eqs. (4.3) and (1.5) that the strong-field limit, low-frequency limit, and classical limit are all governed by large values of the single parameter x appearing in the argument of the Bessel function J_ν. To investigate this limit we recall the Debye asymptotic formulas[4]

$$J_\nu^2(x) \approx \frac{2\cos^2[(x^2-\nu^2)^{1/2} - |\nu\cos^{-1}(\nu/x)| - \frac{1}{4}\pi]}{\pi(x^2-\nu^2)^{1/2}}\,, \qquad 1 - |\nu/x| > \epsilon \quad (4.4a)$$

$$J_\nu^2(x) \approx \frac{2\exp[-2\int_0^\nu \cosh^{-1}(\nu'/|x|)\,d\nu']}{\pi(\nu^2-x^2)^{1/2}}\,, \qquad |\nu/x| - 1 > \epsilon\,. \quad (4.4b)$$

Identifying $\vec{q}$ in Eq. (2.3) with $\vec{q}(\nu)$ and using Eq. (3.13), we see that the classical scattering phase α is related to the quantum parameters via

$$\cos\alpha = \nu/x\,. \quad (4.5)$$

The Eq. (4.4b) form is thus seen to correspond to an energy transfer larger than is classically allowed. As one would expect, it vanishes exponentially as $|x| \to \infty$. Equation (4.4a) can be used to deduce the classical limit. In this limit, the number of photons transferred is typically large. The $\cos^2$ factor indicates large fluctuations as the photon number changes by unity. For a mean behavior, however, we take $\cos^2$ equal to one-half to obtain

$$\frac{d\bar{\sigma}}{d\Omega} = \frac{q(\nu)}{q_0}\frac{d\sigma_B(\vec{Q})}{d\Omega}\left(\pi\left|\frac{e}{mc\hbar\omega}\vec{a}\cdot\vec{Q}\sin\alpha\right|\right)^{-1}. \quad (4.6)$$

This is seen to be identical with the classical formula equation (2.7) when we set $dq^2 = 2m\hbar\omega$.

It is apparent on inspection of x that the classical limit obtains when the classical oscillation amplitude $ea/mc\omega$ is large compared to the interference fringe separation ($\cos\frac{1}{2}\vec{Q}\cdot\vec{\nu}$ modulation factor) between the incident and scattered wave along the direction of oscillation.

V. LOW-FREQUENCY APPROXIMATION

In this section we shall show that in the low-frequency limit [Eq. (4.3)] holds to all orders in the scattering potential provided that the Born approximation for the elastic scattering is replaced by the exact expression for the elastic scattering evaluated at the energy ϵ given by Eq. (1.4).

It is useful to redefine the scattering state $\varphi_{\vec{k}_0}$ so that it becomes time independent in the low-frequency limit. Furthermore, to go beyond the first Born approximation it is more convenient to work in momentum space. Accordingly, we let

$$\bar{\varphi}_{\vec{k}_0}(\vec{k},t) \equiv \frac{1}{(2\pi)^3} e^{i\eta(\vec{k}_0,t)} \int e^{-i\vec{k}\cdot\vec{r}} \varphi_{\vec{k}_0}(\vec{r},t)\, d\vec{r}\,, \tag{5.1}$$

$$\eta(\vec{k},t) = \frac{\hbar k^2}{2m} t - \frac{e}{mc\omega} \vec{k}\cdot\vec{a}\sin\omega t\,. \tag{5.2}$$

Substituting into Eqs. (3.4), (3.6), and (3.7), we find

$$\bar{\varphi}_{\vec{k}_0}(\vec{k},t) = \delta(\vec{k}-\vec{k}_0) - (i/\hbar) e^{-i\Delta(\vec{k},\vec{k}_0,t)} \times \int_{-\infty}^{t} dt'\, dk'\, e^{i\Delta(\vec{k},\vec{k}_0,t')} \times \tilde{V}(\vec{k}-\vec{k}')\bar{\varphi}_{\vec{k}_0}(\vec{k}',t')\,, \tag{5.3}$$

where

$$\Delta(\vec{k},\vec{k}_0,t) = \eta(\vec{k},t) - \eta(\vec{k}_0,t) \tag{5.4}$$

and

$$\tilde{V}(\vec{k}-\vec{k}') = \int e^{-i(\vec{k}-\vec{k}')\cdot\vec{r}} V(\vec{r})\, d_3\vec{r}\,. \tag{5.5}$$

A formal solution to Eq. (5.3) is provided by

$$\bar{\varphi}_{\vec{k}_0} = \sum_{n=0}^{\infty} \bar{\varphi}^{(n)}_{\vec{k}_0}, \tag{5.6}$$

$$\bar{\varphi}^{(0)}_{\vec{k}_0} = \delta(\vec{k}-\vec{k}_0)\,, \tag{5.7}$$

and

$$\bar{\varphi}^{(n+1)}_{\vec{k}_0} = -(i/\hbar) e^{-i\Delta(\vec{k},\vec{k}_0,t)} \int_{-\infty}^{t} dt'\, d\vec{k}' \times e^{i\Delta(\vec{k},\vec{k}_0,t')} \tilde{V}(\vec{k}-\vec{k}')\bar{\varphi}^{(n)}_{\vec{k}_0}(\vec{k}',t')\,. \tag{5.8}$$

In the low-frequency limit,

$$\frac{\partial^2\Delta}{\partial t^2} \approx 0\,, \quad \frac{\partial \bar{\varphi}^{(n)}_{k_0}}{\partial t} \approx 0\,. \tag{5.9}$$

Then

$$\frac{\partial}{\partial t'} \frac{e^{i\Delta(\vec{k},\vec{k}_0,t')}\bar{\phi}^{(n)}_{\vec{k}_0}(k',t')}{i\dot{\Delta}(\vec{k},\vec{k}_0,t')} \approx e^{i\Delta(\vec{k},\vec{k}_0,t')}\phi^{(n)}_{\vec{k}_0}(\vec{k}',t')\,. \tag{5.10}$$

Using the approximation (5.10) in (5.8), we obtain

$$\bar{\phi}^{(n+1)}_{k_0} = -\frac{1}{\hbar\dot{\Delta}(\vec{k},\vec{k}_0,t)} \int d\vec{k}'\, \tilde{V}(\vec{k}-\vec{k}')\bar{\varphi}^{(n)}_{\vec{k}_0}(k',t)\,. \tag{5.11}$$

We note that all time-dependent quantities in Eq. (5.11) refer to the same instant. Consequently, the time may now be regarded as a parameter, and the problem has been reduced to the solution of the elastic scattering problem. Let $\Phi_{\vec{k}_0}(\vec{k})$ be the solution of Eq. (5.3), with the vector potential $\vec{a}$ set equal to zero. Comparison with Eqs. (5.2), (5.4), (5.6), (5.7), and (5.8) shows that, for low frequency ω,

$$\bar{\varphi}_{\vec{k}_0}(\vec{k},t) \approx \Phi_{\vec{K}_0}(\vec{K})\,, \tag{5.12}$$

where

$$\vec{K}_0 = \vec{k}_0 - (e/\hbar c)\vec{a}\cos\omega t\,, \quad \vec{K} = \vec{k} - (e/\hbar c)\vec{a}\cos\omega t\,. \tag{5.13}$$

To compute the scattering[5] we require $[X_{\vec{k}(\nu)}, V\varphi_{\vec{k}_0}]$. Using Eqs. (3.4) and (5.1), we find

$$[X_{\vec{k}(\nu)}, V\varphi_{\vec{k}_0}] = e^{-i\Delta(\vec{k}_0,\vec{k}(\nu),t)} \int d\vec{k}'\, \tilde{V}(\vec{k}(\nu)-\vec{k}')\bar{\varphi}_{\vec{k}_0}(\vec{k}',t) = e^{-i\Delta(\vec{k}_0,\vec{k}(\nu),t)} \int d\vec{K}'\, \tilde{V}(\vec{K}(\nu)-\vec{K}')\Phi_{\vec{K}_0}(\vec{K}')\,. \tag{5.14}$$

The last integral is just the familiar T matrix for elastic scattering, whose arguments we shall take to be momenta rather than wave numbers. Thus, we write

$$\int d\vec{K}\, \tilde{V}(\vec{K}(\nu)-\vec{K}')\Phi_{\vec{K}_0}(\vec{K}') \equiv T(\vec{p}(\nu),\vec{p}_0)\,, \tag{5.15}$$

where the $\vec{p}$ vectors are related to the $\vec{q}=\hbar\vec{k}$ vectors via Eq. (2.1). We note that $p^2(\nu) \neq p_0^2$ except at the classical values of $\cos\omega t \equiv \cos\alpha$ determined by Eq. (2.3), so that (5.15) refers to the T matrix off the energy shell. We shall write T_{el} for the on-shell T matrix which yields the elastic scattering and express it in terms of the momentum transfer and energy.

Finally, we consider [see Eq. (3.14)]

$$\langle X_{\vec{k}(\nu)}, V\varphi_{\vec{k}_0}\rangle = (\omega/2\pi)\int_0^{2\pi/\omega} dt\, e^{-i\Delta(\vec{k}_0,\vec{k}(\nu),t)} T(\vec{p}(\nu),\vec{p}_0)\,. \tag{5.16}$$

For sufficiently small ω, the exponential factor will oscillate many times as t ranges over the integration interval. The principal contribution will come from those values of t for which $\dot{\Delta}$ vanishes. Thus, we approximate (5.16) by removing T from under the integral sign and evaluating it at the stationary phase points, that is at values of t such that $\dot{\Delta}=0$. This condition is immediately seen to be the same as the energy-shell condition with energy determined by Eq. (1.4). Thus, we obtain

$$\langle X_{\vec{k}(\nu)}, V\varphi_{\vec{k}_\nu}\rangle \approx T_{el}(\epsilon,\vec{Q})(\omega/2\pi)\int_0^{2\pi/\omega} dt\, e^{-i\Delta} = T_{el}(\epsilon,\vec{Q})J_\nu(x) \tag{5.17}$$

and hence the final result

$$\frac{d\sigma(\vec{q}(\nu),\vec{q}_0)}{d\Omega} = \frac{q(\nu)}{q_0} J_\nu^2(x) \frac{d\sigma_{el}(\epsilon,\vec{Q})}{d\Omega}\,. \tag{5.18}$$

In applying Eq. (5.18), $\cos\omega t$ is given by Eqs. (2.3) and (4.5). For $\nu > x$, the cosine is greater than 1, and the stationary-phase condition cannot be satisfied. The integral in Eq. (5.16) is then expected to be small, and while the approximation in Eq. (5.17) is no longer justified; it also gives a small result.

It is of interest to note that the weak-field limit of Eq. (5.17) for single-photon processes is valid not only to lowest order in ω ($\sim\omega^{-1}$ for this process) but also to the next higher order ($\sim\omega^0$). To see that this is the case we recall that a form valid to this order has been given by Low.[6] As shown by Brown and Goble,[7] the form given by Low may (again to order ω^0) be written (specialized here to the nonrelativistic limit with spatially constant electromagnetic field)

$$\langle X_{\vec{k}(1)}, V\varphi_{\vec{k}_0}\rangle \approx \frac{e}{2mc\hbar\omega}\left[\vec{a}\cdot\vec{q}_0 T_{el}\left(\frac{q(1)^2}{2m}, \vec{Q}\right) - \vec{a}\cdot\vec{q}(1) T_{el}\left(\frac{q_0^2}{2m}, \vec{Q}\right)\right], \quad (5.19)$$

while (5.17) in the weak-field limit yields

$$\langle X_{\vec{k}(1)}, V\varphi_{\vec{k}_0}\rangle \approx \frac{e}{2mc\hbar\omega}\left[\vec{a}\cdot(\vec{q}_0 - \vec{q}(1))\right] T_{el}(\epsilon, \vec{Q}). \quad (5.20)$$

Using Eq. (1.4) for ϵ one sees that the two formulas are equal to order ω^0.

VI. CONCLUDING REMARKS

The principal result of this paper is Eq. (5.18). It provides a simple and reasonable approximation to multiphoton energy transfers when the frequency of the electromagnetic wave is small or when the scattering potential is weak. From a practical point of view our results suggest that whenever multiphoton effects appear to be significant in electron heating processes, that quantum corrections to a classical discussion are small.

We briefly discuss our results in the context of the gas-breakdown problem. To specify a reasonable power level we recall that the classical low-frequency low-pressure breakdown power P_B for air is determined by $\lambda^2 P_B = 3.16\times 10^5$. This yields $|x_B| = 200\lambda\hat{a}\cdot\vec{Q}/mc$. A typical value for $|\vec{Q}|/mc$ might be 5×10^{-3}, growing by a factor of 3 or so as the electrons are heated. We conclude that multiphoton energy transfers are improbable for say Nd and ruby-laser pulses. On the other hand such processes are important for CO_2 pulses and become dominant as λ is further increased. For a given direction of $\vec{Q}$, the energy transfers peak at $\pm|x_B|\hbar\omega = 248\hat{a}\cdot\vec{Q}/mc$ eV. Thus, the fluctuations in energy transfer tend to be large compared to the mean energy transfer per collision (about 0.06 eV). On the other hand, the peaking is smoothed by averaging the direction of $\vec{Q}$ with respect to $\hat{a}$. The fluctuations due to quantum effects are, of course, not smoothed. In Ref. 2 it was found that quantum effects on electron heating for the case of air are of negligible importance for photons below 0.5 eV. It therefore appears that the neglect of multiphoton processes (or classical energy transfer fluctuations) in the theory of electron heating is of little importance in the theory of gas breakdown.

---

*Work supported in part by the United Stated Atomic Energy Commission.

†Work supported in part by the Air Force Office of Scientific Research, Office of Aerospace Research, U.S. Air Force, under Contract No. F 44620-70-C-0028.

[1]An excellent review of this topic has been given by S. C. Brown, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1956), Vol. XXII.

[2]Norman Kroll and Kenneth M. Watson, Phys. Rev. A **5**, 1883 (1972). This paper contains references to numerous earlier works.

[3]Similar extensions of classical scattering theory appropriate to external electromagnetic wave problems have been given by a number of authors [see, for example, H. R. Reiss, Phys. Rev. A **1**, 803 (1970)]. Because the situation is especially transparent for electron scattering problems, it was felt that a concise derivation of the form we use would be helpful.

[4]See, for example, W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Functions of Mathematical Physics* (Chelsea, New York, 1954), p. 23.

[5]The approximation of Eq. (5.12) is adequate for the evaluation of the scattering via Eq. (5.14). The scattering can, in principle, also be obtained from the residues of $\varphi_{\vec{k}_0}(\vec{k},t)$ at its poles. The approximation Eq. (5.12) is not adequate for this purpose. An adequate approximation can be obtained by using Eq. (5.12) for the right-hand side of Eq. (5.3) and carrying out the indicated time integration exactly.

[6]F. E. Low, Phys. Rev. **110**, 974 (1958).

[7]L. S. Brown and R. L. Goble, Phys. Rev. **173**, 1505 (1968).