Uncertainty Relation for the Discrete Fourier Transform

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We derive an uncertainty relation for two unitary operators which obey a commutation relation of the form $UV = e^{i\phi}VU$. Its most important application is to constrain how much a quantum state can be localized simultaneously in two mutually unbiased bases related by a discrete fourier transform. It provides an uncertainty relation which smoothly interpolates between the well-known cases of the Pauli operators in two dimensions and the continuous variables position and momentum. This work also provides an uncertainty relation for modular variables, and could find applications in signal processing. In the finite dimensional case the minimum uncertainty states, discrete analogues of coherent and squeezed states, are minimum energy solutions of Harper's equation, a discrete version of the harmonic oscillator equation.

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Introduction.—Uncertainty relations provide some of our most fundamental insights into quantum mechanics. They express the fact that noncommuting observables cannot simultaneously have well-defined values. This concept has no classical analogue, and therefore underlies much of the conceptual differences between classical and quantum mechanics. For these reasons, uncertainty relations have attracted a huge amount of attention.

The uncertainty principle was first understood by Heisenberg [1], and formulated precisely by Kennard as [2]

$$\Delta x \Delta p \geq \frac{1}{2}. \tag{1}$$

Here x and p are the position and momentum observables, the variance of observable A in state $|\psi\rangle$ is

$$(\Delta A)^2 = \langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2, \tag{2}$$

and we work in units where $\hbar = 1$, i.e., $[x, p] = i$. This relation was subsequently generalized by Robertson [3] to

$$\Delta A \Delta B \geq \tfrac{1}{2}|\langle\psi|[A, B]|\psi\rangle| \tag{3}$$

for any observables A and B.

The relation Eq. (3) is, however, not always satisfactory. For instance, uncertainty relations for phase and number, or angle and angular momentum, are notoriously tricky; see [4] for an excellent review. In the discrete case there has also been some important work. First of all, note that for spin $1/2$ particles, the uncertainty relations for the Pauli operators [which cannot be deduced from Eq. (3), but can be easily be established from the definition $\Delta\sigma_x^2 = 1 - \langle\sigma_x\rangle^2$ and the constraint $\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 + \langle\sigma_z\rangle^2 \leq 1$ which is saturated for pure states] is

$$\Delta\sigma_x^2 + \Delta\sigma_z^2 \geq 1. \tag{4}$$

An important reinterpretation of Eq. (4) is as an uncertainty relation for Mach-Zehnder interferometers in which one relates the predictability of the path taken by the particle to the visibility of the interference fringes; see, e.g., [5,6]. This has been extended to the case of multipath interferometers; see, e.g., [7,8]. Finally, we mention that other more information theoretic uncertainty relations, such as entropic uncertainty relations, have also been developed [9–12].

In the present work we derive uncertainty relations for two unitary operators that obey the commutation relation $UV = e^{i\phi}VU$. This uncertainty relation has several important applications: it provides an uncertainty relation for the discrete Fourier transform (DFT), and in this context provides a family of uncertainty relations that interpolate between the case of Pauli operators Eq. (4) and that of position and momentum Eq. (1); it also provides an uncertainty relation for modular variables; finally, it should prove useful in signal processing.

We also characterize the quantum states with minimum uncertainty in two bases related by the DFT. These states are discrete analogues of the coherent and squeezed states that are so important in the study of continuous variable systems. They have already been studied previously [13,14]. They are minimum energy eigenstates of Harper's equation [15], a discrete version of the harmonic oscillator Hamiltonian for continuous variables.

We begin by presenting the different applications, before stating and proving our results.

Discrete Fourier transform and modular variables.—Mutually unbiased bases have been extensively studied because of their nice properties and potential applications in quantum information. For instance they can be useful for quantum key distribution [16,17], for locking of quantum information [18], for string committement [19]. A particularly interesting case occurs when the two bases are related by a DFT:

$$|\tilde{\mathbf{k}}\rangle = \sum_{j=-[\frac{d}{2}]}^{[\frac{d-1}{2}]} \frac{e^{+i2\pi jk/d}}{\sqrt{d}}|\mathbf{j}\rangle, \qquad |\mathbf{j}\rangle = \sum_{k=-[\frac{d}{2}]}^{[\frac{d-1}{2}]} \frac{e^{-i2\pi jk/d}}{\sqrt{d}}|\tilde{\mathbf{k}}\rangle$$

with $\langle\mathbf{j}|\mathbf{j}'\rangle = \delta_{jj'}$, $\langle\tilde{\mathbf{k}}|\tilde{\mathbf{k}}'\rangle = \delta_{kk'}$ and j, j', k, $k' = -[\frac{d}{2}], \ldots, [\frac{d-1}{2}]$. This case finds applications in the Pegg-Barnett approach to phase-number uncertainty relations [20], and in multipath interferometers since the "symmetric multiport beam splitter" considered in [8] is just the DFT. The question we ask is, how much can a state be simultaneously localized both in the $|\mathbf{j}\rangle$ and the $|\tilde{\mathbf{k}}\rangle$ bases?

Because of the cyclic invariance of the DFT, it is natural to use a measure of localization which is invariant under cyclic permutations. To this end we introduce the unitary operators

$$U = \sum_{j=-[\frac{d}{2}]}^{[\frac{d-1}{2}]} e^{+i2\pi j/d}|\mathbf{j}\rangle\langle\mathbf{j}|, \quad V = \sum_{k=-[\frac{d}{2}]}^{[\frac{d-1}{2}]} e^{-i2\pi k/d}|\tilde{\mathbf{k}}\rangle\langle\tilde{\mathbf{k}}|. \quad (5)$$

We shall measure the localization in the two bases by the generalization of Eq. (2) to non-Hermitian operators:

$$\begin{aligned}\Delta U^2 &= \langle\psi|U^\dagger U|\psi\rangle - \langle\psi|U^\dagger|\psi\rangle\langle\psi|U|\psi\rangle \\ &= 1 - |\langle\psi|U|\psi\rangle|^2, \\ \Delta V^2 &= 1 - |\langle\psi|V|\psi\rangle|^2.\end{aligned} \quad (6)$$

The uncertainties ΔU^2 and ΔV^2 are the discrete versions of the *dispersion* introduced in [21]; see also [22]. Note that we have $0 \le \Delta U^2 \le 1$ and $0 \le \Delta V^2 \le 1$.

For further use let us collect here some important properties of the operators U and V. They can be written as

$$U = \sum_{k=-[\frac{d}{2}]}^{[\frac{d-1}{2}]} |\widetilde{\mathbf{k}+\mathbf{1}}\rangle\langle\tilde{\mathbf{k}}|, \qquad V = \sum_{j=-[\frac{d}{2}]}^{[\frac{d-1}{2}]} |\mathbf{j}+\mathbf{1}\rangle\langle\mathbf{j}| \quad (7)$$

and obey the commutation relations

$$U^n V^m = V^m U^n e^{+i2\pi nm/d}, \qquad U^{\dagger n} V^m = V^m U^{\dagger n} e^{-i2\pi nm/d}. \quad (8)$$

They also act as translation operators, since if

$$|\psi\rangle \to U^a V^{-b}|\psi\rangle, \quad (9)$$

then $\langle U\rangle \to e^{i2\pi b/d}\langle U\rangle$ and $\langle V\rangle \to e^{i2\pi a/d}\langle V\rangle$.

Our motivation for developing an uncertainty relation for the U and V operators is that the DFT interpolates between two important limits. In the $d = 2$ case we can identify $U = \sigma_x$ and $V = \sigma_z$ and the uncertainty relation Eq. (4) applies.

In supplementary material [23] we discuss in detail how in the limit $d \to \infty$ the DFT approximates the continuous Fourier transform (CFT). The idea is to rewrite $U = e^{iu\sqrt{2\pi/d}}$ and $V = e^{iv\sqrt{2\pi/d}}$, where u and v are Hermitian operators with eigenvalues $n\sqrt{2\pi/d}$, $n \in \{-[\frac{d}{2}], \ldots, [\frac{d-1}{2}]\}$, and to consider the class of states for which $1 - \langle\psi|U|\psi\rangle = \mu$ and $1 - \langle\psi|V|\psi\rangle = \mu'$ are both small complex numbers ($|\mu|, |\mu'| \ll 1$). This implies that $\Delta U^2 = O(|\mu|)$ and $\Delta V^2 = O(|\mu'|)$ are both very small. We then show that on such states one can approximate U and V by their series expansions: $U \simeq 1 + i\sqrt{\frac{2\pi}{d}}u - \frac{\pi}{d}u^2$ and $V \simeq 1 + i\sqrt{\frac{2\pi}{d}}v - \frac{\pi}{d}v^2$. This in turn implies that $\Delta U^2 \simeq \frac{2\pi}{d}(\langle u^2\rangle - \langle u\rangle^2)$ and $\Delta V^2 \simeq \frac{2\pi}{d}(\langle v^2\rangle - \langle v\rangle^2)$; i.e., ΔU^2 and ΔV^2 are proportional to the uncertainty of the operators u and v in the sense of Eq. (2). Furthermore, inserting the joint expansion into Eq. (8) we obtain $uv - vu \simeq i$. Thus, when acting on this class of states, u and v are analogues of the conjugate variables x and p. It then follows from Eq. (1) that $\Delta^2 U$ and ΔV^2 cannot both be made arbitrarily small, since when the above conditions hold they must obey the constraint

$$\Delta U^2 \Delta V^2 \ge \frac{\pi^2}{d^2}. \quad (10)$$

Note, however, that Eq. (10) does not hold when ΔU^2 or ΔV^2 are large. Indeed if we take states that are perfectly localized in one basis or in the other we have

$$|\psi\rangle = |\mathbf{j}\rangle \Rightarrow \Delta U = 0 \quad \text{and} \quad \Delta V = 1 \quad (11)$$

$$|\psi\rangle = |\tilde{\mathbf{k}}\rangle \Rightarrow \Delta U = 1 \quad \text{and} \quad \Delta V = 0. \quad (12)$$

One of our tasks is to find an uncertainty relation that correctly interpolates between the limits Eq. (10)–(12).

Modular variables.—An interesting generalization of the commutation relation Eq. (8) is provided by the translation operators $U = e^{i2\pi x/L}$ and $V = e^{-i2\pi p/P}$ which obey the commutation relations

$$UV = VUe^{i\Phi}, \qquad U^\dagger V = VU^\dagger e^{-i\Phi} \quad (13)$$

with $\Phi = 4\pi^2/LP$. In what follows we shall base our study on unitary operators that obey commutation relations of the type Eq. (13); i.e., we allow arbitrary values of Φ.

The generators $x(\mathrm{mod}L)$ and $p(\mathrm{mod}P)$ of the translation operators U and V are called modular variables. These were introduced in [24] as a tool for understanding nonlocal phenomena in quantum mechanics. Our uncertainty relation for U and V thus also provides an uncertainty relation for the modular variables.

Signal processing.—Uncertainty relations for U and V operators also have implications for signal processing.

On the one hand discrete generalizations of the Q function, the P function, and other discrete phase space functions always refer to a particular state. Minimum uncertainty states are thus natural candidates for these reference states, as discussed in detail in [25,26].

On the other hand we can express the quantum state $|\psi\rangle$ in the $|j\rangle$ basis $|\psi\rangle = \sum_j c_j|\mathbf{j}\rangle$ and reinterpret the c_j as a discrete signal of period d normalized to $\sum_j|c_\mathbf{j}|^2 = 1$. The discrete Fourier transform of the signal c_j is $\tilde{c}_k = \frac{1}{\sqrt{d}}\sum_j e^{-i2\pi jk/d}c_j$.

The fundamental theorem of signal processing, the Wiener-Kinchin theorem, states that the correlation function is the Fourier transform of the spectral intensity:

$$\sum_j c^*_{j+m}c_j = \sum_k e^{-i2\pi km/d}|\tilde{c}_k|^2 = \langle\psi|V^m|\psi\rangle. \tag{14}$$

In the quantum language it corresponds to the two different expressions for V, Eqs. (5) and (7).

Similarly the expectation value of U^n

$$\sum_j |c_j|^2 e^{i2\pi jn/d} = \sum_k \tilde{c}^*_{k+n}\tilde{c}_k = \langle\psi|U^n|\psi\rangle \tag{15}$$

is the Fourier transform of the intensity time series.

In view of this correspondence, our main result stated below provides a constraint between the values of the correlation function (14) and the Fourier transform of the intensity time series (15). This kind of constraint should prove useful in signal processing, as it constrains what kinds of signals are possible, or what kind of wavelet bases one can construct.

Results.—Our main result is

Theorem 1.—Consider two unitary operators U and V which obey

$$UV = VUe^{i\Phi}, \qquad U^\dagger V = VU^\dagger e^{-i\Phi}, \qquad 0 \le \Phi \le \pi \tag{16}$$

and define

$$\Delta U^2 = 1 - |\langle\psi|U|\psi\rangle|^2, \qquad \Delta V^2 = 1 - |\langle\psi|V|\psi\rangle|^2, \tag{17}$$

which are trivially bounded by $0 \le \Delta U^2 \le 1$, $0 \le \Delta V^2 \le 1$, and let

$$A = \tan\frac{\Phi}{2}, \qquad 0 \le A \le +\infty. \tag{18}$$

Then we have the bound

$$(1 + 2A)\Delta U^2\Delta V^2 + A^2(\Delta U^2 + \Delta V^2) \ge A^2. \tag{19}$$

The proof of Theorem 1 is given in the supplementary material [23].

Let us note that Theorem 1 correctly yields the expected asymptotic behaviors. To study the $d \to \infty$ limit, rewrite Eq. (19) as

$$\frac{\Delta U^2\Delta V^2}{A^2} \ge 1 - \left(\Delta U^2 + \Delta V^2 + \frac{2}{A}\Delta U^2\Delta V^2\right).$$

For large d we have $A \simeq \Phi/2 = \pi/d \to 0$. We then recover Eq. (10) when the terms in parenthesis on the right hand side are negligible in front of 1, that is when ΔU and ΔV are both sufficiently small.

In addition Eq. (19) is saturated by the two particular points Eqs. (11) and (12).

Finally, Eq. (19) gives the correct behavior when $d = 2$, Eq. (4). Indeed $d = 2$ is obtained as the limiting case $\Phi \to \pi$, corresponding to $A \to \infty$.

However, numerical investigations for small dimensionality d show that, except for $d = 2$, the bound is not tight, i.e., there are no states which saturate Eq. (19); see Fig. 1. On the other hand, as in [13], a tight bound can be obtained implicitly as the minimum eigenvalue of a Hermitian operator (Harper's equation), and the minimum uncertainty states are the associated eigenstates. To see this we change slightly our point of view, and instead of looking at the accessible region in the ΔU^2, ΔV^2 plane, we look at the accessible region in the $|\langle\psi|U|\psi\rangle|$, $|\langle\psi|V|\psi\rangle|$ plane. We state the following two results for finite dimensional spaces (leaving open the exact way in which they should be formulated for the infinite-dimensional case):

Theorem 2.—Consider a d-dimensional Hilbert space, and two unitary operators U, V acting on that space that obey the conditions of Theorem 1 with $\Phi = 2\pi/d$. Then the maximum of

$$\cos\theta|\langle\psi|U|\psi\rangle| + \sin\theta|\langle\psi|V|\psi\rangle|, \qquad 0 \le \theta \le \pi/2 \tag{20}$$

is given by the smallest eigenvalue of the Hermitian operator

$$H = -\cos\theta C_U - \sin\theta C_V, \qquad 0 \le \theta \le \pi/2 \tag{21}$$

where $C_U = (U + U^\dagger)/2$ and $C_V = (V + V^\dagger)/2$.

Note that Theorem 2 gives implicitly the boundary of the accessible region in the $|\langle U\rangle|$, $|\langle V\rangle|$ space (more precisely the convex hull of the accessible region). A comparison of the bound obtained from Theorem 1 and Theorem 2 in the case $\theta = \pi/4$ is given in Fig. 1.

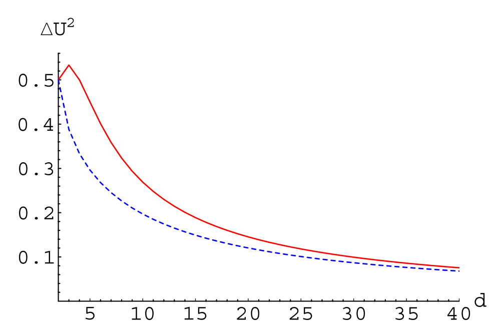

FIG. 1 (color online). Minimum uncertainty ΔU^2 as a function of dimension d when one imposes that $\Delta U^2 = \Delta V^2$. The upper (red and continuous) curve is the exact bound on ΔU^2. It is obtained from the smallest eigenvalue of the operator Eq. (21) when $\theta = \pi/4$. Note that when $d = 2$ and $d = 4$ the exact bound is $\Delta U^2 = 1/2$ and that when $d = 3$ the exact bound is larger than $1/2$, as noted in [13]. The lower (blue and dashed) curve is the bound obtained from the bound Eq. (19) upon imposing that $\Delta U^2 = \Delta V^2$. The two curves coincide when $d = 2$ and have the same asymptotic behavior $\Delta U^2 \ge \pi/d$ for large d.

A slight extension of the proof of Theorem 2 also provides a method to construct the states that saturate the uncertainty relation for U and V:

Theorem 3.—Consider a d dimensional Hilbert space, two unitary operators U, V, and the Hermitian operator H, as described in the statements of Theorems 1 and 2. Denote by $h_{\min}$ the smallest eigenvalue of H. Denote by $|\psi_{\min}\rangle$ the eigenvector corresponding to the smallest eigenvalue of H. Then the unique states that maximize Eq. (20) are the translates $U^a V^{-b}|\psi_{\min}\rangle$. [Remark: in the statement of Theorem 3 we have supposed that the smallest eigenvalue of H is nondegenerate. We expect this to be the case, but have not been able to prove it. If for some values of θ the smallest eigenvalue of H is degenerate, then denote by $|\psi_{\min,\pm,i}\rangle$ the quantum states that are both eigenstates of H with eigenvalue $h_{\min}$ and eigenstates of the operator $P = \sum_{j=-[d/2]}^{[d-1/2]} |-\mathbf{j}\rangle\langle\mathbf{j}|$ with eigenvalues ± 1, and where i labels any additional degeneracy. These states and their translates are the unique states that maximize Eq. (20).]

As discussed above when ΔU^2 and ΔV^2 are both small, and when d is large, the uncertainty relation for U and V reduces to the uncertainty relation for x and p. In this limit the Hamiltonian Eq. (21) reduces to

$$H = -(\cos\theta + \sin\theta)I + \tfrac{1}{2}(\cos\theta u^2 + \sin\theta v^2)$$

and the smallest eigenvalue of H is given by the smallest eigenvalue of $\cos\theta u^2 + \sin\theta v^2$. This suggests that we should interpret the ground states of H as discrete analogues of coherent states (for $\theta = \pi/4$) and squeezed states (for $\theta \neq \pi/4$). It is this correspondence which suggests that the largest eigenvalue of H is nondegenerate, since the smallest eigenvalue of $\cos\theta u^2 + \sin\theta v^2$ is nondegenerate. (This also shows that we can interpret the other eigenstates of H when $\theta = \pi/4$ as discrete analogues of the number states, i.e., the eigenstates of the harmonic oscillator). Note also that in the continuous limit the operator P tends to the parity operator that takes $x \to -x$ and $p \to -p$. This interpretation is discussed in detail in [13,14,25,26]. We refer, in particular, to [14] for plots of the eigenstates of H when $\theta = \pi/4$ and for a discussion of how they tend to the Hermite-Gauss functions in the $d \to \infty$ limit. Note that the equation $H|\psi\rangle = E|\psi\rangle$ is a finite dimensional version of Harper's equation [15], a well-studied equation in mathematical physics.

Conclusion.—In summary we have obtained an uncertainty relation for unitary operators U and V obeying the commutation relation Eq. (16) which has applications to signal processing, modular variables, and the DFT. In particular in the later context this uncertainty relation generalizes to the finite dimensional case the uncertainty relation for position and momentum Eq. (1), and reduces to the uncertainty relation for Pauli operators Eq. (4). We expect that our result will yield insights into other applications of uncertainty relations, such as the precision with which two noncommuting observables can be jointly observed, or the degree to which a ''fuzzy'' measurement of one observable perturbs the other observable.

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