# Full Counting Statistics as the Geometry of Two Planes

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Provided the measuring time is short enough, the full counting statistics (FCS) of the charge pumped across a barrier as a result of a series of voltage pulses are shown to be equivalent to the geometry of two planes. This formulation leads to the FCS without the need for the usual nonequilibrium (Keldysh) transport theory or the direct computation of the determinant of an infinite-dimensional matrix. In the particular case of the application of $N$ Lorentzian pulses, we show the computation of the FCS reduces to the diagonalization of an $N \times N$ matrix. We also use the formulation to compute the core-hole response in the x-ray edge problem and the FCS for a square wave pulse-train for the case of low transmission.

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The experimental and technological importance of a full quantum mechanical treatment of the response of Fermi systems to a time-dependent perturbation, has grown as electronic devices have shrunk. In particular, the quantum statistics of charge transfer events induced by an applied voltage pulse in simple systems such as across a tunneling barrier or along a wire need to be understood if the limits to their capacity to encode or transmit information are to be found [1,2]. They are also excellent examples of nonequilibrium systems for which in some cases complete theoretical treatments are known or are feasible.

Theoretically, interest has concentrated on the full counting statistics (FCS) or generating function $\chi(\lambda)$ for all moments of the charge distribution [3]. There are results for the case of an infinite train of periodic pulses impinging on a tunneling barrier [4–6] and, for the case of a low transparency barrier with a constant bias voltage $V$, the FCS are also known for nonzero temperatures [2]. For general adiabatic pumping, i.e., not just that induced by a voltage pulse applied between two leads, conditions on the form of $\chi(\lambda)$ were reported in [7]. Particular attention has been paid to the case of Lorentzian voltage pulses with "integer area" [$\frac{e}{h}\int_{-\infty}^{\infty} V(t)dt$ is an integer]. If the voltage pulses all have the same polarity they generate so-called "Minimal excitation states" (MES). These have been shown to minimize the number of excitations in a one-channel Fermi gas required to generate a given signal. They have the intriguing property that a train of such pulses retain the minimal noise property [4,8] independent of the separation or width of the generating voltage pulses provided the pulses are all of the same polarity. When such signals impinge on a tunneling barrier their nature is reflected in the FCS [4].

Here we show that the FCS at zero temperature are determined solely by the geometry of two planes. For a train of $N$ Lorentzian pulses these two planes coincide except in an $N$-dimensional subspace. Computing the FCS in this case reduces to the diagonalization of $N \times N$ matrix, which we compute explicitly. The known results for $\chi(\lambda)$ follow directly from this geometric formulation circumventing the need to work with a formulation using Keldysh Green's functions [6] or solving an auxiliary Riemann-Hilbert problem to invert singular operators [5].

The mirror planes are defined by the operators $h = 2n - 1$ and $\tilde{h} = UhU^{\dagger}$, where $n$ is the ground state density matrix and $U$ is the unitary operator which describes the effect of applying a voltage pulse between different components of the system. Directions corresponding to occupied single-particle states are in the corresponding mirror plane (eigenvalue $+1$) while directions corresponding to unoccupied states are in its complement and reflected (eigenvalue $-1$). The planes have the dimension of the number of occupied single-particle states. The intersection of the two planes (and the intersection of the two complements) define the states which are occupied (or unoccupied) in both the initial and transformed states and which therefore contribute nothing to the FCS. The remaining states are rotated with respect to one another by angles which we denote by $\frac{\alpha_k}{2}$. The directions which are rotated can be identified from the commutator $[h, e^{-i\phi(t)}]$, which defines a spanning set of states for the rotated space.

We consider as a model system a quantum point contact between two ideal single channel conductors coupled by a tunneling barrier localized around $x = 0$ and treat the effect of a voltage pulse, $V(t)$, applied between the two conductors (see Fig. 1). We assume that we can neglect the contributions to the noise and other moments of the distribution of transferred electrons which scale as $\log t_0\xi$ where $t_0$ is the observation time and $\xi$ is an ultraviolet cutoff set by the Fermi energy. These are present even in equilibrium and are associated with the fact that the charge number in the left or right electrode is not a conserved quantity [3]. The FCS of transmitted charge across the point contact (which we take to have reflection and transmission probabilities $R$ and $T = 1 - R$) can be obtained from the generating function

$$\chi(\lambda) = \sum_{n=-\infty}^{\infty} P_n e^{i\lambda n}, \tag{1}$$

where $P_n$ is the probability of $n$ particles being transmitted (from the left electrode to the right one).

 

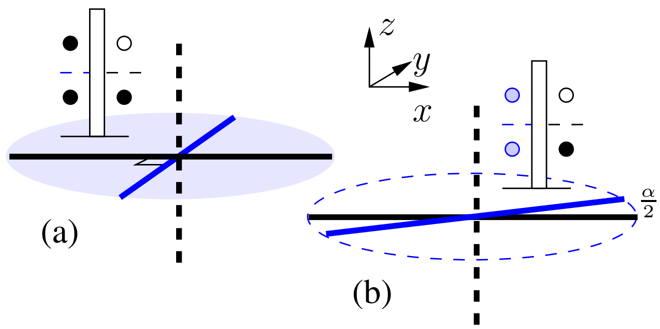


FIG. 1 (color online). The geometry of the FCS. In the space of single-particle states, orbitals occupied at zero temperature correspond to eigenstates of $h$ with eigenvalue 1 and define a mirror plane shown schematically as the $x$ axis. Unoccupied states at zero temperature (with eigenvalue $-1$) are in the complement of plane, one of these is the $y$ axis and the remainder are shown as the $z$ axis. In the insets the dashed line divides states into those above and below the unperturbed Fermi level but no other ordering by energy is implied. Application of a voltage pulse, biasing one electrode with respect to the other, transforms $h \rightarrow \tilde{h} = UhU^{\dagger}$. Case (a): ($N_u = 1$, $N_b = 0$ see text) The corresponding plane has an added dimension and is shown as the shaded plane. Case (b): ($N_u = 0$, $N_b = 1$). The plane rotates by $\alpha/2$ about an axis perpendicular to $h$. The axis of rotation and the new occupied state are found from $[h, e^{-i\phi(t)}]$ (see text). All other states are rotated as a function of time but remain eigenstates of both $h$ and $\tilde{h}$.

The principle result of [6] (see also [7]) is that

$$\chi(\lambda) = (R + Te^{i\kappa_l\lambda})^{N_u} \prod_{j=1}^{N_b} \left(1 + RT\sin^2\frac{\alpha_j}{2}(e^{i\lambda} + e^{-i\lambda} - 2)\right), \tag{2}$$

where $N_u$ is the number of *unidirectional* events with $\kappa_l = \pm 1$ determined by the polarity of the voltage pulse and $N_b$ the number of *bidirectional* events. The geometry for the two types of event is illustrated in Fig. 1. Consider first the case of a simplified system consisting of a single state on either side of the barrier. Suppose now that, after taking account of the voltage pulse, the state on the left side is occupied and empty on the right. This is an example of a single MES [8] or *unidirectional* [6] event and is the case illustrated in Fig. 1(a): the $h$ plane (shown as the $x$ axis) is actually zero dimensional for this simplified one-state system and the $\tilde{h}$ plane is one dimensional. The FCS for this situation is clearly given by $\chi(\lambda) = R + Te^{i\lambda}$: $P_0 = R$ and $P_1 = T$ in (1).

The *bidirectional* event is illustrated in Fig. 1(b). Again consider first the case of a simplified system of a single occupied state on either side of the barrier. If the occupied states are equivalent (i.e., they have the same energy or the same decomposition by energy), then transfer across the barrier is blocked and $\chi(\lambda) = 1$. If the two states are orthogonal to one another (have different energies or orthogonal decompositions by energy) then each state behaves independently. The FCS for this case is a product over that for two independent events each of which is of the type illustrated in Fig. 1(a). This would give $\chi(\lambda) = (R + Te^{i\lambda})(R + Te^{-i\lambda})$. In general, the projection of the two occupied states onto one another is equal to $\cos\frac{\alpha}{2}$, where $\frac{\alpha}{2}$ is the angle between the two planes in Fig. 1. Now, the FCS for this simplified system is just the weighted average of the two previous cases: $\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}(R + Te^{i\lambda})(R + Te^{-i\lambda}) = 1 + RT\sin^2\frac{\alpha}{2}(e^{i\lambda} + e^{-i\lambda} - 2)$.

The general case of a filled Fermi sea is a combination of the two types of events illustrated in Fig. 1. This follows from the fact that the operator $h\tilde{h}$ is unitary and therefore has an orthonormal basis which can be constructed from its eigenvectors [6]. For positive mean transfer from left to right electrode $N_u > 0$ [9], the dimensionality of $\tilde{h}$ is $N_u$ greater than that of $h$. There are therefore $N_u$ directions with eigenvalue of $h\tilde{h}$ equal to $-1$. The remaining eigenvalues come in pairs ($e^{\pm i\alpha_j}$ with $j = 1, \ldots, N_b$) and the 2D plane defined by the corresponding basis vectors, $e_{2j-1} \times e_{2j}$, intersects the two mirror planes defined by $h$ and $\tilde{h}$ in two lines inclined at an angle $\alpha_j/2$. The FCS, $\chi(\lambda)$, is then given by the product over the appropriate factors for the $N_u$ and $N_b$ events to give (2).

The case where $N_b$ and $N_u$ are both finite corresponds to a train of Lorentzian pulses [9] and has attracted a lot of attention [4,8]. Here we show that it is possible to identify explicitly the subspace for which the eigenstates of $h\tilde{h}$ have eigenvalues different from unity. The computation of the FCS reduces to the diagonalization of an $N \times N$ matrix where $N = N_u + 2N_b$ to find the angles $\alpha_i$ from its eigenvalues ($e^{i\alpha_i}$). The flux through the circuit, $\phi(t) = -(e/\hbar)\int^t V(t')dt'$, can be written in this case as

$$e^{i\phi(t)} = \prod_{m=1}^{N} \frac{t - p_m^*}{t - p_m} = 1 + \sum_{m=1}^{N} \frac{A_m}{t - p_m}. \tag{3}$$

The poles at $p_m = t_m + i\tau_m$ have residues $A_m$, which are determined by the system of equations

$$-1 = \sum_{m=1}^{N} \frac{A_m}{p_n^* - p_m}, n = 1, \ldots, N. \tag{4}$$

To within overall phase factors, the effect of the voltage pulse is equivalent to a multiplication of left electrode states by the factor $e^{i\phi(t)}$.

The states that define the space in which the eigenvalues of $h\tilde{h}$ are different from 1 are $|\psi_m(t)\rangle = 1/(t - p_m)$ for $m = 1, \ldots, N$. This follows from considering explicitly the effect of $U$. All single-particle states, $\psi$, satisfying $(h\tilde{h} - 1)|\psi\rangle = he^{i\phi}[h, e^{-i\phi}]|\psi\rangle = 0$, do not change occupancy under the effect of $U$ [and hence make no contribution to $\chi(\lambda)$]. For an arbitrary single-particle state $\psi(t)$, we find using (3) and (4)

$$(h\tilde{h}-1)|\psi\rangle = \sum_{m,m'} \mathrm{sgn}(\tau_{m'}) \frac{2A_{m'}A_m^*}{p_m^* - p_{m'}} \frac{f_m}{t-p_{m'}}. \qquad (5)$$

Here $f_m$ is the projection of $\psi(t)$ onto $|\psi_m(t)\rangle$:

$$f_m = \frac{i}{2\pi} \int dt \frac{1}{(t-p_m)^*} \psi(t). \qquad (6)$$

The right-hand side of (5) vanishes for all $\psi(t)$ orthogonal to the set of states $\{\psi_n(t), n = 1, \ldots, N\}$.

The FCS for the case of a train of $N$ Lorentzian pulses can then be found by considering the effect of $h\tilde{h}$ on the subspace of the states $\{\psi_n(t)\}$. We find $h\tilde{h}|\psi_n\rangle = \sum_m M_{mn}|\psi_m\rangle$, where

$$M_{mn} = \sum_{m'} \frac{\mathrm{sgn}(\tau_m \tau_{m'}) A_m A_{m'}^*}{(p_m - p_{m'}^*)(p_n - p_{m'}^*)}. \qquad (7)$$

The angles $\alpha_i$ which determine the FCS in (2) are found from the eigenvalues ($e^{i\alpha_i}$) of the $N \times N$ matrix $M_{mn}$.

Analytic results for the FCS are known in some simple cases [4]: (i) The $\{\tau_n\}$ all have the same sign and (ii) $N = 2$ with $\mathrm{sgn}(\tau_1\tau_2) = -1$. The case of $N = 4$ has also been solved analytically [10]. Case (i) corresponds to the case of MES. All poles of $e^{i\phi}$ are in the same half-plane and all $N$ states, $|\psi_n(t)\rangle = 1/(t - p_m)$ are eigenstates of $h\tilde{h}$ with eigenvalue $-1$. This can be seen from (7) using $\mathrm{sgn}(\tau_m\tau_m') = 1$, (4) and by considering $\frac{de^{-i\phi(t)}}{dt}$. These states define the $N$ noncoplanar directions which are in one mirror plane and in the complement of the other. The FCS for this case of MES corresponds to $N_u = N$ *unidirectional* events in (2).

Case (ii) corresponds to a single *bidirectional* event. Here the FCS are just found from the eigenvalues of the $2 \times 2$ matrix $M_{mn}$. Explicit calculation gives the result $\sin^2\alpha/2 = |\frac{p_1 - p_2^*}{p_1 - p_2}|^2$ [4]. In this case, because the states $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal to each other, it is also particularly simple to compute the effect of $U$ on the single-particle states explicitly. We work with normalized states $|\bar{\psi}_i\rangle = C_i|\psi_i\rangle$ where $C_i = \sqrt{|p_i - p_i^*|}$. We find that the effect of $U$ on the state $|\bar{\psi}_1^*\rangle = C_1/t - p_1^*$ is to transform it into a linear combination of $\bar{\psi}_1$ and $\bar{\psi}_2$:

$$e^{i\phi(t)}|\bar{\psi}_1^*\rangle = \frac{C_1(p_2^* - p_2)}{C_2(p_1 - p_2)}|\bar{\psi}_2\rangle + \frac{p_1 - p_2^*}{p_1 - p_2}|\bar{\psi}_1\rangle. \qquad (8)$$

If $|\psi_1^*\rangle$ is in the mirror plane corresponding to $h$ then the component of the transformed state proportional to $|\psi_1\rangle$ is in the complement of the mirror plane and its amplitude is given by $\sin\frac{\alpha}{2} e^{i\phi_1} = \frac{p_1 - p_2^*}{p_1 - p_2}$ a result previously obtained using an operator formalism [8].

In the general case when $N_b$ is not finite [9], the geometric approach simplifies calculations when the transparency $T$ (or the scattering phase shifts in other problems) are low, it is possible to deduce the results without the need for any direct diagonalization. We illustrate this by rederiving the standard result for the orthogonality catastrophe or Fermi edge singularity (FES) problem and then use the method to generalize results for the FCS for a system subjected to a sudden pulse of finite duration.

The FES problem arises in the context of the x-ray absorption spectrum of a metal. Here, the form of the spectrum is determined by the response of the conduction electrons to the sudden appearance of a core hole [11] and has been shown to be the consequence of the orthogonality of the ground states of the Fermi gas with and without the local potential due to the core hole [12]. The core-hole spectral function is the Fourier transform with respect to $t_f$ of the overlap $\langle 0|0'\rangle$, where $|0'\rangle = U(t_f)|0\rangle$, $|0\rangle$ is the ground state at $t = 0$ and $U(t_f)$ is the unitary time-development operator of the system. $U(t_f)$ describes the effect of the new potential which switches on at $t = 0$. (All states are written in the interaction basis with $H_0$ the Hamiltonian of the unperturbed system.) This is a one-channel version of the problem we have considered. The overlap $\langle 0|U|0\rangle = \prod_k \cos\alpha_k/2$, where the $e^{\pm i\alpha_k}$ are the eigenvalues of $h\tilde{h}$. When the rotation angles $\alpha_k$ are small, we can expand the cosine to obtain $\langle 0|U|0\rangle \approx \exp[\frac{1}{8}\mathrm{Tr}(h\tilde{h} - 1)]$. In the usual one-channel FES problem, $U = e^{i\phi(t)}$ where $\phi(t) = 2\phi_0\theta(t)\theta(t_f - t)$ and where $\phi_0$ is the phase shift of the core-hole potential computed on the scattering (unperturbed) states of the system. If we work in the time domain $h(t,t') = \frac{i}{\pi}\mathrm{P}(\frac{1}{t-t'})$, where P denotes Cauchy principal part, we find

$$\begin{aligned}\mathrm{Tr}(h\tilde{h} - 1) &= \int dt dt' h(t,t')(e^{i\phi(t)}e^{-i\phi(t')} - 1)h(t',t) \\ &= -\frac{8}{\pi^2}\sin^2\phi_0 \log\xi t_f \end{aligned} \qquad (9)$$

($\xi$ is the usual short-time cutoff of order the Fermi energy) and we recover the standard result for this problem: $\langle 0|U|0\rangle = (\xi t_f)^{-\phi_0^2/\pi^2}$.

The FCS for a train of step pulses in the low transparency limit is known to be Poissonian [13]. Here we will consider a periodic signal with pulses of length $t_p$ and period $S$ applied to the left electrode with respect to the right electrode: $V(t) = 2\phi_0[\delta(t - nS) - \delta(t - nS - t_p)]$. The geometry of this situation is very similar to that of the core-hole Green's function in the FES problem. The general result (2) then gives $\log\chi(\lambda) = \sum_k \log[1 + TR\sin^2\alpha_k/2(e^{i\lambda} + e^{-i\lambda} - 2)]$. In the limit of $TR \ll 1$, we can expand the logarithm and use $\sum_k \sin^2\frac{\alpha_k}{2} = -\frac{1}{4}\mathrm{Tr}(h\tilde{h} - 1)$. We find

$$\begin{aligned}\mathrm{Tr}(h\tilde{h} - 1) = -\frac{4}{\pi^2}\sum_{m,n}\Big[&\log\Big(\frac{t_p + S(m-n-1)}{S(m-n-1)+\tau}\Big) \\ &+ \log\Big(\frac{t_p + S(m-n)}{S(m-n)+\tau}\Big)\Big]\sin^2\phi_0, \end{aligned} \qquad (10)$$

where $\tau$ is a short-time cutoff used to characterize the delta function $\delta(t) = \frac{\tau}{\pi(t^2+\tau^2)}$ and $N_u = 0$ for this periodic train of pulses. In (10), the terms with $m = n$ and $m = n + 1$ are divergent for $\tau \rightarrow 0$, while all other terms are convergent. With a measurement time $NS$ we obtain the usual result $\log\chi(\lambda) = 2N\frac{RT}{\pi^2}\sin^2\phi_0 \log\frac{t_p}{\tau}(e^{i\lambda} + e^{-i\lambda} - 2)$.

Finally we comment on the logarithmic contributions to the noise which are present even in the absence of an applied voltage pulse and limit possible applications and the observability of minimal noise states. When calculating the FCS, the system is assumed to have been prepared in the ground state of the system with zero transmission $T = 0$. The scattering matrix is taken to be different from unity only during the observation time $t_0$ when the transmission becomes nonzero ($T > 0$) and the number of particles on a particular side of the barrier is no longer a good quantum number. The geometry of this situation is similar to the case of the step pulse in voltage except that the unitary transformation acting on the initial states mixes states on both sides of the barrier [14]. In practice, these logarithmic corrections imply that the observation time $t_0$ for the MES and other effects discussed here to be observable above the equilibrium noise must be short. For a system with $\epsilon_F = 10$ meV and setting the equilibrium noise at low temperatures ($\frac{1}{\pi^2}\log\epsilon_F t_0/h$) equal to the minimal noise obtained using from a MES corresponds to working at frequencies $\nu > 300$ MHz.

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[1] C. W. J. Beenakker and H. Schomerus, Phys. Rev. Lett. **86**, 700 (2001).
[2] L. S. Levitov and M. Reznikov, Phys. Rev. B **70**, 115305 (2004).
[3] L. S. Levitov, H. W. Lee, and G. Lesovik, J. Math. Phys. (N.Y.) **37**, 4845 (1996).
[4] D. A. Ivanov, H. W. Lee, and L. S. Levitov, Phys. Rev. B **56**, 6839 (1997).
[5] N. d'Ambrumenil and B. Muzykantskii, Phys. Rev. B **71**, 045326 (2005).
[6] M. Vanevic, Y. V. Nazarov, and W. Belzig, Phys. Rev. Lett. **99**, 076601 (2007).
[7] A. G. Abanov and D. A. Ivanov, Phys. Rev. Lett. **100**, 086602 (2008).
[8] J. Keeling, I. Klich, and L. S. Levitov, Phys. Rev. Lett. **97**, 116403 (2006).
[9] Noninteger $N_u$ would imply a fractional total number of occupied states invalidating the mirror-plane property ($\tilde{h}^2 = 1$) at $T = 0$. In practice, experiments would work with zero mean voltage over some finite measuring time $t_0$ thereby ensuring $N_u = 0$.
[10] A. Pratap, Ph.D. thesis, University of Warwick, 2008 (unpublished).
[11] P. Nozieres and C. T. De Dominicis, Phys. Rev. **178**, 1097 (1969).
[12] P. Anderson, Phys. Rev. Lett. **18**, 1049 (1967).
[13] L. Levitov and G. Lesovik, JETP Lett. **58**, 230 (1993).
[14] The relation of the FCS to the geometry of the two planes corresponding to $\tilde{h}$ and $h$ is more complicated if the transformation mixes left and right states. However, in the equilibrium case it is possible to switch to a basis of scattering states which diagonalizes the scattering matrix $S$ and identify the corresponding angles easily. Constraints on the form of $\chi(\lambda)$ for case of general transformations have also been reported in [7].