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Determination of the amplitude and the phase of the elements of electric cross-spectral density matrix by spectral measurements

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ABSTRACT

The amplitudes and the phases of the elements of electric cross-spectral density matrix are determined experimentally for a pair of points in the cross-section of an expanded laser beam. A modified version of the Young's interferometer is used as an experimental tool, which separates the beams emerging from the double-slit widely and provides ease in insertion of polarizers and half wave rotators in individual beams. To determine these complex elements of the cross-spectral density matrix, the experimentally obtained values of the spectral densities at an off-axis point are put in the mathematical expressions derived by us using the spectral interference law. The four complex generalized Stokes parameters are also determined using the linear combinations of the matrix elements. This unique but simple experimental approach for determining both the two-point parameters might provide a means to investigate the polarization and the coherence properties of the random electromagnetic beams on propagation.

1. Introduction

Recently developed *unified theory of coherence and polarization* proposes a two-point parameter namely the 2×2 electric cross-spectral density matrix [1], which plays an important role in characterizing the polarization and the coherence properties of the random electromagnetic fields simultaneously [2,3]. Extensive studies have been made in the last decade showing many applications of the cross-spectral density matrix in the domain of optics [4,5]. Of late, the spectral generalized Stokes parameters were presented [6], which were claimed to be two-point extension of the single-point usual Stokes parameters [7]. It was shown theoretically [8], that the generalized parameters play a role of modulating parameters to the combined two-beam usual Stokes parameters of a point in the cross-section of an electromagnetic beam. Due to many applications of these quantities in characterizing the vectorial properties of random electromagnetic beams on propagation [9,10], it is natural to propose and investigate experimental methods to determine the elements of these two-point quantities, which are complex in nature.

In the recent past, we proposed a method [11], in which the elements of the cross-spectral density matrix and the generalized Stokes parameters were determined using a modified version of the Young's interferometer and taking spectral measurements at an axial point in the observation plane where the fringe intensity was observed to be maximum. The spectral degree of coherence was found maximum at the axial point, and was equal to its absolute value (magnitude). This reduced both the complex two point parameters into real quantities and thus provided ease in determining them experimentally [11]. In general, it is difficult to define the optic axis and observations are taken at an off-axis point in practical situations. Therefore the parameters remain two-point correlation functions consisting of both the amplitude and the phase parts and complete information of these complex quantities is essential.

In the present paper, making use of the theory presented in Ref. [4] and the spectral interference law, we have derived explicit mathematical expressions for the complex degree of spectral coherence. With the help of these expressions, the magnitude and the phase of the elements of the cross-spectral density matrix are determined for a pair of points in the cross-section of a laser beam by taking spectral measurements at an off-axis point in the observation plane. The present method is an extension of the method proposed recently in Ref. [11] and provides a complete experimental realization of the theoretical approach made in Ref. [4]. A modified version of the Young's interferometer consisting of an additional arrangement of prisms and mirrors is used with a set of polarizers and half wave plates to determine the elements of the electric cross-spectral density matrix experimentally. Using the matrix elements, the complex generalized Stokes parameters are also determined for the same pair of points in the cross-section of the expanded laser beam.

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2. Mathematical preliminary

Let us consider that an electromagnetic beam propagating in z-direction is incident over an opaque screen placed on the plane $z = 0$ with two narrow slits symmetrically positioned about the line perpendicular to the plane as shown in Fig. 1. Let ρ_1 and ρ_2 be two points having position vectors $\mathbf{r_1}$ and $\mathbf{r_2}$, respectively, positioned at different slits. R_1 and R_2 are the distances of the off-axis point P (with position vector $\mathbf{r}$) from the points ρ_1 and ρ_2, respectively (Fig. 1). Considering the fluctuations of the beam stationary in wide sense, the second order coherence properties of the light beam can be given by a 2×2 electric cross-spectral density matrix [1]

$$\overleftrightarrow{W}(\mathbf{r}_1,\mathbf{r}_2,\omega) = W_{\mathrm{ij}}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \langle E_i^*(\mathbf{r}_1,\omega)E_j(\mathbf{r}_2,\omega)\rangle \quad \text{for}(i,j) = (x,y), \tag{1}$$

where the asterisk denotes the complex conjugate and angular brackets denote the ensemble average. According to the spectral interference law [7], for quasimonochromatic light around the frequency ω [4], the spectral density at any arbitrary point P($\mathbf{r}$) in the fringe pattern is given by

$$\varphi(\mathbf{r},\omega) = \varphi^1(\mathbf{r},\omega) + \varphi^2(\mathbf{r},\omega) + 2\sqrt{\varphi^1(\mathbf{r},\omega)}\sqrt{\varphi^2(\mathbf{r},\omega)} \times |\eta(\mathbf{r}_1,\mathbf{r}_2,\omega)|\cos\{\beta(\mathbf{r}_1,\mathbf{r}_2,\omega)+\delta\}, \tag{2}$$

where $\phi^i(\mathbf{r},\omega)$, for $(i = 1,2)$ are the spectral densities at point P due to individual slits ρ_i, respectively, and $\eta(\mathbf{r}_1,\mathbf{r}_2,\omega)$ is the spectral degree of coherence which is a complex parameter and is written as [4]

$$\eta(\mathbf{r}_1,\mathbf{r}_2,\omega) = |\eta(\mathbf{r}_1,\mathbf{r}_2,\omega)|\exp\{i\beta(\mathbf{r}_1,\mathbf{r}_2,\omega)\}. \tag{3}$$

In Eq. (2), $\beta(\mathbf{r}_1,\mathbf{r}_2,\omega)$ is a phase term and $\delta = \frac{\omega(R_1-R_2)}{c}$ is a parameter purely dependent on the path difference $R_1 - R_2$ (see Fig. 1). However, in case observations are made at a distant observation point P (Fig. 1) from the double-slit (in z direction) and slightly off-axis (in x direction) than for all practical purposes, we can assume $R_1 \approx R_2$ or $\delta \approx 0$.

The elements of the electric cross-spectral density matrix can be expressed in terms of the spectral densities at individual slits $\varphi(\mathbf{r}_i,\ \omega)$ for $(i = 1,2)$ and also the spectral degree of coherence $\eta(\mathbf{r}_1,\mathbf{r}_2,\omega)$ as [4,11]

$$\overleftrightarrow{W}(\mathbf{r}_1,\mathbf{r}_2,\omega) = W_{\mathrm{ij}}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \sqrt{\varphi_i(\mathbf{r}_1,\omega)}\sqrt{\varphi_j(\mathbf{r}_2,\omega)}\eta_{\mathrm{ij}}(\mathbf{r}_1,\mathbf{r}_2,\omega), \quad \text{for}(i,j) = (x,y). \tag{4}$$

It is observed from Eq. (4), that the magnitude and phase of the spectral degree of coherence $\eta_{ij}(\mathbf{r}_1,\mathbf{r}_2,\omega)$ determine the complex nature of the elements of the cross-spectral density matrix.

The magnitude of the spectral degree of coherence could be determined by measuring the visibility of the interference fringe around the point P. The visibility of the interference fringes, $v(\mathbf{r})$ is given by the relation [7]

$$v(\mathbf{r}) = |\eta(\mathbf{r}_1,\mathbf{r}_2,\omega)| = \frac{\varphi^{\max}(\mathbf{r}) - \varphi^{\min}(\mathbf{r})}{\varphi^{\max}(\mathbf{r}) + \varphi^{\min}(\mathbf{r})}, \tag{5}$$

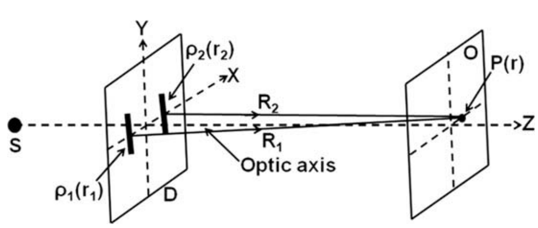

Fig. 1. Figure illustrating the notations. S source, ρ_1 and ρ_2 slits on the plane D, R_1 and R_2 are distances between point P($\mathbf{r}$) and slits, and O is the observation plane.

where $\varphi^{\max}(\mathbf{r})$ and $\varphi^{\min}(\mathbf{r})$ are the maximum and the minimum values of the spectral densities at point P($\mathbf{r}$).

To determine the phase part of the spectral degree of coherence, we use the spectral interference law. From Eqs. (2) and (3), for the condition when spectral densities at point P due to both the beams are approximately same, i.e. $\varphi^1(\mathbf{r},\omega) = \varphi^2(\mathbf{r},\omega)$, we have

$$\mathrm{Re}\eta(\mathbf{r}_1,\mathbf{r}_2,\omega) = \frac{\varphi(\mathbf{r},\omega) - 2\varphi^1(\mathbf{r},\omega)}{2\varphi^1(\mathbf{r},\omega)}, \tag{6}$$

where we have used $\mathrm{Re}\eta(\mathbf{r}_1,\mathbf{r}_2,\omega)=|\eta(\mathbf{r}_1,\mathbf{r}_2,\omega)|\cos\beta(\mathbf{r}_1,\mathbf{r}_2,\omega)$ [see Eq. (2)]. Using Eq. (4), we obtain

$$\cos\beta(\mathbf{r}_1,\mathbf{r}_2,\omega) = \frac{\mathrm{Re}\eta(\mathbf{r}_1,\mathbf{r}_2,\omega)}{v(\mathbf{r})}. \tag{7}$$

The imaginary part of the spectral degree of coherence is given by $\mathrm{Im}\eta(\mathbf{r}_1,\mathbf{r}_2,\omega) = |\eta(\mathbf{r}_1,\mathbf{r}_2,\omega)|\sin\beta(\mathbf{r}_1,\mathbf{r}_2,\omega)$ [see Eq. (2)], which after straight forward calculation gives the sine component as

$$\sin\beta(\mathbf{r}_1,\mathbf{r}_2,\omega) = \frac{\sqrt{\{v(\mathbf{r})\}^2 - \{\mathrm{Re}\eta(\mathbf{r}_1,\mathbf{r}_2,\omega)\}^2}}{v(\mathbf{r})}. \tag{8}$$

Using Eq. (3), the complex value of the spectral degree of coherence is obtained as

$$\eta(\mathbf{r}_1,\mathbf{r}_2,\omega) = v(\mathbf{r}) \times \left[\left\{\frac{\mathrm{Re}\eta(\mathbf{r}_1,\mathbf{r}_2,\omega)}{v(\mathbf{r})}\right\} + i\left\{\frac{\sqrt{\{v(\mathbf{r})\}^2 - \{\mathrm{Re}\eta(\mathbf{r}_1,\mathbf{r}_2,\omega)\}^2}}{v(\mathbf{r})}\right\}\right]. \tag{9}$$

The terms inside the square brackets in Eq. (9) are real (cosine) and imaginary (sine) values of the spectral degree of coherence, which could be obtained using experimental data in Eqs. (5), (7), and (8). The quantities $v_{ij}(\mathbf{r})$ and $\eta_{ij}(\mathbf{r}_1,\mathbf{r}_2,\omega)$ for $(i,j) = (x,y)$, can be determined experimentally by placing polarizers and half wave plates in front of the individual slits with appropriate orientation and by taking spectral measurements at and around the off-axis point in the observation plane. These values can be put in Eq. (4) which gives the complex values of the elements of electric cross-spectral density matrix for a pair of points in the cross-section of the electromagnetic beam.

The four generalized spectral Stokes parameters, which are complex quantities in nature and characterize the polarization properties of electromagnetic field at two spatial points in the cross-section of the beam, are given as

$$S_0(\mathbf{r}_1,\mathbf{r}_2,\omega) = \langle E_x^*(\mathbf{r}_1,\omega)E_x(\mathbf{r}_2,\omega)\rangle + \left\langle E_y^*(\mathbf{r}_1,\omega)E_y(\mathbf{r}_2,\omega)\right\rangle, \tag{10a}$$

$$S_1(\mathbf{r}_1,\mathbf{r}_2,\omega) = \langle E_x^*(\mathbf{r}_1,\omega)E_x(\mathbf{r}_2,\omega)\rangle - \left\langle E_y^*(\mathbf{r}_1,\omega)E_y(\mathbf{r}_2,\omega)\right\rangle, \tag{10b}$$

$$S_2(\mathbf{r}_1,\mathbf{r}_2,\omega) = \langle E_x^*(\mathbf{r}_1,\omega)E_y(\mathbf{r}_2,\omega)\rangle + \left\langle E_y^*(\mathbf{r}_1,\omega)E_x(\mathbf{r}_2,\omega)\right\rangle, \tag{10c}$$

$$S_3(\mathbf{r}_1,\mathbf{r}_2,\omega) = i\left[\left\langle E_y^*(\mathbf{r}_1,\omega)E_x(\mathbf{r}_2,\omega)\right\rangle - \langle E_x^*(\mathbf{r}_1,\omega)E_y(\mathbf{r}_2,\omega)\rangle\right]. \tag{10d}$$

These parameters can also be expressed in terms of the elements of the cross-spectral density matrix using Eq. (1) as [9]

$$S_0(\mathbf{r}_1,\mathbf{r}_2,\omega) = W_{xx}(\mathbf{r}_1,\mathbf{r}_2,\omega) + W_{yy}(\mathbf{r}_1,\mathbf{r}_2,\omega), \tag{11a}$$

$$S_1(\mathbf{r}_1,\mathbf{r}_2,\omega) = W_{xx}(\mathbf{r}_1,\mathbf{r}_2,\omega) - W_{yy}(\mathbf{r}_1,\mathbf{r}_2,\omega), \tag{11b}$$

$$S_2(\mathbf{r}_1,\mathbf{r}_2,\omega) = W_{xy}(\mathbf{r}_1,\mathbf{r}_2,\omega) + W_{yx}(\mathbf{r}_1,\mathbf{r}_2,\omega), \tag{11c}$$

$$S_3(\mathbf{r}_1,\mathbf{r}_2,\omega) = i[W_{yx}(\mathbf{r}_1,\mathbf{r}_2,\omega) - W_{xy}(\mathbf{r}_1,\mathbf{r}_2,\omega)], \tag{11d}$$

Using Eq. (11) along with Eq. (4), the two-point Stokes parameters can be determined easily from the elements of the electric cross-spectral density matrix.

3. Experimental details and results

The schematic of the experimental setup is similar to Fig. 1 of Ref. [11]. A randomly polarized He–Ne laser beam was expanded using a beam expander and was incident over a double-slit D (slit width = 150 μm, slit separation = 200 μm) put symmetrically in the beam path at a distance 20 cm from the expander [Fig. 2(a)]. Both the emerging beams after the double-slit were separated with each other by approximately 9 cm (in *x*-axis) and mixed after traversing a distance of 15 cm from the slit passing through a prism-mirror assembly called as modifying arrangement [Fig. 2(b)]. Approximately straight interference fringes were obtained in the observation plane at a distance of 160 cm from the double-slit, as shown in Fig. 2(c). The spectral measurements were taken at and around an off-axis point in the observation plane using a fibre-coupled spectrometer mounted over a micro-positioner and interfaced with a personal computer [Fig. 1(b)]. The specification and manufacturer of the optical components are the same as mentioned in Ref. [11].

To determine the *xx* element of the cross-spectral density matrix, two polarizers were introduced after the slits separately, both passing the *x* component of the electric field. The maximum and minimum values of the spectral densities were measured with the spectrometer moving the fiber tip around point P [Fig. 1(a)]. The spectral densities due to both the slits as well as due to individual slits were also measured at the off-axis point. Using the experimental data and with the help of Eq. (9), the magnitude and phase of the spectral degree of coherence $\eta_{xx}(\mathbf{r}_1,\mathbf{r}_2,\omega)$ was determined. The value of $\eta_{xx}(\mathbf{r}_1,\mathbf{r}_2,\omega)$ was put in Eq. (4), and using the measured values of the spectral densities at individual slits due to *x*-components of light, the *xx* element of the cross-spectral density matrix was determined. Measurements were taken using the above mentioned procedure for *yy* direction of polarizers also. This gives $\eta_{yy}(\mathbf{r}_1,\mathbf{r}_2,\omega)$ and the *yy* element of the electric cross-spectral density matrix [12].

The *xy* element of the cross-spectral density matrix was determined when one of the polarizers transmitting the *x* component of the electric field was put at one slit (ρ_1) and the other was placed at ρ_2 passing the y component. As orthogonally polarized waves do not interfere, a half wave plate (*W*) having optic axis at 45° with the incident polarization was introduced in the arm of the interferometer after the polarizer P_1 (Fig. 1) which rotated

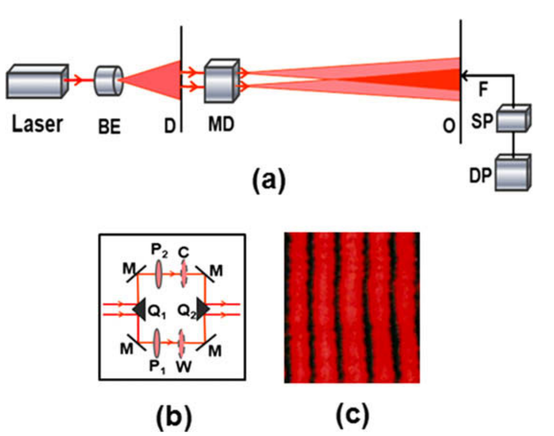

Fig. 2. (a) The schematic diagram of the experimental set up. Notations BE beam expander, MD modifying arrangement, F fiber, S spectrometer and DP personal computer. (b) The modifying arrangement (MD) consisting of mirrors (M) and reflecting prisms (Q). Polarizers (P) and half wave plates (C, W) are placed according to the experimental requirement. (c) Photograph showing interference fringes at the observation plane O obtained using the modified version of the Young's interferometer.

Table 1
Elements of the spectral visibility and the complex values of the spectral degree of coherence.

$v_{xx}(\mathbf{r})$	0.83 ± 0.01	$\eta_{xx}(\mathbf{r}_1,\mathbf{r}_2\omega)$	(0.70 ± 0.01) + i(0.40 ± 0.03)
$v_{xy}(\mathbf{r})$	0.31 ± 0.02	$\eta_{xy}(\mathbf{r}_1,\mathbf{r}_2,\omega)$	(−0.15 ± 0.02) + i(0.28 ± 0.01)
$v_{yx}(\mathbf{r})$	0.25 ± 0.02	$\eta_{yx}(\mathbf{r}_1,\mathbf{r}_2\omega)$	(−0.26 ± 0.02) + i(0.15 ± 0.02)
$v_{yy}(\mathbf{r})$	0.72 ± 0.01	$\eta_{yy}(\mathbf{r}_1,\mathbf{r}_2\omega)$	(0.61 ± 0.01) + i(0.45 ± 0.03)

Table 2
Elements of the electric cross-spectral density matrix and the generalized Stokes parameters.

$W_{xx}(\mathbf{r}_1,\mathbf{r}_2,\omega)$	(622 ± 9.8) + i(353.7 ± 6)	$S_0(\mathbf{r}_1,\mathbf{r}_2,\omega)$	(1107 ± 13) + i(722.7 ± 8.3)
$W_{xy}(\mathbf{r}_1,\mathbf{r}_2,\omega)$	(−115.3 ± 4) + i(210 ± 6.4)	$S_1(\mathbf{r}_1,\mathbf{r}_2,\omega)$	(137 ± 13) + i(−15.3 ± 8.3)
$W_{yx}(\mathbf{r}_1,\mathbf{r}_2,\omega)$	(−193 ± 5.3) + i(108 ± 4.4)	$S_2(\mathbf{r}_1,\mathbf{r}_2,\omega)$	(308.3 ± 6.4) + i(318 ± 7.1)
$W_{yy}(\mathbf{r}_1,\mathbf{r}_2,\omega)$	(485 ± 8.1) + i(369 ± 6.5)	$S_3(\mathbf{r}_1,\mathbf{r}_2,\omega)$	(102 ± 7.1) + i(−78 ± 6.4)

the incident polarization of light by 90°. The path compensation was done by inserting another identical wave plate (C) in the other arm with optic axis along the incident polarization (inactive for polarization rotation). The spectral measurements for the single slits as well as for both the slits were made at the observation plane and the magnitude and the phase of the spectral degree of coherence $\eta_{xy}(\mathbf{r}_1,\mathbf{r}_2,\omega)$ was determined. The *xy* element of the cross-spectral density matrix was obtained from Eq. (4), using $\eta_{xy}(\mathbf{r}_1,\mathbf{r}_2,\omega)$ and the *x* and the *y* components of the spectral densities at individual slits. Rotating both the polarizers by 90°, and taking similar measurements, $\eta_{yx}(\mathbf{r}_1,\mathbf{r}_2,\omega)$ and the *yx* element of the electric cross-spectral density matrix was determined [12].

The magnitudes of the spectral degree of coherence [spectral visibility $v_{ij}(\mathbf{r})$ for $(i,j) = (x,y)$] for different orientations of the polarizers and half wave plates were determined experimentally and are shown in Table 1. The complex values of the elements of the spectral degree of coherence were determined for a pair of points in the cross-section of the laser beam using the above mentioned procedure and are shown in Table 1. It is observed from the experimentally determined values that the visibility of the interference fringes obtained for the principle diagonal elements is higher than that obtained for the off-diagonal elements. This was the same finding as mentioned in Ref. [11] and thus provides validity of the experimental method. This difference in visibility was due to the fact that in principle, for the fully randomly polarized light (laser beam in our case), the orthogonal components of the electric field are completely uncorrelated. However, in practical situations the laser has slight partial polarization, i.e. a little correlation exists between the orthogonal electric field components, resulting into interference fringes with very small visibility [11].

Using the complex values of the spectral degree of coherence and the experimental data, the real and imaginary parts of the elements of the electric cross-spectral density matrix were obtained for a pair of points as shown in Table 2. The measured matrix elements are used here to determine the complex generalized Stokes parameters. Using Eq. (11) and the complex elements of cross-spectral density matrix from Table 2, the four generalized Stokes parameters were determined for the same pair of points. The uncertainty in the measurements was obtained by repeating the experiments three times and is shown with the experimental results in Tables 1 and 2. The usual practice for calculating uncertainty in the measurement is to repeat an experiment more than ten times. In this experiment the uncertainty has been calculated on the basis of the data obtained by three measurements. In this case the uncertainty in the measurement is rather a large number than the usual calculations.

4. Conclusion

To conclude, the complex values of the elements of electric cross-spectral density matrix and the generalized Stokes parameters for a pair of points in the cross-section of an expanded laser beam are determined experimentally using a set of polarizers and phase retarders in a modified version of the Young's interferometer. The coherence theory in space-frequency domain is utilized to derive the mathematical expressions. Determination of the imaginary part along with the real part of these parameters may be useful in determining various vectorial properties of the electromagnetic beams especially for astrophysical measurements. The two-point nature of the parameters also provides method to determine the change in polarization properties of a random electromagnetic beam on propagation.

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[12] In a modified version of the Young's interferometer, the interference fringes obtained at observation plane O, for each of these four conditions were shown in Fig. 2 of Ref. [11].