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SYSTEMS OF TOTAL DIFFERENTIAL EQUATIONS DEFINED OVER SIMPLY CONNECTED DOMAINS

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1. Consider a system of total differential equations

$$d\varphi^i = \sum_{\alpha=1}^{n} \Psi^i_\alpha (x, \varphi)\, dx^\alpha, \qquad (i = 1, \cdots, M), \tag{1.1}$$

where the functions Ψ are defined over an open simply connected domain D of n dimensions, which can be covered by a system of coordinates x^α, and for arbitrary values of the φ's, i.e. for $-\infty < \varphi^i < +\infty$. All quantities involved are real. As a matter of convenience we shall refer to the domain D of the variables x^α and the range $-\infty < \varphi^i < +\infty$ of the variables φ^i, when considered simultaneously, as the domain Δ.

It is assumed that the functions Ψ are continuous and possess continuous first partial derivatives with respect to the x^α and the φ^i. As an additional assumption, the reason for which will be seen immediately, we impose the condition that the derivatives $\partial\Psi^i_\alpha/\partial\varphi^k$ are bounded in Δ, i.e.

$$\left| \frac{\partial \Psi^i_\alpha}{\partial \varphi^k} \right| < N \tag{1.2}$$

in Δ, where N is a sufficiently large positive constant.

The system of equations

$$\frac{\partial \varphi^i}{\partial x^\alpha} = \Psi^i_\alpha (x, \varphi) \tag{1.3}$$

is completely equivalent to (1.1). Owing to the above assumptions we can form the conditions of integrability of (1.3), namely

$$\frac{\partial \Psi^i_\alpha}{\partial x^\beta} - \frac{\partial \Psi^i_\beta}{\partial x^\alpha} + \sum_{k=1}^{M} \frac{\partial \Psi^i_\alpha}{\partial \varphi^k} \Psi^k_\beta - \sum_{k=1}^{M} \frac{\partial \Psi^i_\beta}{\partial \varphi^k} \Psi^k_\alpha = 0, \tag{1.4}$$

the left members of which are defined in Δ; we assume that (1.4) is satisfied identically in this domain.

In this note we give a simple straightforward proof of the existence theorem for the above system. The method used is in the main similar to that employed by E. Cartan in his book *Géométrie des espaces de Riemann*, 1928, pp. 54–57; the system treated by Cartan is, however, less general than the above system (1.1) and is considered from a point of view peculiar to his special problem.

2. Let P and Q be any two distinct points in D with coordinates x_0^α and x_1^α respectively. Join P and Q by a curve C in D, the curve C being defined by $x^\alpha = f^\alpha(t)$, where the functions f are single valued and have continuous first derivatives; this is possible since D is connected. Suppose $f^\alpha(0) = x_0^\alpha$ and $f^\alpha(1) = x_1^\alpha$ so that t has the range $0 \leqq t \leqq 1$. From (1.3) we obtain

$$\frac{d\varphi^i}{dt} = \sum_{\alpha=1}^{n} \Psi_\alpha^i\,(f(t), \varphi)\,\frac{df^\alpha}{dt} \equiv \Psi^i\,(t, \varphi), \tag{2.1}$$

which we regard as equations for the determination of the functions φ^i along C. In consequence of the condition (1.2) and the continuity of the functions Ψ_α^i and f^α we have, for the range $0 \leqq t \leqq 1$ and $-\infty < \varphi^i < +\infty$ of the variables t and φ^i respectively, that (a) the functions Ψ^i are continuous and (b) the derivatives $\partial\Psi^i/\partial\varphi^k$ are bounded. By the theorem of Picard[1] it therefore follows that the equations (2.1) have a solution $\varphi^i(t)$ defined uniquely on C by the arbitrary values $\varphi_0^i = \varphi^i(0)$ of these functions at the point P. The values of the functions $\varphi^i(t)$ at the point Q are then $\varphi^i(1)$. *This proves that if the system* (1.1) *has a solution* $\varphi^i(x)$ *defined over D which takes on the arbitrarily given values* φ_0^i *at the point P it can have at most one such solution.* We proceed to prove the existence of such a solution.

3. Join the above points P and Q by another curve C_1 defined parametrically by the equations $x^\alpha = f_1^\alpha(t)$ where $0 \leqq t \leqq 1$, the functions f_1^α being single valued and continuous with continuous first derivatives. We shall show that from the arbitrarily assigned values φ_0^i of the functions φ^i at the point P we shall obtain, by integration of a system of the type (2.1), the same values of the φ^i at Q regardless of whether the integration is carried out with reference to the curve C or the curve C_1.[2]

Since D is simply connected the closed curve formed by C and C_1 can be shrunk to a point. Hence we can pass from C to C_1 by a continuous one parameter family F of such curves, which we represent by

$$x^\alpha = G^\alpha(t, p)\,, \qquad (0 \leqq t \leqq 1,\ 0 \leqq p \leqq 1)\,,$$

such that $x^\alpha = x_0^\alpha$ for $t = 0$ and $x^\alpha = x_1^\alpha$ for $t = 1$ independently of the parameter p. We select functions G^α which possess continuous derivatives

$$\frac{\partial G^\alpha}{\partial t}\,, \qquad \frac{\partial G^\alpha}{\partial p}\,, \qquad \frac{\partial^2 G^\alpha}{\partial t \partial p}\,,$$

[1] E. Picard, Traité d'Analyse, Vol. II, 1925, p. 373.

[2] It is sufficient to assume that the curves C and C_1 have no points in common except their end points P and Q and that these curves define different directions at each of their end points. In fact if these conditions are not satisfied by C and C_1 we can take a curve C', analogous to C and C_1, such that the above conditions are satisfied by the pair C and C' and also by the pair C_1 and C'. Then the result that the values of the φ^i at the point Q are the same when determined by integration of (2.1) along C as when determined by integration along C_1 will follow from the corresponding result with reference first to the curves C and C' and second to the curves C_1 and C'.

the second derivatives of the G^α being independent of the order of the differentiation.

Now consider the system

$$\frac{d\varphi^i}{dt} = \sum_{\alpha=1}^{n} \Psi^i_\alpha \, (G(t,p), \varphi) \, \frac{\partial G^\alpha}{\partial t} \tag{3.1}$$

along any curve of the family F. Taking $\varphi^i = \varphi^i_0$ for $t = 0$, independently of the parameter p, the equations (3.1) determine a unique set of functions $\varphi^i\,(p, t)$ defined for $0 \leqq t \leqq 1$ and $0 \leqq p \leqq 1$. These functions $\varphi^i(p, t)$ are continuous in the variables p and t and possess continuous derivatives $\partial\varphi^i/\partial t$ and $\partial\varphi^i/\partial p$ for all values of the variables for which they are defined. In fact it can be shown that the derivatives $\partial\varphi^i/\partial p$ satisfy the equations[3]

$$\frac{\partial}{\partial t}\left(\frac{\partial\varphi^i}{\partial p}\right) = \sum_{k=1}^{M}\sum_{\alpha=1}^{n} \frac{\partial\Psi^i_\alpha}{\partial\varphi^k}\frac{\partial\varphi^k}{\partial p}\frac{\partial G^\alpha}{\partial t} + \frac{\partial}{\partial p}\sum_{\alpha=1}^{n} \Psi^i_\alpha \frac{\partial G^\alpha}{\partial t}.$$

Hence the derivatives $\partial^2\varphi^i/\partial p\partial t$ which appear in the left members of these equations exist and are continuous; likewise the existence and continuity of the derivatives $\partial^2\varphi^i/\partial t\partial p$ result by differentiation of the equations (3.1).
Hence it follows that[4]

$$\frac{\partial^2\varphi^i}{\partial p\partial t} = \frac{\partial^2\varphi^i}{\partial t\partial p}, \tag{3.2}$$

for $0 \leqq t \leqq 1$ and $0 \leqq p \leqq 1$, i.e. these second derivatives are independent of the order of differentiation.

Now form the equations

$$\frac{\partial\varphi^i}{\partial t} = \sum_{\alpha=1}^{n} \Psi^i_\alpha \frac{\partial G^\alpha}{\partial t}, \tag{3.3}$$

$$\frac{\partial\varphi^i}{\partial t} = \sum_{\alpha=1}^{n} \Psi^i_\alpha \frac{\partial G^\alpha}{\partial p} + \sigma^i, \tag{3.4}$$

the functions σ^i being defined by these latter equations. Then by differentiation of (3.3) and (3.4) we obtain

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[3] See, Frank-v. Mises, *Differentialgleichungen der Physik*, 2nd Ed., Braumschweig 1930, p. 287, where these results are proved for the case of a single equation. The extension to a system of equations, such as the above equations (3.1), is immediate.

[4] See, for example, E. Goursat, Cours d'Analyse Mathématique, 5th Ed., Paris, 1927, p. 42.

$$\frac{\partial^2 \varphi^i}{\partial t \partial p} = \sum_{k=1}^{M} \left(\sum_{\alpha=1}^{n} \frac{\partial \Psi^i_\alpha}{\partial \varphi^k} \frac{\partial x^\alpha}{\partial t} \right) \left(\sum_{\beta=1}^{n} \Psi^k_\beta \frac{\partial x^\beta}{\partial p} + \sigma^k \right)$$

$$+ \sum_{\alpha=1}^{n} \Psi^i_\alpha \frac{\partial^2 x^\alpha}{\partial t \partial p} + \sum_{\alpha,\beta=1}^{n} \frac{\partial \Psi^i_\alpha}{\partial x^\beta} \frac{\partial x^\alpha}{\partial t} \frac{\partial x^\beta}{\partial p},$$

$$\frac{\partial^2 \varphi^i}{\partial p \partial t} = \sum_{\alpha=1}^{n} \Psi^i_\alpha \frac{\partial^2 x^\alpha}{\partial p \partial t} + \frac{\partial \sigma^i}{\partial t} + \sum_{\alpha,\beta=1}^{n} \frac{\partial \Psi^i_\alpha}{\partial x^\beta} \frac{\partial x^\alpha}{\partial p} \frac{\partial x^\beta}{\partial t}$$

$$+ \sum_{k=1}^{M} \sum_{\alpha,\beta=1}^{n} \frac{\partial \Psi^i_\alpha}{\partial \varphi^k} \Psi^k_\beta \frac{\partial x^\alpha}{\partial p} \frac{\partial x^\beta}{\partial t}.$$

Subtracting corresponding members of these equations leads, on account of (3.2), to the set of equations

$$\frac{\partial \sigma^i}{\partial t} = \sum_{k=1}^{M} \sum_{\alpha=1}^{n} \frac{\partial \Psi^i_\alpha}{\partial \varphi^k} \frac{\partial x^\alpha}{\partial t} \sigma^k, \tag{3.5}$$

when use is made of the integrability conditions (1.4).

Since $\varphi^i = \varphi^i_0$ and $x^\alpha = x^\alpha_0$ for $t = 0$, independently of p, we have from (3.4) that $\sigma^i = 0$ for $t = 0$. Hence from (3.5) it follows that $\sigma^i = 0$ along any curve of parameter p so that, in particular, $\sigma^i = 0$ for $t = 1$. Then from (3.4) the derivatives $\partial\varphi^i/\partial p = 0$ for $t = 1$. Hence the value of φ^i at the point Q is independent of the curve of integration of the family F by which the point P is joined to the point Q. *In other words there exists a set of functions $\varphi^i(x)$ defined throughout the domain D, these functions being uniquely determined by the assignment of their values φ^i_0 at an arbitrary point P and the process of integration of the system* (2.1) *along curves of the type C issuing from P.*

4. Now consider the values $\varphi^i(\bar{x})$ of the above functions $\varphi^i(x)$ at an arbitrary point $\bar{P}$ of the domain D. Starting with the point $\bar{P}$ and the values $\varphi^i(\bar{x})$ we can determine by the above process, i.e. by integration of (2.1) along curves C issuing from P, a set of functions $\bar{\varphi}^i(x)$ analogous to the functions $\varphi^i(x)$. It is then evident that the functions $\bar{\varphi}^i(x)$ will take the values of the corresponding functions $\varphi^i(x)$ at the point P used in the determination of these latter functions; also to determine the values of the functions $\bar{\varphi}^i$ at any point Q of D we can integrate (2.1) along a curve C which passes through the point P. *Hence the functions $\bar{\varphi}^i(x)$ at which we arrive, when we start with any other point $\bar{P}$ of the domain D in the process of the determination of these functions, are identical with the functions $\varphi^i(x)$ provided that the values of the functions $\bar{\varphi}^i(x)$ at the point $\bar{P}$ are the same as the values of the corresponding functions $\varphi^i(x)$ at this point.*

It remains to observe that the above functions $\varphi^i(x)$ satisfy the system (1.1). But this is seen immediately. In fact let Q be any point of the domain D and take the curve C as the curve of parameter x^α passing through Q. Then, in view of the last italicized result, the equations (2.1) are satisfied along this curve C by the functions $\varphi^i(x)$; this gives

$$\frac{\partial \varphi^i(x)}{\partial x^\alpha} = \Psi^i_\alpha\,(x, \varphi(x))$$

at all points of the domain D.

THEOREM. *Let D be an open simply connected domain of n dimensions which can be covered by a system of coordinates x^α and denote by Δ the domain composed of D and the range $-\infty < \varphi^i < +\infty$ simultaneously. Suppose that the functions $\Psi^i_\alpha(x, \varphi)$ defined in Δ are continuous and possess continuous first partial derivatives in this domain; suppose, furthermore, that the equations* (1.4) *are satisfied identically and that the derivatives $\partial\Psi^i_\alpha/\partial\varphi^k$ are bounded in Δ. Then the system* (1.1) *admits a unique solution $\varphi^i(x)$ defined throughout the domain D such that the functions $\varphi^i(x)$ take an arbitrary set of values φ^i_0 at an arbitrary point P of D.*

5. In particular the domain D may be homeomorphic to the interior of an n dimensional Euclidean hypersphere Σ; this is the simplest and undoubtedly most important special case of the above theorem. Going out from this special case we can extend the Theorem to open or closed simply connected spaces S each point of which belongs to a neighborhood capable of being put into one to one reciprocal correspondence with the interior of Σ.[5] Such a space S can therefore be covered by one or more coordinate systems. Assuming the scalar character of the functions φ^i, and the property of differentiability of the coordinate relations throughout portions of S common to two coordinate systems, it is evident that the above discussion will continue to apply since the equations involved are invariant under coordinate transformations. In the statement of the above theorem the domain D can accordingly be replaced by the space S.

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[5] For a precise characterization of the space S see E. Cartan, *La théorie des groupes finis et continus et l'Analysis Situs*, Mém. des Sciences. Math. No. 42, Gauthier-Villars, 1930, p. 3. In the postulates for the space S closed neighborhoods are used by Cartan. The corresponding postulates involving open neighborhoods are given by T. Y. Thomas, *The Differential Invariants of Generalized Spaces*, Cambridge University Press, 1934, p. 1.