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Two-dimensional effects in nonlinear Kronig-Penney models

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An analysis of two-dimensional (2D) effects in the nonlinear Kronig-Penney model is presented. We establish an effective one-dimensional description of the 2D effects, resulting in a set of pseudodifferential equations. The stationary states of the 2D system and their stability is studied in the framework of these equations. In particular it is shown that localized stationary states exist only in a finite interval of the excitation power. [S0163-1829(97)52220-X]

There is a growing interest in the subject of wave propagation in nonlinear photonic band-gap materials and in periodic nonlinear dielectric superlattices.[1] The basic dynamics in these systems is described by the fundamental nonlinear Schrödinger (NLS) equation

$$i\partial_t\psi+\nabla^2\psi+f(\vec{r},|\psi|^2)\psi=0, \tag{1}$$

where $\psi(\vec{r},t)$ is the complex amplitude of quasimonochromatic wave trains, the variable t is time, and $\vec{r}$ is the spatial coordinate. The function $f(\vec{r},|\psi|^2)$ characterizes the nonlinearity of the medium, e.g., the nonlinear corrections of the refractive index of the photonic band-gap materials or the self-interaction of the quasiparticles in the superlattices. In the case of periodic nonlinear superlattices consisting of alternating layers of two dielectrics, it is usually assumed that the nonlinearity of one of the dielectrics is much larger than the nonlinearity of the other, so that the latter can be considered linear. If the thickness of the nonlinear layer is small compared to the de Broglie wavelength within the layer, the problem can be described by the nonlinear Kronig-Penney model[2] with the nonlinearity $f(\vec{r},|\psi|^2)$ in the form

$$f(\vec{r},|\psi|^2)=\sum_n \delta(x-x_n)|\psi(\vec{r},t)|^2 \tag{2}$$

corresponding to a focusing medium with cubic nonlinearity. Here $x_n=nl$ is the coordinate of the nth nonlinear layer and l is the distance between adjacent nonlinear layers. Wave propagation in the framework of the one-dimensional (1D) nonlinear Kronig-Penney model was studied in detail in Refs. 3–5, but in these works the coupling between the longitudinal and transversal degrees of freedom was ignored. It was shown that the transmission properties depend critically on the injected wave power. Additionally, it was shown that these systems exhibit bistability and multistability.

In the present paper we consider 2D effects in the nonlinear Kronig-Penney model given by Eqs. (1) and (2), where the complex amplitude depends on the coordinate x transversal to the nonlinear layers and the longitudinal coordinate z. Denoting with an overbar $\overline{\psi}$, the Fourier transform with respect to t and z, one can represent Eqs. (1) and (2) in the form

$$-(\omega+k^2)\overline{\psi}+\partial_x^2\overline{\psi}+\sum_n \delta(x-x_n)\overline{|\psi|^2\psi}=0. \tag{3}$$

Similarly to the approach used in Refs. 3 and 5, we can solve these equations in the linear medium and thereby express the field $\psi(x,z;t)$ for $nl\leqslant x\leqslant(n+1)l$ in terms of the complex amplitudes $\psi_n(z,t)\equiv\psi(x_n,z;t)$ at the nonlinear layers,

$$\psi(x,z;t)=\frac{\sinh\{\hat{\kappa}[(n+1)l-x]\}}{\sinh(\hat{\kappa}l)}\psi_n(z,t)+\frac{\sinh[\hat{\kappa}(x-nl)]}{\sinh(\hat{\kappa}l)}\psi_{n+1}(z,t), \tag{4}$$

where the complex amplitude $\psi_n(z,t)$ satisfies the set of pseudodifferential equations

$$\frac{\hat{\kappa}}{\sinh l\hat{\kappa}}(\psi_{n+1}+\psi_{n-1})-\frac{2\hat{\kappa}}{\tanh l\hat{\kappa}}\psi_n+|\psi_n|^2\psi_n=0, \tag{5}$$

with periodic boundary conditions $\psi_{n+N}=\psi_n$, where N is the number of layers. In Eqs. (4) and (5) the operator $\hat{\kappa}$ is defined as $\hat{\kappa}\psi=\sqrt{-i\partial_t-\partial_z^2}\psi$ or expressed in the Fourier domain $\hat{\kappa}\overline{\psi}=\sqrt{\omega+k^2}\overline{\psi}$.

In passing it is worth noting the following two limits where the system (5) reduces to systems previously discussed in the literature. First, considering the ordering

$$\partial_t\sim\partial_z^2\sim\epsilon,\quad \psi_n\sim\sqrt{\epsilon},\quad \psi_{n+1}+\psi_{n-1}-2\psi_n\sim\epsilon\psi_n$$

for $\epsilon\to 0$, Eq. (5) reduces to

$$l^2(i\partial_t+\partial_z^2)\psi_n+\psi_{n+1}+\psi_{n-1}-2\psi_n+l|\psi_n|^2\psi_n=0, \tag{6}$$

which is the so-called discrete-continuum NLS equation introduced by Aceves *et al.*[6] to describe soliton dynamics in nonlinear optical fiber arrays. Second, increasing the distance l between the nonlinear layers, the interlayer coupling [the first term on the left-hand side of Eq. (5)] vanishes and the equation takes the form

$$2\sqrt{-i\partial_t-\partial_z^2}\psi_n-|\psi_n|^2\psi_n=0, \tag{7}$$

which for static distributions $\partial_t\psi=0$ reduces to the nonlinear Hilbert NLS equation introduced recently by Gaididei *et al.*[7]

In what follows we will be interested in stationary states of the system and therefore study solutions of the form

$$\psi_n(z,t)=\phi_n(z)e^{i\lambda^2 t}, \tag{8}$$

where λ^2 is the nonlinear frequency and $\phi_n(z)$ the amplitude in the nth nonlinear layer. Since Eq. (5) is Galilean invariant, standing excitations can always be Galileo boosted to any velocity in the z direction. Introducing the ansatz (8) into Eq. (5), we obtain

$$\frac{\sqrt{\lambda^2-\partial_z^2}}{\sinh(l\sqrt{\lambda^2-\partial_z^2})}(\phi_{n+1}+\phi_{n-1}) -2\frac{\sqrt{\lambda^2-\partial_z^2}}{\tanh(l\sqrt{\lambda^2-\partial_z^2})}\phi_n+|\phi_n|^2\phi_n=0. \tag{9}$$

Note that for z-independent amplitudes ϕ_n and $\lambda^2=-k^2$ the set of equations (9) reduces to the algebraic equations considered in Refs. 3–5. We shall consider spatially localized (in the z direction) solutions ($\lambda^2>0$).

Equation (1) has as an integral of the motion the power

$$P=\int_{-\infty}^{\infty}dx\,dz|\psi|^2=\frac{1}{2}\sum_n\frac{\partial}{\partial\lambda^2}\int_{-\infty}^{\infty}dz|\phi_n|^4, \tag{10}$$

where the last equality in Eq. (10) is obtained using Eqs. (1), (2) and (8). Considering a finite number N of nonlinear layers, a physically reasonable excitation pattern is $\phi_n(z)=\phi(z)$, where the complex amplitudes ψ_n are the same in all nonlinear layers. For this excitation pattern the real-valued profile $\phi(z)$ should satisfy the equation

$$\mathcal{L}(\partial_z)\phi+\phi^3=0, \tag{11}$$

where the dispersion operator $\mathcal{L}$ has the form

$$\mathcal{L}(\partial_z)=-2\sqrt{\lambda^2-\partial_z^2}\tanh\left(\frac{l}{2}\sqrt{\lambda^2-\partial_z^2}\right). \tag{12}$$

The natural longitudinal extension of the excitation in this system will clearly be determined by the values q_j where the dispersion operator $\mathcal{L}(q)$ vanishes. From Eq. (12) it follows that

$$q_j^2=\lambda^2+\left(\frac{2j\pi}{l}\right)^2,\quad j=0,1,2,\ldots. \tag{13}$$

In the limit where the nonlinear frequency λ^2 is small we therefore expect nonlinear excitations with an extension much larger than $1/q_j$ ($j\geq 1$) so that only the scale q_0^{-1} is important. A Padé approximation of degree (2,2) (Ref. 8) for the operator $\mathcal{L}$ with respect to the variable $\lambda^2-\partial_z^2$ is therefore appropriate and yields

$$-\frac{12}{l}\frac{\lambda^2-\partial_z^2}{\frac{12}{l^2}+\lambda^2-\partial_z^2}\phi+\phi^3=0. \tag{14}$$

Gaididei *et al.*[9] have previously investigated this type of equation and shown that under the boundary conditions $\phi(z)\to 0$ for $z\to\pm\infty$ the solution only exists for $\lambda\leq\lambda_c\equiv\sqrt{3/2l^2}$. From the analysis of Ref. 9 the power P can be found analytically and it can be seen that $P(\lambda)$ is a nonmonotonic function with a local maximum. The conclusion that stationary states exist only in a finite interval $0\leq P\leq P_m=P(\lambda_m)$ is confirmed by direct numerical simulation of Eq. (9). Figure 1 shows that the agreement with the numerically obtained result is good for $\lambda^2\leq 1/2$ (in the simulations the lattice spacing was $l=1$), but for intermediate values of the nonlinear frequency λ^2 there is only qualitative agreement. The discrepancy for larger values of the nonlinear frequency and, in particular, the existence of the limiting value λ_c is due to the approximate character of Eq. (14) since it was obtained from Eqs. (11) and (12) in the limit of small λ. The numerical solution of Eqs. (11) and (12) shows that the stationary states exist for any values of λ, but for $\lambda>\lambda_m$ the power P is a monotonically decreasing function of λ. This asymptotic behavior can easily be understood since the scaling transformation $\phi=\lambda^{1/2}R(\zeta)$, $\zeta=\lambda z$, together with the assumption $\lambda l\gg 1$, reduces Eqs. (11) and (12) to

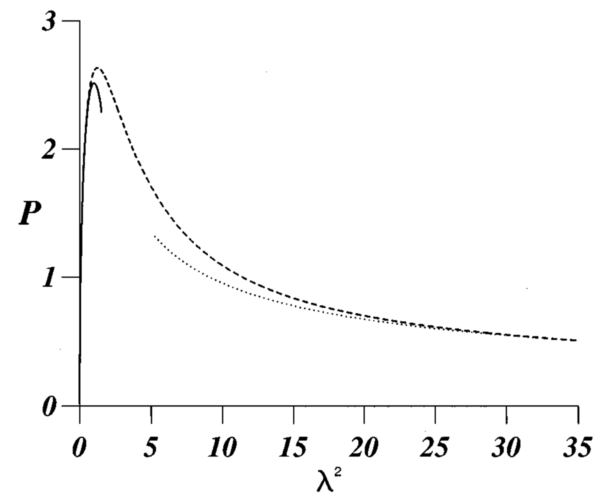

FIG. 1. Power P of the stationary state $\psi_n(z,t)=\phi(z,t)e^{i\lambda^2 t}$ versus the nonlinear frequency λ^2. Numerical result from Eq. (11) (dashed line), Padé approximation (14) (full line), and the asymptotic relation $P\sim 1/\lambda$ as $\lambda\to\infty$ (dotted line). The spacing between the nonlinear layers is $l=1$.

$$-2\sqrt{1-\partial_\zeta^2}R+R^3=0, \tag{15}$$

which is independent of λ. The applied scaling therefore yields $P\sim 1/\lambda$ as $\lambda\to\infty$, which agrees with the results of the numerical simulations (see Fig. 1). A numerically obtained example of the excitations described by Eqs. (11) and (4) is shown for $\lambda^2=5.0$ in Fig. 2.

Discussing the stability of the stationary states satisfying Eq. (11), there are two sources of instability to be considered: longitudinal and transversal perturbations. The perturbations of the first type are of the same symmetry with respect to the transversal degrees of freedom as the stationary state Eq. (11), while the second type of perturbations breaks this symmetry. The role of transversal perturbations is considered in the small amplitude limit [Eq. (6)] of Eq. (5). Linearizing Eq. (6) around the solution $\psi_n(z,t)=e^{i\lambda^2 t}\phi(z)$, the spectral problem

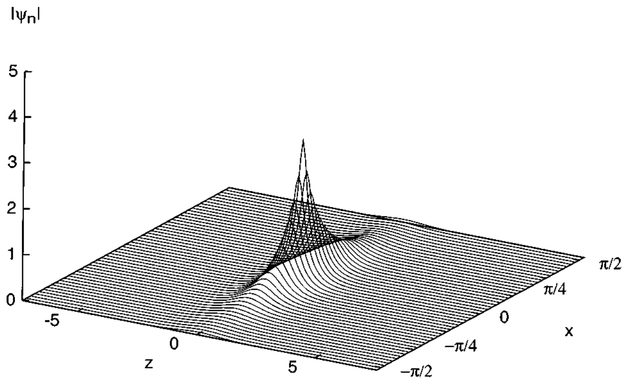

FIG. 2. Stationary state localized in the longitudinal direction with $\lambda^2=5.0$ The spacing between the nonlinear layers is $l=\pi$, so the nonlinear layers are positioned at $x=\ldots,-\pi,0,\pi,\ldots$.

$$\left[-l^2\partial_z^2+\lambda^2+l\phi^2+4\sin^2\left(\frac{ql}{2}\right)\right]u_n+\mu l^2 v_n=0,$$

$$\left[-l^2\partial_z^2+\lambda^2+3l\phi^2+4\sin^2\left(\frac{ql}{2}\right)\right]v_n-\mu l^2 u_n=0 \quad (16)$$

governs the evolution of perturbations $\psi_n(z,t)=[\phi(z)+u_n(z,t)+iv_n(z,t)]\exp(i\lambda^2 t)$, $u_n,v_n\sim\exp(iqn+\mu t)$. These equations are similar to those determining the soliton stability in 2D NLS models.[10] Using the results of Ref. 10 we get

$$\mu^2=\frac{12}{(\pi^2-6)l^2}\left[3\lambda^2 l^2-4\sin^2\left(\frac{ql}{2}\right)\right]. \quad (17)$$

Thus the growth rate of the perturbations is real and an instability occurs for perturbations with wave numbers q satisfying $4\sin^2(ql/2)<3\lambda^2 l^2$.

Studying the role of longitudinal perturbations, we use the fact that Eq. (1) is the Euler-Lagrange equation for the action

$$S=\int_{-\infty}^{\infty}dt\int d\vec{r}\left(\frac{i}{2}(\partial_t\psi^*\psi-\partial_t\psi\psi^*)+\psi^*\nabla^2\psi+\frac{1}{2}\sum_n\delta(x-x_n)|\psi|^4\right). \quad (18)$$

Assuming that $\Phi(\vec{r},\Lambda)e^{i\Lambda t}$ is the stationary solution of Eq. (1), we shall investigate the longitudinal stability of the stationary state using a variational approach with a trial function in the form

$$\psi(\vec{r},t)=\sqrt{\frac{\mathcal{P}}{P}}\Phi(\vec{r},\Lambda(t))e^{i\alpha(t)|\vec{r}|}, \quad (19)$$

where $\Lambda(t)$ and $\alpha(t)$ are real time-dependent variational parameters, $|\vec{r}|=\sqrt{x^2+z^2}$, and $P(t)=\int d\vec{r}\Phi^2(\vec{r},\Lambda(t))$ is the power that corresponds to the real-valued state $\Phi(\vec{r},\Lambda)$ with Λ being an arbitrary function of t. Note that the function (19) automatically satisfies the normalization condition (10) with a power $\mathcal{P}$. Inserting Eq. (19) into Eq. (18), we obtain the effective action

$$S=\mathcal{P}\left[J(\Lambda)\frac{d\alpha}{dt}-\alpha^2+\Lambda-\frac{W}{P}\right]+\frac{1}{2}\frac{\mathcal{P}^2}{P^2}W, \quad (20)$$

where

$$W=\sum_n\int d\vec{r}\,\delta(x-x_n)\Phi^4,\quad J=\int d\vec{r}|\vec{r}|\Phi^2. \quad (21)$$

W is the effective nonlinear interaction, while the parameter J characterizes the spatial distribution of the excitation. The Euler-Lagrange equations for the action (20) can be reduced to

$$M(\Lambda)\frac{d^2\Lambda}{dt^2}+\frac{1}{2}\frac{dM(\Lambda)}{d\Lambda}\left(\frac{d\Lambda}{dt}\right)^2=k(\Lambda)\left(\frac{\mathcal{P}}{P}-1\right), \quad (22)$$

where

$$M(\Lambda)=\frac{1}{2}\left(\frac{dJ}{d\Lambda}\right)^2,\quad k(\Lambda)=1+W\frac{d}{d\Lambda}\frac{1}{P}. \quad (23)$$

The stationary points Λ_s of Eq. (22) are then determined by the equality $P=\mathcal{P}$. For small deviations $\delta=\Lambda-\Lambda_s$ from the stationary state we obtain from Eq. (22)

$$M(\Lambda_s)\frac{d^2\delta}{dt^2}-\mathcal{P}k(\Lambda_s)\left(\frac{d}{d\Lambda}\frac{1}{P}\right)_{\Lambda_s}\delta=0, \quad (24)$$

and for a positive-definite nonlinear term W the condition for instability reads

$$\left(\frac{dP}{d\Lambda}\right)_{\Lambda_s}<0. \quad (25)$$

An equation of the same structure as Eq. (22) was recently obtained for a 1D NLS equation by Pelinovsky *et al.*[11] in the framework of perturbation theory, which is valid near the threshold of the soliton instability. Here and in our approach the equations describe in an approximate way the dynamics of the excitations.

At this point it is appropriate to note that Eq. (22) can be considered as an equation of motion for an effective particle with the kinetic energy T and potential energy U given by

$$T=\frac{1}{2}M(\Lambda)\left(\frac{d\Lambda}{dt}\right)^2,\quad U=\frac{1}{\mathcal{P}}H\{\Phi\}, \quad (26)$$

where

$$H\{\Phi\}=-\Lambda\mathcal{P}+\frac{W\mathcal{P}}{P}-\frac{1}{2}\frac{W\mathcal{P}^2}{P^2} \quad (27)$$

is the Hamiltonian of the system in the stationary state $\Phi(\vec{r},\Lambda)$. In accordance with Eqs. (26) and (27), the minima of the effective potential U correspond to the stable stationary states and the dynamics governed by Eq. (22) has an oscillatory character, while the maxima of the function U correspond to unstable stationary states.

Returning to the system described by Eq. (1) with a nonlinear term given by Eq. (2), we see (Fig. 1) that stationary states defined by Eqs. (8) and (11) are unstable with respect to longitudinal perturbations for

$$\lambda^2 l^2 > \lambda_m^2, \tag{28}$$

where the right-hand side of this inequality represents the value of the nonlinear frequency for which the power P in the system with the interlayer spacing $l=1$ reaches its maximum value $\lambda_m^2 \simeq 1.25$. Combining the inequality given in connection with Eq. (17) and the inequality (28), we expect stable stationary solutions of the form given in Eqs. (8) and (11) for nonlinear frequencies satisfying the condition

$$\lambda^2 < \frac{1}{l^2}\min\left\{\frac{4}{3}\sin^2\left(\frac{\pi}{N}\right), \lambda_m^2\right\}. \tag{29}$$

In particular, this means that the stationary state $\psi_n(z,t) = e^{i\lambda^2 t}\phi(z)$ is stable neither in the case of only one nonlinear layer $(l \to \infty)$ nor in the quasicontinuum limit $(N \to \infty)$.

Assuming a large number of nonlinear layers, we now consider the general properties of the stationary states localized in the transversal as well as the longitudinal direction. Considering in Eq. (9) the nonlinear term as an inhomogeneity, we obtain

$$\overline{\phi_n}(k) = \frac{1}{2}\sum_m G_{n-m}(k,\lambda)\overline{|\phi_m|^2\phi_m}(k), \tag{30}$$

where the lattice Green's function is

$$G_{n-m}(k,\lambda) = \frac{\sinh(l\sqrt{\lambda^2+k^2})}{\sqrt{\lambda^2+k^2}}\frac{1}{N} \times \sum_q \frac{e^{iql(n-m)}}{\cosh(l\sqrt{\lambda^2+k^2}) - \cos(ql)} \to \frac{1}{\sqrt{\lambda^2+k^2}} e^{-|n-m| l \sqrt{\lambda^2+k^2}} \tag{31}$$

as $N \to \infty$. In terms of the spatial variables z and n we therefore obtain

$$\phi_n(z) = \frac{1}{2}\sum_m \int_{-\infty}^{\infty} dy K_0(\lambda\sqrt{(n-m)^2 l^2 + (z-y)^2}) \times |\phi_m(y)|^2 \phi_m(y), \tag{32}$$

where $K_0(z)$ is the modified Bessel function of the second kind. The properties of the soliton in the 2D NLS model are recovered for small λ since the sum in Eq. (32) in this limit can be approximated by an integral, where the kernel $K_0(\lambda\sqrt{x^2+z^2})$ is the Green's function for the 2D Helmholtz equation $(\nabla^2 - \lambda^2)g(x,z) = 0$.[12] In particular, this means that the power P has a finite value as $\lambda \to 0$.

For finite nonlinear frequencies λ^2 the asymptotic far field $(\lambda\sqrt{l^2n^2+z^2} \gg 1)$ behavior of the stationary solution $\phi_n(z)$ is given by

$$\phi_n(z) \sim K_0(\lambda\sqrt{l^2n^2+z^2}) \sim \left(\frac{\pi}{2\lambda\sqrt{l^2n^2+z^2}}\right)^{1/2} \exp(-\lambda\sqrt{l^2n^2+z^2}), \tag{33}$$

so the excitation is exponentially localized in both spatial directions. Using a scaling argument similar to that used in Eq. (15), we again obtain that the power P is monotonically decreasing, $P \sim \lambda^{-1}$ as $\lambda \to \infty$. This result points to longitudinal instability of the localized stationary state.

In summary, we have studied 2D effects in the nonlinear Kronig-Penney model and shown that the problem can be reduced to a set of 1D pseudodifferential equations that are generalizations of the equations obtained in the equivalent 1D problem. We have shown that the system has stationary states that are uniform in all nonlinear layers, and the constraints under which these stationary states are stable have been determined. In particular we have found that the localized stationary states only exists in a finite interval of the excitation power P. Finally, we have shown that the system also permits states that are localized in both spatial direction.

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[1] *Photonic Band Gaps and Localization*, edited by C.M. Soukoulis (Plenum, New York, 1993); *Photonic Band Gap Materials*, edited by C.M. Soukoulis (Kluwer Academic, London, 1996).

[2] C.M. Soukoulis, J.V. Jose, E.N. Economou, and P. Sheng, Phys. Rev. Lett. **50,** 764 (1983).

[3] Q. Li, C.T. Chan, K.M. Ho, and C.M. Soukoulis, Phys. Rev. B **53,** 15 577 (1996); E. Lidorikis, Q. Li, and C.M. Soukoulis, *ibid.* **54,** 10 249 (1996).

[4] N.G. Sun and G.P. Tsironis, in *Photonic Band Gap Materials* (Ref. 1), p. 535.

[5] D. Hennig, H. Gabriel, G.P. Tsironis, and M. Molina, Appl. Phys. Lett. **64,** 2934 (1994).

[6] A.B. Aceves, G.G. Luther, C. De Angelis, A.M. Rubenchik, and S.K. Turitsyn, Phys. Rev. Lett. **75,** 73 (1995).

[7] Yu.B. Gaididei, S.F. Mingaleev, P.L. Christiansen, and K.Ø. Rasmussen, Phys. Rev. E **55**, 6136 (1997).

[8] *Padé Approximants Method and its Applications to Mechanics*, edited by H. Cabannes (Springer-Verlag, Berlin, 1976).

[9] Yu.B. Gaididei, S.F. Mingaleev, P.L. Christiansen, and K.Ø. Rasmussen, Phys. Lett. A **222,** 152 (1996).

[10] E.A. Kuznetsov, A.M. Rubenchik, and V.E. Zakharov, Phys. Rep. **142,** 103 (1986).

[11] D.E. Pelinovsky, V.V. Afanasjev, and Yu.S. Kivshar, Phys. Rev. E **53**, 1940 (1996).

[12] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953).