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RINGS OF ANALYTIC FUNCTIONS*

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1. This paper is a sequel to the author's paper "Function Rings and Riemann Surfaces", listed as [1] in the References at the end of the paper. The reader will need to refer to [1] at several places in our argument.

Let E be the open unit disk in the z-plane and let $\overline{E}$ be its closure. Let φ be an analytic function on E extendable to all of $\overline{E}$ to be continuous. Assume :

(1.1) *If* $z_1, z_2 \in \overline{E}, z_1 \neq z_2,$ *then* $\varphi(z_1) \neq \varphi(z_2)$.

Under this hypothesis, a well-known theorem of Walsh [2, p. 36] allows us to conclude that every function analytic on E and continuous on $\overline{E}$ is uniformly approximable on $\overline{E}$ by polynomials in φ.

Our object in this paper is to consider the analogous approximation problem when the single function φ is replaced by a pair of analytic functions φ, f and the disk E is replaced by a finite region on a Riemann surface. We restrict ourselves to the case when φ and f are both analytic on the boundary of the region considered as well as on the region itself.

Let $\mathscr{S}$ be a Riemann Surface, Γ_0 a simple closed analytic curve on $\mathscr{S}$ such that Γ_0 is the boundary of a region D_0 with $D_0 \cup \Gamma_0$ compact.

DEFINITION 1.1. $\mathfrak{A}(D_0)$ is the ring of all functions analytic on $D_0 \cup \Gamma_0$.

DEFINITION 1.2. For g_1, g_2 in $\mathfrak{A}(D_0)$, $[g_1, g_2]$ is the subring of $\mathfrak{A}(D_0)$ consisting of all polynomials, including constants, in g_1 and g_2.

Let φ, f be a pair of functions in $\mathfrak{A}(D_0)$ neither of which is a constant. In the following two theorems we assume :

(1.2) The differential $d\varphi$ does not vanish on Γ_0.

THEOREM 1.1. *In order that every function in* $\mathfrak{A}(D_0)$ *be uniformly approximable on* $D_0 \cup \Gamma_0$ *by functions in* $[\varphi, f]$ *it is necessary and sufficient that*

(1.3) *If* p_1, p_2 *are distinct points in* $D_0 \cup \Gamma_0$, *either* φ *or* f *takes on different values at* p_1 *and* p_2.

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(1.4) *If* $p \in D_0$, $d\varphi(p) \neq 0$ *or* $df(p) \neq 0$.

We now drop hypothesis (1.4) and replace (1.3) by the condition that φ and f separate points only on the boundary Γ_0. Then :

THEOREM 1.2. *Assume*

(1.5) φ *and* f *together separate points on* Γ_0 .

Then there exists a finite subset T *of* $D_0 \cup \Gamma_0$ *and an integer* $\bar{n}$ *such that if* $g \in \mathfrak{A}(D_0)$ *and* g *vanishes at each point of* T *to an order no less than* $\bar{n}$, *then* g *is approximable on* $D_0 \cup \Gamma_0$ *uniformly by functions in* $[\varphi, f]$.

We choose a simple closed curve J_1 on $\mathscr{S}$ bounding a region $\mathscr{S}_1$ which contains $D_0 \cup \Gamma_0$ such that $\mathscr{S}_1 \cup J_1$ is compact and φ and f are analytic on $\mathscr{S}_1 \cup \Gamma_1$. Set $\Lambda = \{p \in \mathscr{S}_1 | \exists q$ *in* $\mathscr{S}_1$ *with* $\varphi(p) = \varphi(q),\ d\varphi(q) = 0\}$. Then Λ is finite. Let Γ_1 be a simple closed analytic curve in $\mathscr{S}_1$ which bounds a region D_1 containing $D_0 \cup \Gamma_0$ such that Γ_1 does not meet Λ and such that

(1.6) $d\varphi \neq 0$ *on* Γ_1, *and*

(1.7) φ and f together separate points on Γ_1 .

Such a curve exists because of (1.2) and (1.5). Write $\overline{D}_1$ for $D_1 \cup \Gamma_1$.

DEFINITION 1.3. A point p in D_1 is φ-singular if either

(i) There is some q in D_1 with $\varphi(p) = \varphi(q)$ and $d\varphi(q) = 0$, or

(ii) There exists q_1, q_2 in D_1 with $\varphi(q_1) = \varphi(q_2)$ and $f(q_1) = f(q_2)$ and $\varphi(p) = \varphi(q_1)$

LEMMA 1.1. *The set of* φ*-singular points in* D_1 *is finite.*

PROOF. Since φ is analytic in $D_1 \cup \Gamma_1$, only finitely many points in D_1 can satisfy (i) in Definition. 1.3. Assume now infinitely many points in D_1 satisfy (ii) in that Definition. Then there exist points p_n, q_n, $n = 1, 2, \cdots$, with $p_n \neq q_n$ and $\varphi(p_n) = \varphi(q_n)$ and $f(p_n) = f(q_n)$. It easily follows from this that there exists a pair of distinct points a, b in D_1 and a one-one conformal map τ of a neighborhood U of a on a neighborhood of b such that for q in U

$$\varphi(\tau(q)) = \varphi(q) \tag{1.8}$$

$$f(\tau(q)) = f(q) \tag{1.9}$$

Let η: $z = z(t)$, $0 \leqq t \leqq 1$, be a Jordan arc lying in D_1 except for its endpoint $z(1)$ which is on Γ_1, with $z(0) = a$ and such that η does not meet Λ. We assert that one of the following two cases must occur. Either

(1.10) τ can be analytically continued along η up to $z(1)$ and $\tau(z(1)) \in D_1$, or

(1.11) continuation is possible up to some point $z(t')$ on η with $\tau(z(t')) \in \Gamma_1$

and $\tau(z(t)) \in D_1$, $t < t'$.

Assume first that continuation is possible up to $z(1)$ for τ as a map from $\mathscr{S}$ in to $\mathscr{S}$. If $\tau(z(1)) \notin D_1$, then for some $t_0 \leqq 1$, $\tau(z(t_0)) \in \Gamma_1$ and so (1.11) holds. If $\tau(z(1)) \in D_1$, (1.10) holds.

It remains to assume that for some $c \leqq 1$, continuation is possible up to $z(t)$ if $t < c$ but not for $t = c$. Assume now (1.11) does not hold. Then for each $t < c$, $\tau(z(t)) \notin \Gamma_1$ and so $\tau(z(t)) \in D_1$ for all $t < c$. Choose a sequence $\{t_n\}$ converging to c from below and let T be a limit point of the image sequence $\{\tau(z(t_n))\}$. Then $T \in D_1 \cup \Gamma_1$. Now formula (1.8) remains true under continuation. Hence

$$\varphi(\tau(z(t_n))) = \varphi(z(t_n)) \qquad \text{for all } n,$$

whence

$$\varphi(T) = \varphi(z(c)) .$$

Since η does not meet Λ, $d\varphi \neq 0$ at T. Hence in a neighborhood U of T the restriction φ_U of φ is one-one. Choose n with $\tau(z(t_n))$ in U and let α be an are on η containing $z(t_n)$ with $\tau(\alpha) \subset U$. For z in α we then have

$$\varphi_U(\tau(z)) = \varphi(z) \qquad \text{or} \qquad \tau(z) = \varphi_U^{-1}(\varphi(z))$$

But $\varphi_U^{-1}(\varphi)$ is analytic at $z(c)$ and so τ admits an analytic continuation along η beyond $z(c)$. This is a contradiction. Hence (1.11) holds. Our assertion is thus proved.

Assume now (1.10) holds. By an argument just like the preceding with Γ_1 replacing η we see that either

(1.12) τ is continuable along all of Γ_1 starting at $z(1)$ on $\eta \cap \Gamma_1$ with $\tau(p) \in D_1$ for all p in Γ_1, or

(1.13) τ is continuable along Γ_1 up to some point p_0 with $\tau(p_0) \in \Gamma_1$ such that if p on Γ_1 precedes p_0 in this continuation, $\tau(p) \in D_1$.

Choose now p'' on Γ_1 such that for all p in D_1, $\varphi(p) \neq \varphi(p'')$. If (1.12) holds, we have $\tau(p'') \in D_1$ and $\varphi(\tau(p'')) = \varphi(p'')$. This is impossible. Hence (1.13) holds. If now $\tau(p_0) \neq p_0$, the fact that (1.8) and (1.9) remain true under continuation gives that (1.7) is violated, while if $\tau(p_0) = p_0$, (1.6) is violated. Thus if (1.10) holds, we reach a contradiction. If (1.11) holds, a similar argument using τ^{-1} instead of τ produces a contradiction. Thus the assumption that infinitely many points in D_1 satisfy (ii) is untenable. So the Lemma is proved.

Because of the lemma, we can find a simple closed analytic curve Γ lying in D_1, which satisfies the following definition :

DEFINITION 1.4. Γ bounds a region D which contains $D_0 \cup \Gamma_0$ and

(i) No φ-singular point lies on Γ

(ii) φ takes only finitely many values more than once on Γ

NOTE in particular :

(1.14) $d\varphi$ does not vanish on Γ

(1.15) If $p \in \Gamma$ and $q \in D \cup \Gamma$, $p \neq q$, then $\varphi(p) \neq \varphi(q)$ or $f(p) \neq f(q)$.

2. Let φ, f be the functions of the last Section and let Γ, D be as given in Definition 1.4. Write γ for the image of Γ under φ and write $\Omega(\varphi)$ for the complement of γ in the plane. Fix a measure $d\sigma$ on Γ with

$$\int_\Gamma g(t)d\sigma(t) = 0, \qquad \text{if } g \in [\varphi, f], \tag{2.1}$$

and

(2.2) $d\sigma$ has no point mass at any point which φ maps into a multiple point of γ.

DEFINITION 2.1. $d\mu$ is the measure on γ defined as follows : if S is a subset of γ containing no multiple points, $d\mu(S) = d\sigma(\varphi^{-1}(S))$, and $d\mu = 0$ at each multiple point.

DEFINITION 2.2. For each g in $[\varphi, f]$, $g^*(\lambda) = g(\varphi^{-1}(\lambda))$ for λ in γ, where λ is not a multiple point.

Then

$$\int_\gamma g^*(\lambda)d\mu(\lambda) = 0 \qquad \text{if } g \in [\varphi, f] . \tag{2.3}$$

For each component W of $\Omega(\varphi)$ we set

$$\Phi(W, g, z) = \frac{1}{2\pi i}\int_\Gamma \frac{g(t)d\sigma(t)}{\varphi(t) - z}, \qquad z \in W, g \in [\varphi, f] . \tag{2.4}$$

Then we also have

$$\Phi(W, g, z) = \frac{1}{2\pi i}\int_\gamma \frac{g^*(\lambda)d\mu(\lambda)}{\lambda - z} . \tag{2.5}$$

For W a component of $\Omega(\varphi)$, $\varphi^{-1}(W)$ means the set of points in D which φ maps into W. Let us use annular coordinates z, $a < |z| < b$, in a neighborhood of Γ on $\mathscr{S}$ in which φ and f are analytic such that Γ gets the equation : $|z| = 1$. Then φ and f may be regarded as functions analytic in the annulus $a < |z| < b$. Because of (1.14) and (1.15) we have

(2.6) $\varphi' \neq 0$ on $|z| = 1$ and

(2.7) φ and f together separate points on $|z| = 1$.

Thus φ and f satisfy the hypotheses (a), (b), (c) made in the Introduction of [1]. The other assumptions made in the Introduction of [1] are that φ maps $|z| = 1$ on a curve with only finitely many multiple points and that the closed algebra A generated by φ and f on $|z| = 1$ is a proper subalgebra of the algebra C of all continuous functions on $|z| = 1$.

The first of these assumptions is satisfied here because of Definition 1.4, (ii). Further, if a continuous function on Γ lies in the closed algebra generated by φ and f, then by the maximum principle it is the boundary function of a function analytic in D, and so A is a proper subset of C. Thus φ and f satisfy all the assumptions made in the Introduction of [1], and so the results of [1] are valid for this pair of functions. We shall use results in [1] to prove the following theorem, which is the goal of the present section.

THEOREM 2.1. *For each component W of $\Omega(\varphi)$ there exists a unique meromorphic function $k(W)$ on $\varphi^{-1}(W)$, $k(W) \not\equiv 0$, such that if $z \in W$, z not a branch-point for φ^{-1}, and $q_1, \cdots, q_m$ are the points on $\varphi^{-1}(W)$ which φ maps into z, then for all g in $[\varphi, f]$ we have*

$$\sum_{i=1}^{m} g(q_i)k(W, q_i) = \Phi(W, g, z)\ . \tag{2.8}$$

If $\varphi^{-1}(W)$ is empty for some W, we interpret the left side in (2.8)*as* 0, i.e., *we have $\Phi(W, g) \equiv 0$ for each g.*

In the notation of the Introduction of [1] we write $\mathscr{F}$ for the Riemann surface of the function $f(\varphi^{-1})$ and $\bar{\gamma}$ for the simple closed curve on $\mathscr{F}$ obtained by continuing a fixed element of $f(\varphi^{-1})$ along γ. Here φ^{-1} means the inverse to a function-element of φ which maps a neighborhood of a point on Γ one-one on a neighborhood of a point on γ. Then $\bar{\gamma}$ projects on γ. Let $\mathscr{D}$ be the component of the complement of $\bar{\gamma}$ on $\mathscr{F}$ which is defined in Definition 4.1 of [1]. By the proof of Theorem 1 of [1], $\mathscr{D} \cup \bar{\gamma}$ is compact and by Lemma 4.4 of [1], no point of $\mathscr{D}$ projects into the unbounded component of $\Omega(\varphi)$.

We now define a map χ from $D \cup \Gamma$ to places over points in the plane as follows: For each p in $D \cup \Gamma$, let φ_p denote the restriction of φ to a neighborhood of p in $\mathscr{S}$ and let φ_p^{-1} be the inverse element of φ_p in a neighborhood of $\varphi(p)$ in the plane. φ_p^{-1} may be branched.

DEFINITION 2.3. $\chi(p)$ is the pair $(\varphi(p), f(\varphi_p^{-1}))$ considered as a place over the point $\varphi(p)$ in the plane, where $f(\varphi_p^{-1})$ means the function-element.

LEMMA 2.1. *The map χ maps Γ homeomorphically on $\bar{\gamma}$ and is a one-one conformal map of D onto $\mathscr{D}$.*

PROOF. The first assertion follows at once from the definition of $\bar{\gamma}$ and the fact that φ and f together separate points on Γ. We next claim that χ is one-one in D. For let p_1, p_2 be distinct points in D with $\chi(p_1) = \chi(p_2)$. Then $\lambda = \varphi(p_1) = \varphi(p_2)$ and $f(\varphi_{p1}^{-1}) = f(\varphi_{p2}^{-1})$ in a neighborhood of λ. This contradicts Lemma 1.1, and so χ is one-one, as asserted.

Next we show that for some q in D, $\chi(q)$ is in $\mathscr{D}$. To this end fix a_0 on Γ with $\varphi(a_0)$ lying on the boundary of the unbounded component of $\Omega(\varphi)$. Call this component W_∞ and let W_1 be the other component on whose boundary $\varphi(a_0)$ lies. Assume $\varphi(a_0)$ is a simple point on γ. Choose a neighborhood U of a_0 on $\mathscr{S}$ on which φ is one-one and such that $\varphi(U)$ is the union of $\varphi(U) \cap W_1$ and $\varphi(U) \cap W_\infty$ and a simple arc on γ. Then $\varphi(U \cap D) = \varphi(U) \cap W_1$. For else $\varphi(U \cap D) = \varphi(U) \cap W_\infty$ and then $\varphi(D)$ must cover W_∞, contradicting the compactness of $D \cup \Gamma$.

Let now $\overline{p}$ be the place $(\varphi(a_0), f(\varphi_{a_0}^{-1}))$, which lies on $\overline{\gamma}$. Let $\overline{U}$ be a neighborhood of $\overline{p}$ on $\mathscr{F}$ projecting one-one into $\varphi(U)$. Fix $\overline{q}$ in $\overline{U} \cap \mathscr{D}$ and let its projection be z_0 with $z_0 \notin \gamma$. Then $z_0 \in W_1$, since points in $\mathscr{D}$ never project into W_∞. Hence $z_0 \in \varphi(U) \cap W_1$, whence there is a unique q in $U \cap D$ with $\varphi(q) = z_0$. It follows directly from the definitions that $\chi(q) = \overline{q}$ and so that $\chi(q) \in \mathscr{D}$.

Fix now p in D. Because of the definition of χ, $\chi(p) \in \mathscr{F}$. Join p to q, where $q \in D$ and $\chi(q) \in \mathscr{D}$, by an arc lying in D. If $\chi(p) \notin \mathscr{D}$, then for some r on the arc $\chi(r) \in \overline{\gamma}$. But then there is an r' in Γ with $\chi(r') = \chi(r)$. This implies that r' is φ-singular and so contradicts the choice of Γ. Hence $\chi(p) \in \mathscr{D}$.

It is easily verified that χ is conformal on D and it remains to be shown that χ maps D onto $\mathscr{D}$. Assume the contrary. Then $\chi(D)$ is a proper open subset of $\mathscr{D}$. Then $\chi(D)$ has a boundary point b in $\mathscr{D}$. Hence there exists a sequence of points p_n in D with $\chi(p_n)$ converging to b. Let $\tilde{p}$ be a limit point of the p_n in $D \cup \Gamma$. If $\tilde{p} \in \Gamma$, then $\chi(\tilde{p}) \in \overline{\gamma}$. But also $\chi(\tilde{p}) = b \in \mathscr{D}$ and this is a contradiction. Hence $\tilde{p} \in D$. It follows that χ maps a neighborhood of $\tilde{p}$ on a neighborhood of b in $\mathscr{D}$ whence b was not a boundary point of $\chi(D)$. Hence $\chi(D) = \mathscr{D}$. This completes the proof of the lemma.

DEFINITION 2.4. Z is the function on $\mathscr{D}$ which assigns to each place its projection in the plane. For each component W of $\Omega(\varphi)$, $Z^{-1}(W)$ denotes the set of points p on $\mathscr{D}$ with $Z(p)$ in W.

DEFINITION 2.5. For each g in $[\varphi, f]$ we define G on $\mathscr{D} \cup \gamma$ by

$$G(p) = g(\chi^{-1}(p)) \,. \tag{2.9}$$

LEMMA 2.2. *Let W be a component of $\Omega(\varphi)$ with $Z^{-1}(W)$ non-empty. Then there exists a meromorphic function $\overline{k}$ on $Z^{-1}(W)$, $\overline{k} \not\equiv 0$, with*

$$\textstyle\sum_{i=1}^{m} G(p_i)\,\overline{k}(p_i) = \Phi(W, g, z) \tag{2.10}$$

for all g in $[\varphi, f]$ and z in W, where $p_1, \cdots, p_m$ are the points on $Z^{-1}(W)$ with $Z(p_i) = z$, and G is given by Definition 2.5 and Φ by formula (2.4).

NOTE. To prove this lemma, we need the notion of a "regular Riemann surface" over a component W of $\Omega(\varphi)$ which was given in [1]. Definition 3.1. This definition is given in terms of a measure satisfying

(2.3) and so is applicable to our $d\mu$. In [1], Lemma 4.4, it is shown that if $\overline{W}$ is the regular surface over W, then $Z^{-1}(W) \subset \overline{W}$. We need here that $Z^{-1}(W) = \overline{W}$, and the following two lemmas are proved to this end.

LEMMA 2.3. *Let W, W' be two components of $\Omega(\varphi)$ having a common boundary arc α. Assume that $Z^{-1}(W)$ and $Z^{-1}(W')$ are non-empty. Then the regular surfaces $\overline{W}$ and $\overline{W}'$ over W and W' exist by Lemma 4.4 of* [1]. *If $\overline{W}'$ is contained in $\mathscr{D}$, then also $\overline{W}$ is contained in $\mathscr{D}$.*

PROOF. Since $Z^{-1}(W)$ is non-empty, W is a bounded component of $\Omega(\varphi)$ by Lemma 4.4 of [1]. The hypotheses of our lemma are then just those of Lemma 4.3 of [1].

Let β: $z = z(t)$, $a \leqq t \leqq b$ be an arc in the plane such that for some c, $a < c < b$, $z(t) \in W$ for $a \leqq t < c$, $z(c) \in \alpha$, and $z(t) \in W'$, for $c < t \leqq b$. Let $z(a)$ not be a branch-point for $\overline{W}$ and let $\overline{W}$ have m sheets. Let $h_1, \cdots, h_m$ be the function-elements of the places on $\overline{W}$ over $z(a)$. By Lemma 4.3 of [1], h_1 can be continued along β up to c, giving rise there to an algebraic function element $h_1(c)$. By abuse of language, we shall identify places with their function-elements. By Lemma 4.3 again, if $h_1(c) \notin \overline{\gamma}$, then $h_1(c)$ may be continued along β for $t > c$ and for each such t gives rise to a place $h_1(t)$ over $z(t)$ belonging to $\overline{W}'$. By hypothesis $\overline{W}' \subset \mathscr{D}$ and so $h_1(t) \in \mathscr{D}$. Continuing $h_1(t)$ backward along β till $t = a$ and using that $h_1(c) \notin \overline{\gamma}$ we get that h_1 belongs to $\mathscr{D}$.

It remains to consider the case that $h_1(c) \in \overline{\gamma}$. Then if $|t - c|$ is small, either $h_1(t) \in \mathscr{D}$ for $t < c$ or $h_1(t) \in \mathscr{D}_1$ for $t < c$, where $\mathscr{D}_1$ is the component of the complement of $\overline{\gamma}$ on $\mathscr{F}$ which is not $\mathscr{D}$. In the second case there exists t_0 with $h_1(t_0) \in \mathscr{D}_1 \cap \overline{W}$.
We can now choose a path $\overline{\eta}$ on $\mathscr{D}_1$ whose initial point is $h_1(t_0)$ and whose endpoint projects into the unbounded component W_∞ of $\Omega(\varphi)$. Let η be the projection of $\overline{\eta}$ in the plane. By application of Lemma 4.3 of [1] to the components of $\Omega(\varphi)$ traversed by η, using the fact that $\overline{\eta}$ does not meet $\overline{\gamma}$, we conclude that each component traversed by η is bounded.* But by choice of $\overline{\eta}$, η penetrates into W_∞. This is a contradiction. Hence $h_1(t) \in \mathscr{D}$ for $t < c$, $|t - c|$ small, whence by continuation backward along β we get that $h_1 \in \mathscr{D}$. Similarly each h_i, $i = 2, \cdots, m$ is in $\mathscr{D}$. Hence every place of $\overline{W}$ belongs to $\mathscr{D}$ and the lemma is proved.

LEMMA 2.4. *Let W be a component of $\Omega(\varphi)$ such that $Z^{-1}(W)$ is non-empty and so the regular surface $\overline{W}$ over W exists. Then $\overline{W} \subset \mathscr{D}$.*

* For a detailed argument of this type, cf. [1], proof of Lemma 4.4.

PROOF. Let W_0 be the region entering in the definition of $\mathscr{D}$, (Definition 4.1 of [1]). Fix a point q_0 in $\overline{W}_0$. By Definition 4.1, $q_0 \in \mathscr{D}$. Since $Z^{-1}(W)$ is non-empty and $Z^{-1}(W) \subset \overline{W}$, we have that there is some q in $\overline{W} \cap \mathscr{D}$. Join q_0 to q by a path $\overline{\beta}$ in $\mathscr{D}$ having the following properties: if β is the projection of $\overline{\beta}$ in the plane, then β meets γ only finitely often and β contains no multiple point of γ. Then there exists a finite sequence of components of $\Omega(\varphi)$; $W_0, W_1, \cdots W_n = W$, such that $W_{i+1} \neq W_i$, all i, and β traverses this sequence in succession. Then W_i and W_{i+1} have a common boundary arc for each i. For each i there is some point on $\overline{\beta}$ and so on $\mathscr{D}$ projecting into W_i, whence by Lemma 4.4 of [1] the regular surface $\overline{W}_i$ over W_i exists.

Now $\overline{W}_0 \subset \mathscr{D}$, by Definition 4.1. Hence $\overline{W}_1 \subset \mathscr{D}$, by our Lemma 2.3. Repeatedly using this lemma, we arrive at the conclusion that $\overline{W}_n = \overline{W} \subset \mathscr{D}$, as asserted.

PROOF OF LEMMA 2.2. Combining Lemma 2.4 with the fact that $Z^{-1}(W) \subset \overline{W}$, we get that $Z^{-1}(W) = \overline{W}$.

By definition of $\overline{W}$, (Definition 3.1 of [1]), there exists a meromorphic function $\overline{k}$, $\overline{k} \not\equiv 0$, defined on $\overline{W}$, and there exists a homomorphism L of $[\varphi, f]$ into functions meromorphic on $\overline{W}$, such that, with g^* as in Definition 2.2:

$$\sum_{i=1}^{m} Lg(p_i)\overline{k}(p_i) = \frac{1}{2\pi i}\int_\gamma \frac{g^*(\lambda)d\mu(\lambda)}{\lambda - z}, \qquad z \in W \tag{2.11}$$

where $p_1, \cdots, p_m$ are the places on $\overline{W}$ lying over z and further Lf is the function on $\overline{W}$ assigning to a place (b, h) the value $h(b)$. Since $\overline{W} = Z^{-1}(W)$, the p_i are the points on $Z^{-1}(W)$ with $Z(p_i) = z$.

Then $Lf = F$ on $\overline{W}$, where $F(p) = f(\chi^{-1}(p))$. Also $L\varphi = Z$, which we see as follows: Applying (2.11) to $g = \varphi \cdot f^\nu$ we get

$$\sum_{i=1}^{m} (L\varphi)(p_i)F^\nu(p_i)\overline{k}(p_i) = \frac{1}{2\pi i}\int_\gamma \frac{\lambda(f^{\nu *}(\lambda))d\mu(\lambda)}{\lambda - z}.$$

But (2.3) gives that the right side equals

$$z \cdot \frac{1}{2\pi i}\int_\gamma \frac{f^{\nu *}(\lambda)d\mu(\lambda)}{\lambda - z}$$

and this again, by (2.11), is

$$z \cdot \sum_{i=1}^{m} F^\nu(p_i)\overline{k}(p_i).$$

Hence we have

$$\sum_{i=1}^{m}(L\varphi(p_i)-z)\overline{k}(p_i)F^{\nu}(p_i)=0\,. \tag{2.12}$$

This last equality now holds for $\nu=0, 1, 2, \cdots$. Consider the system of equations, for the x_i:

$$\sum_{i=1}^{m} x_i\overline{k}(p_i)F^{\nu}(p_i),\ \nu=0, 1, 2, \cdots, m-1\,. \tag{2.13}$$

The determinant of this system is

$$\pm\prod_{i=1}^{m}\overline{k}(p_i)\prod_{i,j=1}^{m}(F(p_i)-F(p_j))\,.$$

From condition (3.10) in the definition of $\overline{W}$ in [1], and (3.18) in [1], we see that this determinant vanishes only for isolated values of z in W. It follows from (2.12) that $L\varphi(p_i)=z=Z(p_i)$, $i=1, \cdots m$, except for isolated values of z in W. Hence $L\varphi\equiv Z$, as asserted.

Since now L is a homomorphism, we get for every polynomial P in two variables

$$\sum_{i=1}^{m}P(Z, F)(p_i)\overline{k}(p_i)=\frac{1}{2\pi i}\int_{\gamma}\frac{P(\lambda, f^*(\lambda))d\mu(\lambda)}{\lambda-z}\,. \tag{2.14}$$

Thus for all g in $[\varphi, f]$ we have

$$\sum_{i=1}^{m}G(p_i)\overline{k}(p_i)=\frac{1}{2\pi i}\int_{\gamma}\frac{g^*(\lambda)d\mu(\lambda)}{\lambda-z}$$

where G is defined by Definition 2.5, and this is exactly the equation (2.10) which was to be proved. Lemma 2.2 is thus established.

PROOF OF THEOREM 2.1. Let χ be the map defined in Definition 2.3. Define a function $k(W)$ on $\varphi^{-1}(W)$ for each component W of $\Omega(\varphi)$ with $\varphi^{-1}(W)$ non-empty by

$$k(W, p)=\overline{k}(\chi(p))$$

where $\overline{k}$ is the function on $Z^{-1}(W)$ obtained in Lemma 2.2. Then Lemma 2.2 gives for g in $[\varphi, f]$, and setting $z=\varphi(q_i)=Z(\chi(q_i))$:

$$\sum_{i=1}^{m}g(q_i)k(W, q_i)=\sum_{i=1}^{m}G(\chi(q_i))\overline{k}(\chi(q_i))=\Phi(W, g, z)\,.$$

Thus we have proved (2.8). Finally, it is easy to see that (2.8) determines $k(W)$ uniquely for we may set $g=f^{\nu}$, $\nu=0, 1, \cdots, m-1$ in (2.8) and solve the resulting system for $k(W, q_i)$, $i=1, \cdots, m$.

It remains to consider a component W for which $\varphi^{-1}(W)$ is empty. We have to show that $\Phi(W, g)\equiv 0$ for all g in $[\varphi, f]$.

If φ is one-one on all of Γ, then $\varphi^{-1}(W)$ empty implies that W is the unbounded component, whence $\Phi(W, g)=0$, all g, by (2.3). We may hence assume that at least one multiple point λ^* of γ lies on the boundary of W. We can then choose a curvilinear triangle Δ whose interior lies in W which has λ^* as a vertex and two of whose sides, α_1 and α_2, are arcs on γ with endpoint λ^*. We can choose Δ so small that there exist halfopen arcs β_1 and β_2 on Γ with distinct endpoints p_1 and p_2 such that $\varphi(p_1) = \varphi(p_2) = \lambda^*$ and φ maps $\beta_1 - p_1$ one-one on $\alpha_1 - \lambda^*$ and $\beta_2 - p_2$ one-one on $\alpha_2 - \lambda^*$, and that no points outside β_1 or β_2 map on $\alpha_1 - \lambda^*$ or $\alpha_2 - \lambda^*$. Since φ and f together separate points on Γ, $f(p_1) \neq f(p_2)$. Hence the function $f^* = f(\varphi^{-1})$ has different limits as $\lambda \to \lambda^*$ along α_1 and along α_2.

Let W_1 be the component of $\Omega(\varphi)$ other than W which has α_1 as boundary arc. Since $\varphi^{-1}(W)$ is empty and $\alpha_1 - \lambda^*$ is covered just once by φ on Γ, W_1 is covered exactly once by φ in D. We now apply (2.8) to W_1 and get a function k_1 on $\varphi^{-1}(W_1)$ such that for all z in W_1

(i) $$g(\varphi^{-1}(z))k_1(\varphi^{-1}(z)) = \Phi(W, g, z)$$

for all g in $[\varphi, f]$. Let $C[\varphi, f]$ be the Banach algebra of all functions on Γ which are uniform limits of functions in $[\varphi, f]$. If $g \in C[\varphi, f]$, then g has a unique continuous extension to $D \cup \Gamma$ which is analytic in D. Call this extension again g. Then (i) clearly remains true for all g in $C[\varphi, f]$. For each λ in $\alpha_1 - \lambda^*$ let $p(\lambda)$ be the unique point in Γ which φ maps on λ. Set $g^* = g(\varphi^{-1})$. Then (i) yields:

(ii) $$g^*(\lambda)k_1(p(\lambda)) = \Phi(W_1, g, \lambda) \qquad \text{a.e. on } \alpha_1,\ g \in C[\varphi, f]$$

On the other hand, we know (Lemma 2.3 of [1]):

(iii) $$\Phi(W_1, g, \lambda) = \Phi(W, g, \lambda) + g^*(\lambda)\rho(\lambda) \qquad \text{a.e. on } \alpha_1,$$

where $\rho(\lambda)$ is a suitably defined derivative of $d\mu$ at λ. Hence

(iv) $$g^*(\lambda)(k_1(p(\lambda)) - \rho(\lambda)) = \Phi(W, g, \lambda) \qquad \text{a.e. on } \alpha_1$$

whence, using (iv) with $g = 1$, we get:

(v) $$g^*(\lambda)\Phi(W, 1, \lambda) = \Phi(W, g, \lambda) \qquad \text{a.e. on } \alpha_1.$$

Assume now that it is false that $\Phi(W, g) \equiv 0$ for all g in $[\varphi, f]$. Because of (v), then, $\Phi(W, 1) \not\equiv 0$. Hence by (v):

(vi) $$g^*(\lambda) = \Phi(W, g, \lambda)\Phi(W, 1, \lambda)^{-1} \qquad \text{a.e. on } \alpha_1, \text{ all } g \text{ in } C[\varphi, f].$$

By an argument we have used earlier (proof of Lemma 4.2 of [1]), this implies that the function F defined on W by

(vii) $$F(z) = \Phi(W, g, z) \cdot \Phi(W, 1, z)^{-1}$$

is a bounded analytic function in W assuming the boundary values f^* continuously on α_1. A parallel argument shows that F also takes the boundary values f^* on α_2. But f^* has a jump discontinuity as we go from α_1 to α_2 at λ^*. The boundary function of a bounded analytic function in the Jordan region Δ cannot have such a jump. We thus have a contradiction. This forces us to conclude that $\Phi(W, g) \equiv 0$ for all g in $[\varphi, f]$. This was what we had to prove. Theorem 2.1 is thus established.

3. For each component W of $\Omega(\varphi)$, set

$$\Delta(W, z) = \prod_{i,j=1}^{m} (f(p_i) - f(p_j))^2, \qquad z \in W$$

where $p_1, \cdots p_m$ are those points in $\varphi^{-1}(W)$ which φ maps on z. Because of Lemma 1.1, $\Delta \not\equiv 0$. Since φ and f are analytic on Γ, and Γ contains no φ-singular points, $\Delta(W)$ is analytic on the boundary of W and does not vanish there. By setting $\Delta(W, p) = \Delta(W, z)$ where $z = \varphi(p)$, $\Delta(W)$ becomes defined on $\varphi^{-1}(W)$.

DEFINITION 3.1. For each W, $Z(W, \varphi)$ is the set of zeros of $\Delta(W)$ in $\varphi^{-1}(W)$ and $n(W)$ is the maximal order of these zeros.

$Z(\varphi)$ is the union of the sets $Z(W, \varphi)$ over all components W of $\Omega(\varphi)$ and n is the maximum of the $n(W)$.

DEFINITION 3.2. $E(\varphi) =$ the set of points p in D such that there exists q_1, q_2 on Γ, $q_1 \neq q_2$, with $\varphi(q_1) = \varphi(q_2) = \varphi(p)$.

DEFINITION 3.3. $M(\varphi) =$ the set of point q in Γ such that there exists q' in Γ with $\varphi(q) = \varphi(q')$, $q \neq q'$. By (ii) of Definition 1.4, $M(\varphi)$ is finite.

Fix annular coordinates $re^{i\theta}$ in a neighborhood of Γ on $\mathscr{S}$ such that $r = 1$ is the equation of Γ and $r < 1$ in D. Let k^* be a measureable function defined on the unit circle with

$$\int_0^{2\pi} |k^*(e^{i\theta})|^2 d\theta < \infty \tag{3.1}$$

and assume that the measure $d\sigma = k^*(e^{i\theta})d\varphi$ on Γ satisfies (2.1), where $d\varphi$ is the differential of φ, and Γ is oriented positively with respect to D.

DEFINITION 3.4. For W a component of $\Omega(\varphi)$, $p \in \varphi^{-1}(W)$, set $K_\varphi(p) = k(W, p)$ where $k(W)$ is defined by (2.8) relative to the measure $k^* d\varphi$. K_φ is then defined on the union of the sets $\varphi^{-1}(W)$.

THEOREM 3.1.[1] K_φ can be extended to all of D to be analytic everywhere on D except for possible poles of order $\leqq n$ at points of $Z(\varphi)$ and possible isolated singularities at the points of $E(\varphi)$. Furthermore:

(i) $\lim_{p \to t} K_\varphi(p) = k^*(t)$ for a.a. t on Γ, the limit being nontangential.

(ii) If $e^{i\theta_0}$ is any point on Γ and not in $M(\varphi)$, there exist numbers s_1, s_2, $s_1 < \theta_0 < s_2$, with

$$\int_{s_1}^{s_2} |K_\varphi(re^{i\theta})|^2 d\theta = 0(1) \qquad \text{as } r \longrightarrow 1 .$$

[1] I am indebted to Professor Beurling for a discussion which led me to a simpler proof for this theorem.

PROOF. Denote by D^* the union of all the sets $\varphi^{-1}(W)$ with W a component of $\Omega(\varphi)$. Fix $\overline{p}$ in $D - D^*$ and $\overline{p} \notin E(\varphi)$. Set $\overline{\lambda} = \varphi(\overline{p})$.

Let $\overline{q}$ be the unique point in Γ with $\varphi(\overline{q}) = \overline{\lambda}$, and let $\overline{p}_1, \cdots, \overline{p}_m$ be the points in D with $\varphi(\overline{p}_i) = \overline{\lambda}$ and $\overline{p} = \overline{p}_1$. Since $\overline{q}$ is not φ-singular, $d\varphi \neq 0$ at each $\overline{p}_i$, and also at $\overline{q}$. Hence there exists a neighborhood U of $\overline{\lambda}$ and single-valued maps $p_1, \cdots, p_m$ with $p_i(\overline{\lambda}) = \overline{p}_i$, $i = 1, \cdots, m$, from U into D with $\varphi(p_i(z)) = z$ if $z \in U$, and a single-valued map p_{m+1} from U into $\mathscr{S}$ with $p_{m+1}(\overline{\lambda}) = \overline{q}$ and $\varphi(p_{m+1}(z)) \equiv z$.

Let $\overline{\alpha}$ denote an arc on Γ containing $\overline{q}$ and set $\alpha = \varphi(\overline{\alpha})$. We assume $\alpha = U \cap \gamma$ and α is free of multiple points of $\varphi(\Gamma)$.

Let W and W' be the two components of $\Omega(\varphi)$ which have α as a boundary arc. Since Γ contains no φ-singular points, $\Delta(W)$ and $\Delta(W')$ do not vanish on α. We may assume U so small that $\Delta(W) \neq 0$ in $U \cap W$ and $\Delta(W') \neq 0$ in $U \cap W'$. Let $\varphi^{-1}(W)$ be m-sheeted and $\varphi^{-1}(W')$ be $m+1$ sheeted. Then $m \geqq 1$. Let $k(W)$, $k(W')$ be the functions associated to the measure $k^* d\varphi$ by Theorem 2.1. We claim: For a.a. λ on α, the functions $k(W, p_i(z))$ have non-tangential limits $k(W, p_i(\lambda))$ as $z \to \lambda$ from within W, $i = 1, \cdots m$, and similarly the limits $k(W' \; p_i(\lambda))$ exist as $z \to \lambda$ from within W' for $i = 1, \cdots m + 1$. Further, a.e. on α

(3.2) $$k(W, p_i(\lambda)) = k(W', p_i(\lambda)),\ i = 1, \cdots, m$$

and

(3.3) $$k(W', p_{m+1}(\lambda)) = k^*(p_{m+1}(\lambda)) .$$

The existence of non-tangential limits follows from (2.8) and the existence of such limits for the functions $\Phi(W, g, z)$. Using (2.8) with $g = f^\nu$ and taking limits we get

(3.4) $$\sum_{i=1}^{m} f(p_i(\lambda))^\nu k(W, p_i(\lambda)) = \Phi(W, f^\nu, \lambda) \quad \text{a.e. on } \alpha,\ \nu \geqq 0.$$

(3.5) $$\sum_{i=1}^{m+1} f(p_i(\lambda))^\nu k(W', p_i(\lambda)) = \Phi(W', f^\nu, \lambda) \quad \text{a.e. on } \alpha,\ \nu \geqq 0.$$

By Lemma 2.3 of [1] we get

(3.6) $$\Phi(W', f^\nu, \lambda) = \Phi(W, f^\nu, \lambda) + k^*(p_{m+1}(\lambda)) f^\nu(p_{m+1}(\lambda)) \quad \text{a.e. on } \alpha,\ \nu \geqq 0.$$

Hence we get from (3.4)

(3.7) $$\begin{aligned} &\sum_{i=1}^{m} f(p_i(\lambda))^\nu k(W, p_i(\lambda)) + f(p_{m+1}(\lambda))^\nu k^*(p_{m+1}(\lambda)) \\ &= \Phi(W', f^\nu, \lambda) \quad \text{a.e. on } \alpha,\ \nu \geqq 0. \end{aligned}$$

From (3.5) and (3.7) we get the asserted equations (3.2) and (3.3) by solving for $k(W, p_i(\lambda))$ the two systems of equations with $\nu = 0, 1, \cdots, m$ and using the fact that $\prod_{i,j=1}^{m+1} (f(p_i(\lambda)) - f(p_j(\lambda)) \neq 0$ on α.

We can now choose a region R in the complex x-plane, $x = s + iv$, defined by

$$a \leqq s \leqq b, \quad -c < v < c, \quad c > 0$$

having the following properties, up to and including (3.9):

The arc α' on Γ consisting of points with annular coordinates e^{is}, $a \leqq s \leqq b$, is contained in $\overline{\alpha}$ and $\overline{q}$ has coordinate $e^{i\overline{s}}$, $a < \overline{s} < b$. Set

$$t(x) = \varphi(e^{ix}) = \varphi(e^{-v}e^{is}) \qquad \text{for } x \text{ in } R\,.$$

Then t is a one-one analytic map from R to a neighborhood of α in the plane contained in U. Set

$$Q(x, s) = \frac{1}{t(s) - t(x)} - \frac{1}{t'(x)}\frac{1}{s - x} \qquad \text{for } s, x \text{ in } R\,. \tag{3.8}$$

Then

(3.9) Q is analytic in both variables for x, s in some region containing the closure of R.

Write R^+ for $R \cap \{x \mid v > 0\}$, R^- for $R \cap \{x \mid v < 0\}$. Then t maps R^+ into W' and R^- into W and the segment $a \leqq s \leqq b$, $v = 0$, into α. Set $k_1(x) = k(W, p_1(t(x)))$ for $x \in R^-$ and $k_1'(x) = k(W', p_1(t(x))$ for $x \in R^+$. Fix real numbers s_1, s_2 with $a < s_1 < \overline{s} < s_2 < b$ and define $H^2(R^+)$ as the class of functions h analytic in R^+ and satisfying:

$$\int_{s_1}^{s_2} |h(s + iv)|^2 ds = 0(1) \qquad \text{as } v \to 0\,. \tag{3.10}$$

Define $H^2(R^-)$ similarly. We shall show

$$k_1 \in H^2(R^-) \text{ and } k_1' \in H^2(R^+)\,. \tag{3.11}$$

We observe that for $x \in R^-$

$$\Phi(W, f^\nu, t(x)) = \frac{1}{2\pi i}\int_{\Gamma - \alpha'} \frac{f^\nu(p)k^*(p)d\varphi(p)}{\varphi(p) - t(x)} + \frac{1}{2\pi i}\int_a^b \frac{f^\nu(e^{is})k^*(e^{is})\tilde{\varphi}(s)ds}{t(s) - t(x)} \tag{3.12}$$

where $\tilde{\varphi}(s) = \dfrac{d}{ds}(\varphi(e^{is}))$. Set $g_\nu(s) = f^\nu(e^{is})k^*(e^{is})\tilde{\varphi}(s)$. The first term on the right in (3.12) is then analytic for x in all of R. The other term equals

$$\frac{1}{t'(x)}\int_a^b \frac{g_\nu(s)ds}{s - x} + \int_a^b Q(x, s)g_\nu(s)ds\,. \tag{3.13}$$

Because of (3.9) the second term in (3.13) is analytic in all of R. Also, $g_\nu \in L^2(a, b)$, since $k^*(e^{is}) \in L^2(0, 2\pi)$ by hypothesis and f and $\tilde{\varphi}$ are continuous. By a well-known property of the Cauchy integral, then

$$\int_a^b \frac{g_\nu(s)ds}{s - x} \text{ belongs to } H^2(R^-)\,.$$

Also $\frac{1}{t'(x)}$ is analytic in R. Hence $\Phi(W, f^\nu, t(x)) \in H^2(R^-)$ for each ν.

Now it follows from (2.8) (compare [1], (3.17)) that $k(W)$ is a quotient whose denominator is $\Delta(W)$ and whose numerator is a finite linear combination of functions $\Phi(W, f^\nu)$ with coefficients which are polynomials in f and in symmetric functions of $f(p_1), \cdots, f(p_m)$. Since $\Delta(W) \neq 0$ on α, we get from this and the fact that $\Phi(W, f^\nu, t(x)) \in H^2(R^-)$ that $k_1 \in H^2(R^-)$. A parallel argument gives $k_1' \in H^2(R^+)$, so that (3.11) holds.

We now appeal to the following known result: Let two functions h and h' be analytic, respectively, in domains V and V' where V lies in the upper and V' in the lower half-plane and V and V' have a common boundary segment I on the real axis. Let h and h' have coinciding boundary-values a.e. on I. Assume the L^2-means of h over segments I_n in V parallel to I and approaching I as $n \to \infty$ remain bounded, and assume the analogous situation for h'. Then h and h' provide analytic continuations of each other across I.

By (3.2) and (3.11) the functions k_1 and k_1' satisfy the hypothesis of this theorem, and so these functions continue each other analytically across the segment $s_1 \leqq s \leqq s_2$. Hence there exists a function analytic in a neighborhood $\overline{U}$ of $\overline{p}$ on D which coincides on $\overline{U} \cap \varphi^{-1}(W)$ with $k(W)$ and on $\overline{U} \cap \varphi^{-1}(W')$ with $k(W')$.

Since this is so for an arbitrary point $\overline{p}$ in $D - D^* - E(\varphi)$, we have the following consequence:

If K_φ is given on D^* by Definition 3.4 and if we define K_φ by analytic continuation on the rest of D, except for points in $E(\varphi)$, then K_φ is a single-valued function on $D - E(\varphi)$ analytic except for the poles of $k(W)$ in $\varphi^{-1}(W)$ for each W. Since $k(W)$ is the quotient of two analytic functions in $\varphi^{-1}(W)$ with denominator $\Delta(W)$, K_φ has no poles in $D - E(\varphi)$ except possibly at points of $Z(\varphi)$ and of order $\leqq n$. (Recall Definition 3.1).

Because of (3.3) we also have

$$\lim_{p \to t} K_\varphi(p) = k^*(t) \qquad \text{a.e. on } \overline{\alpha}$$

It now $\overline{q}_1$ is any point on Γ, a similar argument shows that the preceding relation is valid in a neighborhood of $\overline{q}_1$ on Γ, and so assertion (i) is proved.

Further, arguing with $k(W', p_{m+1}(z))$ as we did with $k(W', p_1(z))$, we find for $x = s + iv$, $v > 0$:

(3.14) $$\int_{s_1}^{s_2}\left|k(W', p_{m+1}(t(x)))\right|^2 ds = \int_{s_1}^{s_2}\left|K_\varphi(e^{-v}e^{is})\right|^2 ds = 0(1)$$

as $v \to 0$. Thus Theorem 3.1 is proved, except for assertion (ii) for points $\bar{q}$ on Γ, $\bar{q} \notin M(\varphi)$, for which there does not exist $\bar{p}$ in D with $\varphi(\bar{q}) = \varphi(\bar{p})$. But this case can be treated in a quite similar manner and so Theorem 3.1 is established.

4. In this section we consider the exceptional points in $E(\varphi)$ and $M(\varphi)$ appearing in Theorem 3.1. We shall do this by replacing φ by suitable other functions in Theorem 3.1.

THEOREM 4.1. *Let k^* be as in Theorem* 3.1. *Then there exists a function K analytic on D except for possible poles of orders $\leqq n$ at points of $Z(\varphi)$ such that*:

(4.1) $$\int_0^{2\pi} |K(\mathrm{re}^{i\theta}) - k^*(e^{i\theta})|^2 d\theta \longrightarrow 0 \text{ as } r \longrightarrow 1,$$

and

(4.2) $$\lim_{r\to 1} K(\mathrm{re}^{i\theta}) = k^*(e^{i\theta}) \qquad \text{a. e. on } (0, 2\pi).$$

LEMMA 4.1. *Fix $\bar{r} > 0$. Fix $\bar{q} \in \Gamma$. Then we can find ε, $0 < |\varepsilon| < \bar{r}$, such that if $\Psi = \varphi + \varepsilon f$, then for all q' in Γ with $q' \neq \bar{q}$, $\Psi(q') \neq \Psi(\bar{q})$.*

PROOF. Set
$$Q(t) = \frac{\varphi(t) - \varphi(\bar{q})}{f(t) - f(\bar{q})}\,.$$

Then Q is meromorphic on Γ and hence maps Γ on a finite sum of analytic curves. We can hence find $\varepsilon, |\varepsilon| < r$, such that Q does not take the value $-\varepsilon$ on Γ. Set $\Psi = \varphi + \varepsilon f$. We claim Ψ satisfies the assertion of the lemma. For, assume the contrary. Then there is some q' in Γ, $q' \neq \bar{q}$, such that $\Psi(\bar{q}') = \Psi(q)$, or

(4.3) $$\varphi(q') + \varepsilon f(q') = \varphi(\bar{q}) + \varepsilon f(\bar{q}),$$

whence

$$\varphi(q') - \varphi(\bar{q}) = -\varepsilon(f(q') - f(\bar{q}))\,.$$

If $f(q') - f(\bar{q})$ were zero, then φ and f would take the same values at $\bar{q}$ and q', contrary to Definition 1.4. Hence (4.3) gives

$$Q(q') = -\varepsilon\,.$$

But this contradicts the choice of ε. Hence Ψ does satisfy the assertion and the lemma holds.

LEMMA 4.2. *There is an $\bar{r} > 0$ such that if $|\varepsilon| < \bar{r}$ and $\Psi = \varphi + \varepsilon f$, then Γ contains no Ψ-singular points (as defined in Definition* 1.3 *with Ψ replacing φ) and Ψ takes only finitely many values more than once on Γ.*

PROOF. Assume first that for each $\bar{r}$ there is an $\varepsilon, |\varepsilon| < \bar{r}$, such that for $\Psi_\varepsilon = \varphi + \varepsilon f$ there are Ψ_ε-singular points on Γ. Then there exist $\varepsilon_n \to 0$ such that one of the following two cases occurs:

(i) For each n there exist p_n in Γ, q_n in $\overline{D}_1$ (as defined in § 1) with $\Psi_{\varepsilon_n}(p_n) = \Psi^{\varepsilon}{}_n(q_n)$ and $d\,\Psi_{\varepsilon_n}(q_n) = 0$. By passing to a suitable subsequence, we get $p_n \to p$ in Γ, $q_n \to q$ in $\overline{D}_1$. Then, since f is bounded on $\overline{D}_1$, $\varphi(p) = \varphi(q)$, and similarly $d\varphi(q) = 0$. Hence p is φ-singular, contrary to our choice of Γ.

(ii) For each n there exist p_n in Γ, q_n', q_n^2 distinct points in $\overline{D}_1$, with $\Psi_{\varepsilon_n}(p_n) = \Psi_{\varepsilon_n}(q_n') = \Psi_{\varepsilon_n}(q_n^2)$ and $f(q_n') = f(q_n^2)$. Hence $\varphi(q_n') = \varphi(q_n^2)$. Again we may assume $p_n \to p$ in Γ, $q_n' \to q_1$ and $q_n^2 \to q_2$ with q_1, q_2 in $\overline{D}_1$. Then $\varphi(p) = \varphi(q_1) = \varphi(q_2)$ and $f(q_1) = f(q_2)$. If $q_1 \neq q_2$, p is φ-singular. If $q_1 = q_2$, $d\varphi(q_1) = 0$. Hence again p is φ-singular. But this contradicts our choice of Γ. Hence for some $\overline{r}$, Γ contains no Ψ-singular points if $\Psi = \varphi + \varepsilon f$, $|\varepsilon| < \overline{r}$. Finally, for $\overline{r}$ sufficiently small, we have that if $|\varepsilon| < \overline{r}$, $\Psi = \varphi + \varepsilon f$ takes only finitely many values more than once on Γ, as is seen by use of (ii) in Definition 1.4. This proves our lemma.

PROOF OF THEOREM 4.1. We take $K = K_\varphi$, as given in Definition 3.4. Let $\overline{r}$ be the number introduced in Lemma 4.2. Fix $\overline{q} \in M(\varphi)$. By Lemma 4.1 we can choose ε, $|\varepsilon| < \overline{r}$, such that if $\Psi = \varphi + \varepsilon f$, then $\overline{q} \notin M(\Psi)$, where $M(\Psi)$ is given by Definition 3.3 with Ψ replacing φ. By Lemma 4.2, Γ satisfies conditions (i), (ii) of Definition 1.4 relative to Ψ. Also clearly $[\varphi, f] = [\Psi, f]$. Since the measure $k^* d\,\varphi$ on Γ satisfies (2.1), we have also

$$\int_\Gamma g(t)k^*(t)\frac{d\varphi}{d\Psi}(t)d\Psi(t) = 0 \qquad \text{if } g \in [\Psi, f]. \tag{4.4}$$

We can hence apply Theorem 3.1 to the pair Ψ, f and the function $k^*\dfrac{d\varphi}{d\Psi}$. Define K_Ψ on D in accordance with Definition 3.4 applied to Ψ. Then Theorem 3.1, applied to Ψ yields:

$$\lim_{p\to t} K_\Psi(p) = k^*(t)\frac{d\varphi}{d\Psi}(t) \qquad \text{a. e. on } \Gamma. \tag{4.5}$$

Combining (4.5) and (i) of Theorem 3.1 as applied to φ, we conclude that $K_\Psi\dfrac{d\Psi}{d\varphi}$ and K_φ have the same non-tangential boundary values a. e. on Γ. Hence

$$K_\varphi \equiv K_\Psi\frac{d\Psi}{d\varphi} \text{ in } D\,. \tag{4.6}$$

Let now $\bar{q}$ have coordinate $e^{i\bar{\theta}}$. Then since $\bar{q} \notin M(\Psi)$, by (ii) of Theorem 3.1 applied to Ψ there exist numbers (s_1, s_2), $s_1 < \bar{\theta} < s_2$, with

$$\int_{s_1}^{s_2} |K_\Psi(\mathrm{re}^{i\theta})|^2 d\theta = 0(1) \qquad \text{as } r \longrightarrow 1.$$

Since also $\dfrac{d\Psi}{d\varphi}$ is analytic on Γ, (4.6) gives

$$(4.7) \qquad \int_{s_1}^{s_2} |K_\varphi(\mathrm{re}^{i\theta})|^2 d\theta = 0(1) \qquad \text{as } r \longrightarrow 1.$$

The same now holds for each point in $M(\varphi)$. Hence every point on Γ, without exception, lies on an open arc for which (ii) of Theorem 3.1 holds for K_φ. Since Γ is compact, this implies that

$$(4.8) \qquad \int_0^{2\pi} |K_\varphi(\mathrm{re}^{i\theta})|^2 d\theta = 0(1) \qquad \text{as } r \longrightarrow 1.$$

Because of (4.8) and the fact that by (i) of Theorem 3.1 K_φ has k^* as boundary-function, a classical result gives assertion (4.1) of our theorem. Also (4.2) holds by Theorem 3.1. It remains to prove that the isolated singularities of K_φ in $E(\varphi)$ are removable.

Fix $\bar{p}$ in $E(\varphi)$. By Definition 3.2, $\bar{p} \in D$ and the point $\varphi(\bar{p})$ lies on $\varphi(\Gamma)$. Since Γ contains no φ-singular point we have for each q in Γ either $\varphi(\bar{p}) \neq \varphi(q)$ or $f(\bar{p}) \neq f(q)$. From this we obtain, arguing as in the proof of Lemma 4.1 that, given $\bar{r} > 0$, we can find ε', $|\varepsilon'| < \bar{r}$, so that if $\Psi' = \varphi + \varepsilon' f$, then $\Psi'(q') \neq \Psi'(\bar{p})$ for every $q' \in \Gamma$. Using Lemma 4.2 we can show, as earlier, that Theorem 3.1 is applicable to the pair Ψ', f.

Form $K_{\Psi'}$ in accordance with Definition 3.4, applied to Ψ'. Theorem 3.1 then yields that $K_{\Psi'}$ has no singularities except at points of $E(\Psi')$ and $Z(\Psi')$. Now $\bar{p} \notin E(\Psi')$, since we chose Ψ' so that $\Psi'(\bar{p}) \notin \Psi'(\Gamma)$. Also, since $\bar{p} \in E(\varphi)$, $\varphi(\bar{p}) \in \varphi(\Gamma)$. Since Γ contains no φ-singular point, there is a constant $\delta > 0$ with $|\varphi(\bar{p}) - \varphi(p')| \geqq \delta$ for every φ-singular point p' in $D \cup \Gamma$. By choosing ε' sufficiently near zero, now, we can assume that if b is any Ψ'-singular point, then there exists a φ-singular point c with $|\varphi(b) - \varphi(c)| < \delta$. Hence $\bar{p}$ is not Ψ'-singular. Hence $\bar{p} \notin Z(\Psi')$. Hence $K_{\Psi'}$ is regular at $\bar{p}$. But by (4.6) applied to Ψ', this yields that K_Ψ is regular at $\bar{p}$, unless $d\varphi(\bar{p}) = 0$. But that would imply that Γ contains a φ-singular point, since $\bar{p} \in E(\varphi)$. Hence $d\varphi(\bar{p}) \neq 0$ and so K_φ is regular at $\bar{p}$.

Thus K_φ is regular at each point of $E(\varphi)$, and our theorem is completely proved.

5. In this final section we prove Theorems 1.1 and 1.2 by use of Theorem 4.1.

PROOF OF THEOREM 1.1. The necessity of conditions (1.3) and (1.4) is obvious.

Assume now these conditions hold. Fix $g \in \mathfrak{A}(D_0)$. Then there is some region $\mathscr{S}g$ containing $D_0 \cup \Gamma_0$ in which g is analytic. We can assume without loss of generality that the curve Γ and the region D of Definition 1.4 lie in $\mathscr{S}g$ and that no φ-singular points lie in $D - (D_0 \cup \Gamma_0)$. Then $Z(\varphi) \subset D_0 \cup \Gamma_0$. We assert that for each $\eta > 0$, there is some G in $[\varphi, f]$ with

$$\int_0^{2\pi} |g(e^{i\theta}) - G(e^{i\theta})|^2 d\theta < \eta \,, \tag{5.1}$$

where $e^{i\theta}$ are coordinates on Γ. For choose $k^* \in L^2(0, 2\pi)$ such that

$$\int_0^{2\pi} G(e^{i\theta})k^*(e^{i\theta})d\varphi = 0 \,, \qquad \text{for all } G \in [\varphi, f]. \tag{5.3}$$

Because of (5.3), $k^*d\varphi$ satifies (2.1). Hence Theorem 4.1 is applicable and it yields a function K on D satisfying (4.1) and (4.2) and analytic on D except for poles in $Z(\varphi)$ of order $\leqq n$.

We claim that the residue of $Kd\varphi$ vanishes at each $\bar{p} \in Z(\varphi)$. For let $p_1, \cdots, p_s$ be the remaining points in $Z(\varphi)$. Because of (1.3) and the fact that $Z(\varphi) \subset D_0 \cup \Gamma_0$, the points $(\varphi(\bar{p}), f(\bar{p}))$, $(\varphi(p_1), f(p_1))$, $\cdots$, $(\varphi(p_s), f(p_s))$, are distinct in the space of two complex variables. Hence there is a polynomial P such that $P(\varphi, f)$ vanishes at $p_1, \cdots p_s$ and equals 1 at $\bar{p}$.

Set $Q = P^n$. Then for $\nu \geqq 1$, $Q^\nu K$ is analytic at every point of D except at $\bar{p}$. Because of (4.1) and Cauchy's theorem, we have for $\nu \geqq 1$, if β is a simple closed curve in D containing in its interior $\bar{p}$ but none of the p_i, $1 \leqq i \leqq s$, that

$$\frac{1}{2\pi i}\int_\Gamma Q^\nu k^* d\varphi = \frac{1}{2\pi i}\int_\beta Q^\nu K d\varphi \,. \tag{5.4}$$

It is easy to see that there exits constants $C_0, C_1, \cdots, C_m$ with

$$\frac{1}{2\pi i}\int_\beta Q^\nu K d\varphi = C_0 + C_1\nu + C_2\nu^2 + \cdots + C_m\nu^m \tag{5.5}$$

for $\nu = 1, 2, \cdots$, where $C_0 =$ residue of $Kd\varphi$ at $\bar{p}$. But now $Q^\nu \in [\varphi, f]$ for each ν and so (5.3) yields that the left side in (5.4) vanishes for $\nu \geqq 1$. Hence all the C_i must vanish and in particular C_0, the residue of $Kd\varphi$ at $\bar{p}$ vanishes.

Now for any G_0 in $[\varphi, f]$, the measure $G_0 k^* d\varphi$ also satisfies (2.1) and

$$\int_0^{2\pi} |G_0(e^{i\theta})k^*(e^{i\theta})|^2 d\theta < \infty \,.$$

Hence we may apply the preceding to conclude that the residue of $G_0Kd\varphi$ vanishes at $\bar{p}$.

Consider now the function g in $\mathfrak{A}(D_0)$ with which we began and fix $\bar{p}$ in $Z(\varphi)$. Because of (1.2) and (1.4), either $d\varphi(\bar{p}) \neq 0$ or $df(\bar{p}) = 0$. Assume the first. Then in some neighborhood of $\bar{p}$, g is a uniform limit of polynomials in φ. Since the residue of $G_0Kd\varphi$ at $\bar{p}$ is 0 for each G_0 in $[\varphi, f]$, we get that the residue of $gKd\varphi$ is zero at $\bar{p}$ also. If $df(\bar{p}) = 0$, the same will hold.

This now holds for each point $\bar{p}$ in $Z(\varphi)$. If $\bar{p} \notin Z(\varphi)$, $gKd\varphi$ is analytic at $\bar{p}$. Hence by the residue theorem applied to $gKd\,\varphi$, and using (4.1) we get

$$\int_0^{2\pi} g(e^{i\theta})k^*(e^{i\theta})d\varphi = 0 . \tag{5.6}$$

We have then seen that for k^* in $L^2(0, 2\pi)$, (5.3) implies (5.6). By a well-known property of the space L^2, this implies that (5.1) holds for some $G \in [\varphi, f]$. On the other hand, a sequence of analytic functions converging in the L^2-sense on Γ converges uniformly on every compact set in D. Hence there exists a sequence of functions in $[\varphi, f]$ converging uniformly to g on the compact set $D_0 \cup \Gamma_0$. Thus Theorem 1.1 is proved.

Proof of Theorem 1.2. Fix g in $\mathfrak{A}(D_0)$. Let $\mathscr{S}g$ be a region in which g is analytic containing $D_0 \cup \Gamma_0$ and choose Γ as in the preceding proof. Let k^* satisfy (5.3) and lie in $L^2(0, 2\pi)$.

Set $T = Z(\varphi)$ and set $\bar{n} = n$, as given in Definition 3.1. Then T is a finite subset of $D_0 \cup \Gamma_0$. Then, by Theorem 4.1, K has poles only at points of $Z(\varphi)$ and there of order $\leqq n$. Hence if g vanishes at each point on $Z(\varphi)$ with an order $\geqq \bar{n}$, the differential $gK^*d\varphi$ is analytic everywhere in D. Hence

$$\int_\Gamma gk^*d\varphi = 0 .$$

From this it follows as in the proof of the last theorem that g is approximable uniformly on $D_0 \cup \Gamma_0$ by functions in $[\varphi, f]$. This completes the proof of Theorem 1.2.

Appendix

Let A be any subalgebra of $\mathfrak{A}(D_0)$ which contains a function satisfying (1.2). It is then easy to prove the following generalization of Theorem 1.1,

Theorem. *Assume that*

(a) *A separates points on* $D_0 \cup \Gamma_0$.

(b) *If* $p \in D_0$, *there is some* g_p *in* A *with* $dg_p \neq 0$ at p. *Then every function in A can be approximated uniformly on* $D_0 \cup \Gamma_0$ *by functions of* A.

It is easy to construct two functions φ and f in A which satisfy (1.2) and (1.5). We may then apply Theorem 4.1 to the algebra $[\varphi, f]$ which is contained in A, and finally argue as in the proof of Theorem 1.1. We shall not enter into the details of the argument.

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REFERENCES

1. J. WERMER, *Function rings and Riemann surfaces*, Ann of Math., 67 (1958), 45-71.
2. J. L. WALSH, Interpolation and Approximation by rational functions in the complex domain, Amer. Math. Soc. Colloquium Publications, Vol. 20.