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PLASMA OSCILLATIONS AND WAVES

Theory of Nonlinear Space Charge Waves in Neutralized Electron Flows: Gas-Dynamic Approach

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Abstract—The problem of the structure and dynamics of nonlinear longitudinal space charge waves in an electron flow that is treated as a gas and is described by an adiabatic equation of state with an arbitrary adiabatic index is solved exactly, and the solution obtained is examined in detail. The isothermal case is considered separately. It is shown that the waves in question can occur in the form of a fast or a slow space charge wave. The boundary values of the velocity and amplitude of these waves are obtained. The effect of the finite transverse dimensions of the electron flow on the structure of the wave and its dynamics is investigated.

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1. INTRODUCTION

Longitudinal electrostatic space charge waves (SCWs) in charge-neutralized electron flows play a key role in beam–plasma interactions. Such waves are also of fundamental importance in the processes occurring in beam–plasma microwave oscillators and amplifiers with long-term interaction and in plasma-based collective charged-particle accelerators.

Numerous experiments on SCWs in electron flows that have recently been carried out in many laboratories of the world (see, e.g., [1, 2]) force us to take a closer look at the existing theory and to consider whether everything has been done or whether there are some unclarified and unresolved issues.

Today, it is generally recognized that the linear theory of neutralized electron flows has been fully developed, and its description can be found in many monographs (see, e.g., [3, 4]).

The present status of the nonlinear theory is elucidated in review [5], which was published as early as 1993 but reflects the current state of affairs in the theory of nonlinear steady longitudinal SCWs, and the material presented there was repeated in a recent textbook [6] essentially without additional information.

According to [5, 6], the structure and dynamics of nonlinear steady SCWs in charge-neutralized cold electron flows unbounded in the transverse direction has been investigated in detail [7], the effect of the finite transverse dimensions of an electron flow on the parameters of SCWs has been considered [5], and the effect of the waveguide structure on the structure and stability of SCWs has been examined too [8, 9]. In all of the cited papers, the analysis was carried out using the hydrodynamic approach, which is valid for cold electron flows.

Fenstermacher and Seyler [10] made an attempt to consider SCWs in a warm electron flow by applying a gas-dynamic approach. They described the flow by the equation of state of an electron gas with the adiabatic index $\gamma = 2$. It was noted (see also [5]) that this choice of the γ value was not physically justified and was made only for mathematical convenience in obtaining an analytic solution.

In recent years, substantial progress has been achieved in the theory of nonlinear longitudinal electrostatic waves (specifically, ion acoustic plasma waves [11] and dust acoustic plasma waves [12, 13]) in other physical systems. In those papers, the problem of the dynamics of nonlinear waves was solved using a gas-dynamic approach in which the ion (or dust) plasma component was treated as a gas and was described by the adiabatic equation of state with an arbitrary adiabatic index γ. The objective of the present study, which was stimulated by the results achieved in [11–13], is to construct a theory of nonlinear steady SCWs in a charge-neutralized electron flow using a gas-dynamic approach in which the electrons are also treated as a gas and are described by an adiabatic equation of state with an arbitrary adiabatic index γ. Here, as is usually done in describing a collisionless plasma (see [14]), it is assumed that there is enough time for the flow to relax to a local thermodynamically equilibrium state due to uncorrelated Coulomb interactions between electrons.

The paper is organized as follows. In Section 2, the problem of the structure of a steady SCW is solved for a transversely bounded neutralized electron flow at a constant temperature ($\gamma = 1$). The reason for considering this simplest case is twofold. First, this is interesting from a methodological standpoint because it makes possible a detailed and comprehensive analysis of the mathematical solution method. Second, the solution for nonlinear isothermal SCWs in an electron flow is of interest by itself; to the best of my knowledge, it was not presented in the literature. Section 3 gives a complete solution of the main problem of SCWs in a transversely unbounded electron flow with an arbitrary adiabatic index γ. In Section 4, the results obtained are generalized to a transversely bounded electron flow. It is everywhere assumed that the neutralizing background is immobile and the flow is collisionless. The final section summarizes the main conclusions following from an analysis of the solutions derived.

2. ISOTHERMAL SPACE CHARGE WAVES

Let us consider a transversely unbounded steady uniform electron flow with the charge density ρ_0 that propagates along the x axis with the velocity v_0 through an immobile neutralizing background. We assume that the magnetic field applied in the x direction is strong enough to suppress the transverse electron motion.

We start with the following standard set of equations for the electron velocity v in the flow, the electron charge density ρ, the electron gas pressure P, and the electric potential φ:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0, \tag{1}$$

$$m\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) = e\frac{\partial \varphi}{\partial x} - \frac{e}{\rho}\frac{\partial P}{\partial x}, \tag{2}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = 4\pi(\rho - \rho_0), \tag{3}$$

where e and m are the charge and mass of an electron. Assuming that the electron gas is isothermal, we substitute the equation of state

$$P = \frac{\rho kT}{e} \tag{4}$$

into the equation of motion (2). We also introduce the notation

$$\eta = \frac{e}{m}, \quad \omega_p = \sqrt{4\pi\eta\rho_0}, \quad v_T = \sqrt{\frac{kT}{m}}, \quad \tau = \left(\frac{v_T}{v_0}\right)^2 \tag{5}$$

and normalize the variables and quantities as follows:

$$t = \frac{1}{\omega_p}t', \quad x = \frac{v_0}{\omega_p}x', \quad \rho = \rho_0\rho', \quad v = v_0 v', \quad \varphi = \frac{v_0^2}{\eta}\varphi'. \tag{6}$$

With the equation of state (4) and relationships (5) and (6), the basic equations become (hereafter, the primes by the dimensionless variables are omitted)

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0, \tag{7}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{\partial \varphi}{\partial x} - \tau\frac{\partial(\ln\rho)}{\partial x}, \tag{8}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \rho - 1. \tag{9}$$

We seek a solution to these equations in the form of a steady SCW running with the velocity u. To do this, we introduce the new self-similar variable

$$\xi = x - ut, \quad \frac{\partial}{dt} = -u\frac{d}{d\xi}, \quad \frac{\partial}{\partial x} = \frac{d}{d\xi}, \tag{10}$$

with which the set of partial differential equations (7)–(9) can be reduced to the ordinary differential equations

$$-u\frac{d\rho}{d\xi} + \frac{d(\rho v)}{d\xi} = 0, \tag{11}$$

$$-u\frac{dv}{d\xi} + v\frac{dv}{d\xi} = \frac{d\varphi}{d\xi} - \tau\frac{d(\ln\rho)}{d\xi}, \tag{12}$$

$$\frac{d^2\varphi}{d\xi^2} = \rho - 1. \tag{13}$$

Continuity equation (11) and equation of motion (12) can be simply solved by integration with allowance for the conditions $\rho = 1$ and $v = 1$ at $\varphi = 0$:

$$v = \frac{1}{\rho}(1 - u + \rho u), \tag{14}$$

$$u(1 - v) + \frac{1}{2}(v^2 - 1) = \varphi - \tau\ln\rho. \tag{15}$$

Substituting solution (14) into solution (15), we obtain the function $\varphi(\rho)$, which will play an important role in the analysis to follow:

$$\varphi = u\left[1 - \frac{1}{\rho}(1 - u + \rho u)\right] + \frac{1}{2}\left[\frac{1}{\rho^2}(1 - u + \rho u) - 1\right] + \tau\ln\rho. \tag{16}$$

The plot of this function is represented by a curve having a minimum. The function also should have a root at $\rho = 1$, indicating that the unperturbed "electron flow + neutralizing background" system is quasineutral. Figure 1 illustrates the dependence $\varphi(\rho)$ by two typical plots, which are such that the quasineutrality point lies on the right branch of one of the plots and on the left branch of the second. The branches that intersect the abscissa at other points are unphysical and should be discarded; in Fig. 1, they are shown by dots.

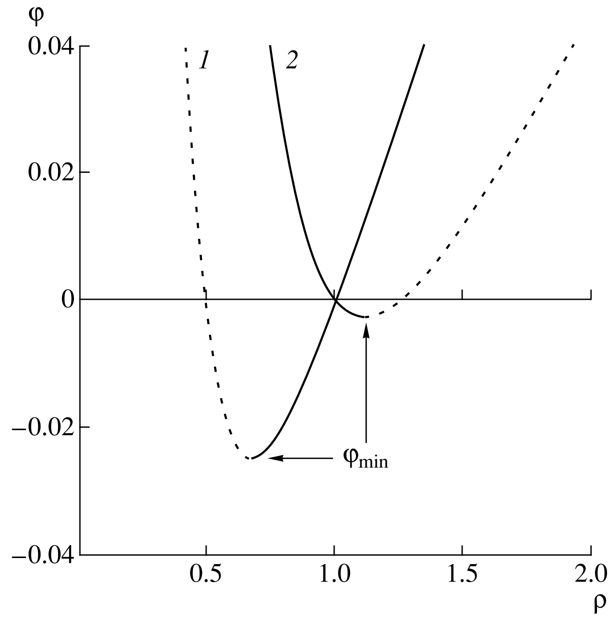

Fig. 1. Plots of the dependence $\varphi(\rho)$ (16) for (*1*) $u = 1.3$ and $\tau = 0.2$ and for (*2*) $u = 0.5$ and $\tau = 0.2$. The branches that are discarded are shown by dots.

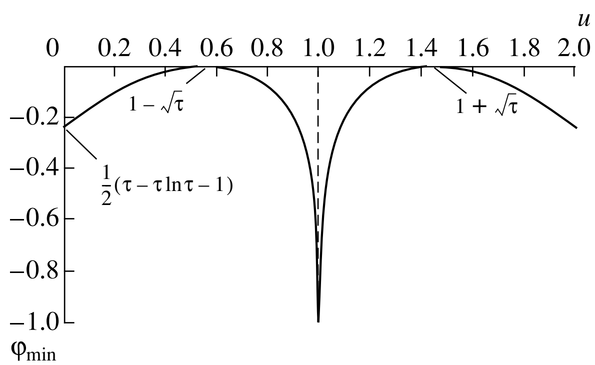

Fig. 2. Minimum value of the potential in an isothermal SCW vs. wave velocity, $\varphi_{min}(u)$, for $\tau = 0.2$.

In order to determine the position of the minimum in the function $\varphi(\rho)$, we find its derivative and equate it to zero:

$$\frac{d\varphi}{d\rho} = \frac{\rho^2\tau - (u-1)^2}{\rho^3} = 0. \tag{17}$$

As a result, we obtain

$$\rho_{max} = \begin{cases} \dfrac{u-1}{\sqrt{\tau}} & \text{for} \quad u > 1 \\[2ex] \dfrac{1-u}{\sqrt{\tau}} & \text{for} \quad u < 1 \end{cases} \tag{18}$$

and

$$\varphi_{min} = \begin{cases} -\dfrac{1}{2}u^2 + u + \tau\ln\left(\dfrac{u-1}{\sqrt{\tau}}\right) - \dfrac{1}{2} + \dfrac{\tau}{2} \\ \text{for} \quad u > 1 \\ -\dfrac{1}{2}u^2 + u + \tau\ln\left(\dfrac{1-u}{\sqrt{\tau}}\right) - \dfrac{1}{2} + \dfrac{\tau}{2} \\ \text{for} \quad u < 1. \end{cases} \tag{19}$$

Note that, in expression (18) for the electron charge density, we use the subscript max, because the density is maximum at the point where the electrostatic potential is minimum. We will demonstrate this again later.

In what follows, we will need the expression for the second derivative of function (16):

$$\frac{d^2\varphi}{d\rho^2} = -\frac{\rho^2\tau - 3(u-1)^2}{\rho^4}. \tag{20}$$

Figure 2 shows the dependence $\varphi_{min}(u)$ for a moderate value of τ ($\tau = 0.2$), i.e., for a warm electron flow. We can see that the entire range of possible values of the wave velocity u is divided into four subranges:

(i) $0 < u < 1 - \sqrt{\tau}$,

(ii) $1 - \sqrt{\tau} < u < 1$,

(iii) $1 < u < 1 + \sqrt{\tau}$,

(iv) $1 + \sqrt{\tau} < u < +\infty$,

the first two of which correspond to a slow SCW (SSCW), whereas the last two, to a fast SCW (FSCW). The coordinates of the characteristic points of the dependence $\varphi_{min}(\rho)$ are given in Fig. 2. One can readily see how the dependence $\varphi_{min}(\rho)$ changes as τ increases. For $\tau > 1$, i.e., when the "electron flow + neutralizing background" system can already be regarded as a plasma with a slowly drifting hot electron gas, subrange (i) disappears and the lower boundary of subrange (iv) is greatly displaced rightward.

Let us now consider Poisson's equation (13). Using the rule for differentiating a composite function,

$$\frac{d^2\varphi}{d\xi^2} = \frac{d\varphi}{d\rho}\frac{d^2\rho}{d\xi^2} + \frac{d^2\varphi}{d\rho^2}\left(\frac{d\rho}{d\xi}\right)^2, \tag{21}$$

and taking into account expressions (17) and (20), we reduce Poisson's equation (13) to an autonomous second-order differential equation for $\rho(\xi)$:

$$\rho[\rho^2\tau - (u-1)^2]\frac{d^2\rho}{d\xi^2} - [\rho^2\tau - 3(u-1)^2]\left(\frac{d\rho}{d\xi}\right)^2 = \rho^4(\rho - 1). \tag{22}$$

The order of the equation can be lowered by making the replacement $p(\rho) = d\rho/d\xi$. As a result, we arrive at Bernoulli's equation

$$\frac{dp}{d\rho} = f_1(\rho)p + f_n(\rho)p^n, \tag{23}$$

in which the terms on the right-hand side are defined as

$$\begin{aligned} n = -1, \quad f_1(\rho) &= \frac{1}{\rho}\frac{\rho^2\tau - 3(u-1)^2}{\rho^2\tau - (u-1)^2}, \\ f_{-1}(\rho) &= \frac{\rho^3(\rho - 1)}{\rho^2\tau - (u-1)^2}. \end{aligned} \tag{24}$$

Using the solution to Bernoulli's equation (see [15], Section 1.1.5) and returning to the sought-for variable $\rho(\xi)$, we can write the general solution to Eq. (22) in terms of two integration constants $C_{1,2}$:

$$\begin{aligned} &\xi + C_2 \\ &= \pm \int\limits_{r=\rho} \frac{[r^2\tau - (u-1)^2]dr}{r^2\sqrt{r^2(2r\tau - 2\tau\ln r + C_1) + (2r-1)(u-1)^2}}. \end{aligned} \tag{25}$$

Thus, we have obtained the general solution to the problem of the wave structure in an isothermal electron flow, with the integration constants $C_{1,2}$ being determined from the initial conditions corresponding to the amplitude of the initial perturbation.

Figure 3 shows two examples of the structure of a periodic SCW for the same values of the parameters u and τ but for different amplitudes of the initial perturbation. We can see that small perturbations evolve into nearly periodic, almost sinusoidal waves, while large perturbations grow into asymmetric waves with prolonged rarefaction and short compression phases. The waves resulting from larger perturbations can even break; in this case, steady solutions (25) do not exist. The phase diagram of an SCW in Fig. 4 shows how much the charge density profile $\rho(\xi)$ departs from a sinusoidal shape, i.e., how much closed phase trajectories departs from ellipses.

The maximum electron charge density in the compression phase, ρ_{max}, is easy to determine: it is exactly equal to the amplitude given by formula (18). The minimum electron charge density in the rarefaction phase, ρ_{min}, was calculated numerically.

The results of numerical analysis are illustrated in Fig. 5. We can see that, in subranges (ii) and (iii) of the values of the velocity u, steady SCWs cannot exist and that the steady waves in subranges (i) and (iv) are SSCWs and FSCWs, respectively. The maximum values of the electron charge density in the compression phase lie in the hatched regions above unity (i.e., above the horizontal lines) and cannot exceed the value ρ_{max}, while the minimum values in the rarefaction phase lie in the hatched regions below unity (i.e., below the horizontal lines) and cannot be smaller than the value ρ_{min}.

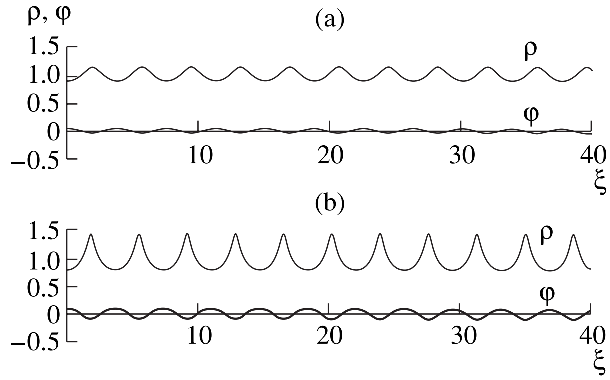

Fig. 3. Profiles $\rho(\xi)$ and $\varphi(\xi)$ in an isothermal SCW for $u = 1.75$ and $\tau = 0.2$ in the case of (a) a small-amplitude initial perturbation and (b) an initial perturbation with the maximum possible amplitude.

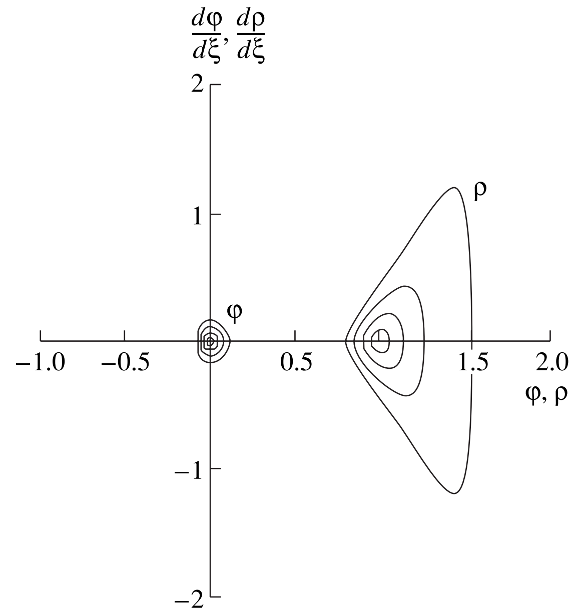

Fig. 4. Phase portrait of an isothermal SCW for $u = 1.75$ and $\tau = 0.2$.

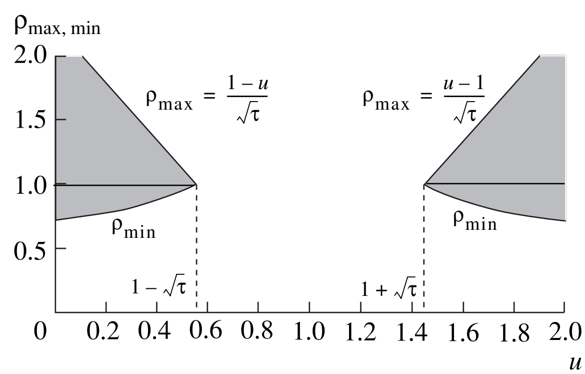

Fig. 5. Allowed amplitudes of an isothermal SCW vs. velocity u for $\tau = 0.2$ (the ranges of allowed amplitudes are hatched).

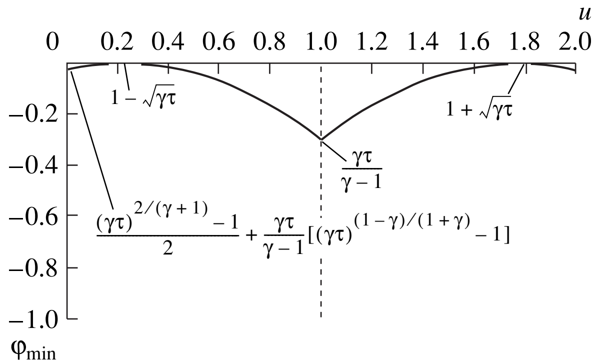

Fig. 6. Minimum value of the potential in an adiabatic SCW vs. wave velocity, $\varphi_{min}(u)$, for $\gamma = 3$ and $\tau = 0.2$.

We can readily see that the higher the temperature, the gentler the slope of the straight lines (18), with a resulting decrease in the wave amplitude ρ_{max}, and that SSCWs cannot exist in a plasma with hot electrons ($\tau > 1$). In the opposite limit $\tau \longrightarrow 0$, the hatched regions in Fig. 5 merge and steady SCWs can have arbitrary velocities u.

3. ADIABATIC SPACE CHARGE WAVES

Here, we consider adiabatic SCWs in an unbounded steady uniform electron flow propagating against an immobile neutralizing background. We make the same physical assumptions as in the previous section and also assume that the electron gas in the wave satisfies not Eq. (4) but the adiabatic equation

$$P = \frac{\rho_0}{e}kT_0\left(\frac{\rho}{\rho_0}\right)^{\gamma}, \tag{26}$$

where T_0 is the unperturbed electron gas temperature. For a thermal process, it is more justified to use this equation than Eq. (4) because it eliminates questions of external energy supply and energy sink for SCWs.

Keeping the notation and normalizing constants of Section 2, we immediately write out the following basic equations for the independent self-similar variable ξ, which are analogous to Eqs. (11)–(13):

$$-u\frac{d\rho}{d\xi} + \frac{d(\rho v)}{d\xi} = 0, \tag{27}$$

$$-u\frac{dv}{d\xi} + v\frac{dv}{d\xi} = \frac{d\varphi}{d\xi} - \gamma\tau\rho^{\gamma-2}\frac{d\rho}{d\xi}, \tag{28}$$

$$\frac{d^2\varphi}{d\xi^2} = \rho - 1. \tag{29}$$

The solution to continuity equation (27) has the form (14), and the equation of motion has the solution

$$u(1-v) + \frac{1}{2}(v^2 - 1) = \varphi - \frac{\gamma\tau}{\gamma-1}(\rho^{\gamma-1} - 1). \tag{30}$$

Substituting expression (14) into solution (30) yields the dependence $\varphi(\rho)$. For the allowed γ values, $1 \le \gamma \le 3$, the plot of the function $\varphi(\rho)$ is similar in shape to that in Fig. 1: it has a minimum and two rising branches, one of which is discarded and, accordingly, is not shown here.

The derivatives of the function $\varphi(\rho)$ have the form

$$\frac{d\varphi}{d\rho} = \frac{\gamma\tau\rho^{\gamma+1} - (u-1)^2}{\rho^3}, \tag{31}$$

$$\frac{d^2\varphi}{d\rho^2} = -\frac{\gamma\tau\rho^{\gamma+1}(2-\gamma) - 3(u-1)^2}{\rho^4}. \tag{32}$$

By analyzing formula (31), we can find ρ_{max} and φ_{min}:

$$\rho_{max} = \frac{(u-1)^{2/(\gamma+1)}}{(\gamma\tau)^{1/(\gamma+1)}}, \tag{33}$$

$$\begin{aligned}\varphi_{min} &= u(1-u) + \frac{1}{2}(u^2 - 1)\\ &+ \frac{\gamma\tau}{\gamma-1}\left\{\left[\frac{(u-1)^2}{\gamma\tau}\right]^{(\gamma-1)/(\gamma+1)} - 1\right\}\\ &+ \frac{(u-1)^2}{2}\left[\frac{(u-1)^2}{\gamma\tau}\right]^{-2/(\gamma+1)}.\end{aligned} \tag{34}$$

The plot of the dependence $\varphi_{min}(u)$ is displayed in Fig. 6, with the coordinates of the characteristic points indicated. The dependence is similar to that in Fig. 2.

We transform Poisson's equation (29) in the same way as was done with Eq. (9). Specifically, using derivatives (31) and (32), we convert it to an equation for $\rho(\xi)$:

$$\begin{aligned}&\rho[\gamma\tau\rho^{\gamma+1} - (u-1)^2]\frac{d^2\rho}{d\xi^2}\\ &- [\gamma\tau\rho^{\gamma+1}(2-\gamma) - 3(u-1)^2]\left(\frac{d\rho}{d\xi}\right)^2 = \rho^4(\rho - 1).\end{aligned} \tag{35}$$

This equation can in turn be reduced to Bernoulli's equation (23), in which the terms on the right-hand side have the form

$$\begin{aligned}n = -1, \quad f_1(\rho) &= \frac{1}{\rho}\frac{\gamma\tau\rho^{\gamma+1}(2-\gamma) - 3(u-1)^2}{\gamma\tau\rho^{\gamma+1} - (u-1)^2},\\ f_{-1}(\rho) &= \frac{\rho^3(\rho - 1)}{\gamma\tau\rho^{\gamma+1} - (u-1)^2}.\end{aligned} \tag{36}$$

Turning again to [15], we finally write the general solution to Eq. (35):

$$\xi + C_2 = \pm \int\limits_{r=\rho} \frac{\sqrt{1-\gamma}[\gamma\tau\rho^{\gamma+1} - (u-1)^2]^2 dr}{\sqrt{G(r, \gamma, \tau, u)}}, \quad (37)$$

where, for brevity, we have introduced the notation

$$\begin{aligned} G(r, \gamma, \tau, u) &= -2r^5 + \gamma^3 R_3(r, \gamma, \tau, u) \\ &+ \gamma^2 R_2(r, \gamma, \tau, u) + \gamma R_1(r, \gamma, \tau, u) + R_0(r, \gamma, \tau, u), \end{aligned} \quad (38)$$

$$\begin{aligned} R_3(r, \gamma, \tau, u) &= 2(r^{3\gamma+8} - r^{3\gamma+7})\tau^3 \\ &+ [(2r^{2\gamma+7} - r^{2\gamma+6})(u-1)^2 + C_1 r^{2\gamma+8}]\tau^2, \end{aligned} \quad (39)$$

$$\begin{aligned} R_2(r, \gamma, \tau, u) &= -2r^{3\gamma+8}\tau^3 \\ &+ [(5r^{2\gamma+6} - 6r^{2\gamma+7})(u-1)^2 - C_1 r^{2\gamma+8}]\tau^2 \\ &+ [2(r^{\gamma+5} - 2r^{\gamma+6})(u-1)^4 - C_1 r^{\gamma+7}(u-1)^2], \end{aligned} \quad (40)$$

$$\begin{aligned} R_1(r, \gamma, \tau, u) &= 4r^{2\gamma+7}(u-1)^2\tau^2 \\ &+ 2[(3r^{\gamma+6} - 2r^{\gamma+5})(u-1)^4 - C_1 r^{\gamma+7}(u-1)^2]\tau \\ &- (2r^5 - r^4)(u-1)^6 + C_1 r^6 (u-1)^4 - r^4, \end{aligned} \quad (41)$$

$$\begin{aligned} R_0(r, \gamma, \tau, u) &= -2r^{\gamma+6}(u-1)^4 \\ &+ (r^4 - 2r^5)(u-1)^6 - C_1 r^6 (u-1)^4 + r^4. \end{aligned} \quad (42)$$

Using the general solution just obtained, we can plot the profiles of an SCW (see Fig. 7a) and investigate the characteristic properties of the wave. Note that, now, the electron gas temperature in the wave is not constant: in the rarefaction phase, the electron temperature is lower than the initial temperature, and in the compression phase, it is higher. The temperature profile in the SCW was obtained by using the dimensionless adiabatic equation of state $T = \tau\rho^{\gamma-1}$. The phase diagram of the SCW is presented in Fig. 8.

Figure 9, which shows possible minimum and maximum values of the electron density in an SCW, is qualitatively similar to Fig. 5. However, quantitatively, the range of velocities u in which steady SCWs cannot exist is broader by an amount equal to the adiabatic index $\sqrt{\gamma}$. Note that, in accordance with expression (33), the upper boundary of the maximum values of the electron charge density in the compression phase, ρ_{max}, is not a straight line, in contrast to that in Fig. 5.

4. ADIABATIC SPACE CHARGE WAVES IN A TRANSVERSELY BOUNDED ELECTRON FLOW

Here, we consider SCWs in a transversely bounded electron flow. We begin by noting that the basic electron continuity equation and the basic equation of electron motion have the same form as those in Section 3. This is why all the conclusions of Section 3 that concern the boundaries of the ranges of existence of SSCWs and FSCWs, the maximum electron charge density, and the potential and temperature in the wave remain valid, as well as the plots shown in Figs. 6 and 9.

The problem in question differs from the previous one only in the form of the dimensionless Poisson's equation, which now reads

$$\Delta_\perp \varphi + \frac{\partial^2 \varphi}{\partial x^2} = \rho - 1, \quad (43)$$

where $\Delta_\perp$ is the transverse Laplace operator. Under certain conditions for the transverse dimensions of the flow, we can approximately set $\Delta_\perp \varphi \approx -k_\perp^2 \varphi$. Going through the same sequence of mathematical transformations as in the previous section, we reduce Poisson's equation (43) to Bernoulli's equation (23), in which the terms on the right-hand side have the form

$$n = -1, \quad f_1(\rho) = \frac{1}{\rho}\frac{\gamma\tau\rho^{\gamma+1}(2-\gamma) - 3(u-1)^2}{\gamma\tau\rho^{\gamma+1} - (u-1)^2},$$

$$f_{-1}(\rho) = \frac{\rho^3(\rho-1) + \rho^3 k_\perp^2 \left\{ u\left[1 - \frac{1}{\rho}(1-u+\rho u)\right] + \frac{1}{2}\left[\frac{1}{\rho^2}(1-u+\rho u)^2 - 1\right] + \frac{\gamma\tau}{\gamma-1}(\rho^{\gamma-1} - 1)\right\}}{\gamma\tau\rho^{\gamma+1} - (u-1)^2}. \quad (44)$$

Now, we can easily solve Poisson's equation, but the resulting general solution is rather involved, so we do not write it out here. An analysis of the solution shows that the transverse wavenumbers of the steady SCWs that can exist in a transversely bounded electron flow are such that $k_\perp < k_{\perp max}$, where $k_{\perp max}$ depends on the parameters u, γ, and τ, as well as on the perturbation amplitude. Figure 10 shows an example of the depen-

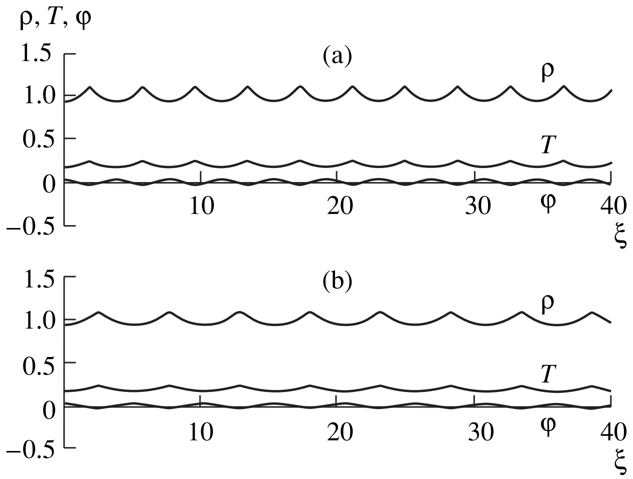

Fig. 7. Profiles $\rho(\xi)$, $T(\xi)$, and $\varphi(\xi)$ in an adiabatic SCW for $u = 2$, $\gamma = 3$, and $\tau = 0.2$ in the case of (a) a transversely unbounded electron flow and (b) a transversely bounded electron flow ($k_\perp = 1$).

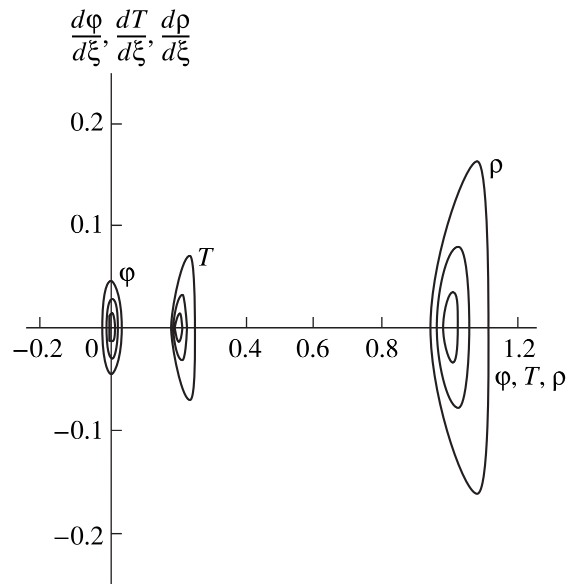

Fig. 8. Phase portrait of an adiabatic SCW for $u = 2$, $\gamma = 3$, and $\tau = 0.2$.

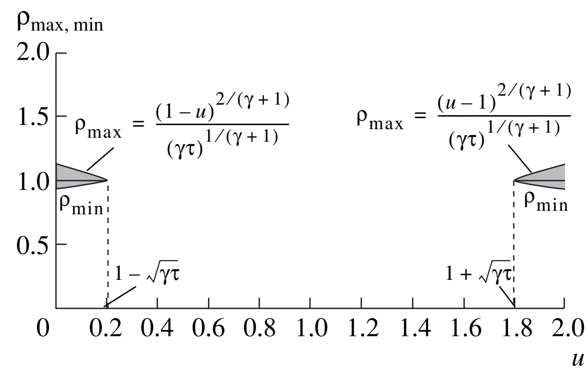

Fig. 9. Allowed amplitudes of an adiabatic SCW vs. velocity u for $\gamma = 3$ and $\tau = 0.2$ (the ranges of allowed amplitudes are hatched).

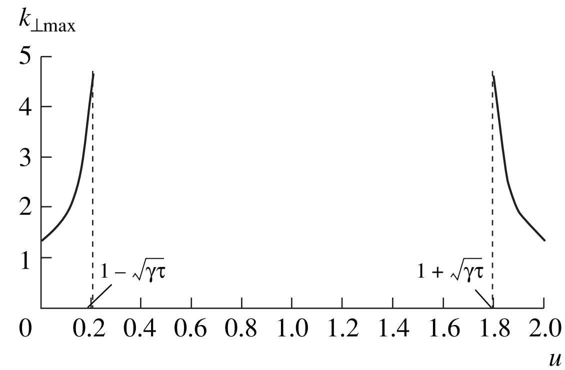

Fig. 10. Dependence $k_{\perp\max}(u)$ in an adiabatic SCW in a transversely bounded electron flow for $\gamma = 3$ and $\tau = 0.2$.

dence $k_{\perp\max}(u)$ for particular values of γ and τ and for the maximum possible perturbation amplitude determined from Fig. 9.

For comparison, Fig. 7b displays the structure of an SCW with the transverse wavenumber $k_\perp = 1$, calculated from solution (44) to Poisson's equation (43) for the same parameters and the same amplitude of the initial perturbation as those of a transversely unbounded electron flow in Fig. 6a. We can see that, in the range of the allowed transverse wavenumbers $k_\perp < k_{\perp\max}$, the only effect of the finite transverse dimensions of the flow is to increase the wavelength of the SCW without changing its shape.

5. CONCLUSIONS

In the present paper, the problem of the structure and dynamics of nonlinear steady longitudinal SCWs in an electron flow treated as a gas and described by an adiabatic equation of state with an arbitrary adiabatic index γ has been solved exactly and the solutions obtained have been examined in detail (isothermal SCWs with $\gamma = 1$ have been considered separately). The minimum and maximum electron densities as functions of the wave propagation velocity u have been determined. The analysis has revealed the following characteristic features of SCWs:

(i) there are two types of waves: SSCWs with velocities $u < 1 - \sqrt{\gamma\tau}$ and FSCWs with velocities $u > 1 + \sqrt{\gamma\tau}$, and there are no SCWs with velocities within the range $1 - \sqrt{\gamma\tau} < u < 1 + \sqrt{\gamma\tau}$;

(ii) the SCWs are asymmetric in shape: the deviations of the electron charge density from its equilibrium value in the compression and rarefaction phases are not equal to one another, $1 - \rho_{min} \neq \rho_{max} - 1$;

(iii) in a hot electron flow such that $\gamma\tau > 1$, steady SSCWs do not exist at all;

(iv) the transverse wavenumbers of the steady SCWs that can exist in a transversely bounded electron flow are bounded from above, $k_\perp < k_{\perp max}$; and

(v) in a transversely bounded electron flow, the wavelength of the SCWs with transverse wavenumbers within the range $0 < k_\perp < k_{\perp max}$ is longer than that of the waves in a transversely unbounded flow.

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