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# Quantization of the elastic modes in an isotropic plate

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## Abstract

We quantize the elastic modes in a plate. For this, we find a complete orthogonal set of eigenfunctions of the elastic equations and normalize them. These are the phonon modes in the plate and their specific forms and dispersion relations are manifested in low-temperature experiments in ultra-thin membranes.

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## 1. Introduction

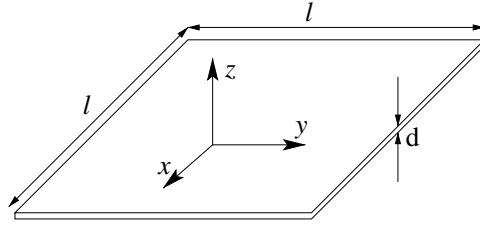
Nowadays, the devices used in many high-sensitivity applications reach such a level of miniaturization that the wavelength of the quantum quasiparticles used in their modeling is comparable to the dimensions of the device. The examples we are most familiar with are the ultra-sensitive electromagnetic radiation detectors. In a very general way of speaking, such detectors consist of some thin metallic films, a few tens of nanometers in thickness, deposited on a dielectric membrane. The dielectric membrane is usually made of  $\text{SiN}_x$  and has a thickness of the order of 100 nm. To reach the level of sensitivity and speed required for applications, these detectors have to work at sub-Kelvin temperatures and at such temperatures the dominant phonon wavelength is comparable to the devices' thickness [1–9].

To describe the thermal properties of such membranes or detectors, the electron–phonon interaction, or in general any interaction of phonons with impurities or disorder in the membrane, we have to know the phonon modes in the membrane. For this, we have to find the eigenmodes of the elastic equations and quantize the elastic field.

For infinite half-spaces, the quantization has been carried out by Ezawa [10].

### 1.1. Elastic eigenmodes in plates with parallel surfaces

The dielectric membranes that we discuss here are actually plates with parallel surfaces in the usual nomenclature of elasticity theory. Therefore, to make the paper more readable to the mathematically inclined reader, as well as to the people working in elasticity theory, we shall



**Figure 1.** Typical plate, or membrane, like those used to support mesoscopic detectors. The thickness  $d$  is of the order of 100 nm and  $l \gg d$ . The plate surfaces are parallel to the  $xy$  plane and cut the  $z$  axis at  $\pm d/2$ .

apply here this nomenclature in spite of the fact that experimentalists seem to prefer the term ‘membrane’.

The elastic eigenmodes of 3D bulk systems are simple plane waves of three different polarizations ( $\sigma$ ), two transversal ( $\sigma = t$ ) and one longitudinal ( $\sigma = l$ ). In the presence of boundaries these bulk modes are coupled to each other and form a new set of eigenmodes.

The elastic eigenmodes in plates with parallel surfaces have been studied for a long time, mostly in connection with sound propagation and earthquakes (see, for example, [11]). To introduce these modes, let us consider a plate of thickness  $d$  and area  $l^2$ , with  $l \gg d$ . The two surfaces of the plate are parallel to the  $(xy)$  plane and cut the  $z$  axis at  $\pm d/2$  (see figure 1). Throughout the paper we shall use  $V$  for the volume of the plate (or in general of the solid that we describe—see section 2.1) and  $\partial V$  for its surface. We shall assume that  $l$  is much bigger than any wavelength of the elastic perturbations considered here. The *displacement field* at position  $\mathbf{r}$  is going to be denoted by  $\mathbf{u}(\mathbf{r})$  or  $\mathbf{v}(\mathbf{r})$ . The unit vectors along the coordinate axes are denoted by  $\hat{\mathbf{x}}$ ,  $\hat{\mathbf{y}}$  and  $\hat{\mathbf{z}}$ .

The displacement fields obey the dynamic equation

$$\rho \partial^2 u_i / \partial t^2 = c_{ijkl} \partial_j \partial_k u_l \quad \forall i = 1, 2, 3. \quad (1)$$

Here, as everywhere in this paper we shall assume summation over repeated indices. Assuming that the medium is isotropic (see, for example, [11] for the constraints on the tensor  $[c]$ ), equation (1) is reduced to  $\partial^2 u_i / \partial t^2 = \partial_j p_{ij}$ , with  $p_{ij}$  defined as in [10],  $p_{ij} = \rho^{-1} c_{ijkl} \partial_k u_l = (c_l^2 - 2c_t^2)(\partial_k u_k) \delta_{ij} + c_t^2(\partial_i u_j + \partial_j u_i)$ , and  $c_l$ ,  $c_t$  the longitudinal and transversal sound velocities, respectively.

Introducing the operator  $\tilde{L}$  (we shall use *tilde* for operators and *hat* for unit vectors) by  $\tilde{L}\mathbf{u} \equiv \rho^{-1} c_{ijkl} \partial_j \partial_k u_l = c_l^2 \text{grad div } \mathbf{u} - c_t^2 \text{curl curl } \mathbf{u} \equiv c_l^2 \nabla \cdot \nabla \cdot \mathbf{u} - c_t^2 \nabla \times \nabla \times \mathbf{u}$ , the wave equation (1) becomes

$$\partial^2 \mathbf{u} / \partial t^2 = \tilde{L}\mathbf{u}. \quad (2)$$

The surface is free, so the stress should be zero there. This amounts to the boundary conditions [10, 11]

$$p_{ij} n_j = 0, \quad \text{on } \partial V, \quad \forall i = 1, 2, 3, \quad (3)$$

where  $\hat{\mathbf{n}}$  is the unit vector normal to the surface of components  $n_1$ ,  $n_2$  and  $n_3$ —we shall use this notation throughout the paper.

Applying equations (2) and (3) to the plate we obtain the elastic eigenmodes, which are classified into three groups, according to their symmetry or polarization direction: the *horizontal shear* ( $h$ ), the *symmetric* ( $s$ ) and the *antisymmetric* ( $a$ ) Lamb waves. We call these different groups ‘polarizations’ and shall denote them by  $\sigma$ , as in the case of the bulk phonons. All these waves are propagating (or decaying, if the wave vector is complex) along the plate

and have a stationary form in the direction perpendicular to the surfaces. The  $h$  wave is polarized parallel to the surfaces and perpendicular to the propagation direction. The  $s$  and  $a$  waves are superpositions of longitudinal and transversal waves, polarized in a plane that is perpendicular to the surfaces and contains the propagation direction. The difference between the  $s$  and the  $a$  waves comes from the fact that the displacement field along the  $z$  direction is symmetric for the  $s$  wave and antisymmetric for the  $a$  wave, while the displacement field along the propagation direction is antisymmetric for the  $s$  wave and symmetric for the  $a$  wave (see below). Explicitly, the three types of modes are

$$\mathbf{u}_h = (\hat{\mathbf{k}}_{\parallel} \times \hat{\mathbf{z}}) \cdot N_h \cos \left[ \frac{m\pi}{b} \left( z - \frac{b}{2} \right) \right] e^{i(\mathbf{k}_{\parallel} \cdot \mathbf{r} - \omega t)}, \quad m = 0, 1, 2, \dots, \quad (4a)$$

$$\begin{aligned} \mathbf{u}_s = N_s \left\{ \hat{\mathbf{z}} \cdot \mathbf{k}_{\parallel} \left[ -2k_t k_l \cos \left( k_t \frac{b}{2} \right) \sin(k_l z) + [k_t^2 - k_{\parallel}^2] \cos \left( k_l \frac{b}{2} \right) \sin(k_t z) \right] \right. \\ \left. + \hat{\mathbf{k}}_{\parallel} \cdot i k_t \left[ 2k_{\parallel}^2 \cos \left( k_t \frac{b}{2} \right) \cos(k_l z) + [k_t^2 - k_{\parallel}^2] \cos \left( k_l \frac{b}{2} \right) \cos(k_t z) \right] \right\} e^{i(\mathbf{k}_{\parallel} \cdot \mathbf{r} - \omega t)}, \quad (4b) \end{aligned}$$

$$\begin{aligned} \mathbf{u}_a = N_a \left\{ \hat{\mathbf{z}} \cdot \mathbf{k}_{\parallel} \left[ 2k_t k_l \sin \left( k_t \frac{b}{2} \right) \cos(k_l z) - [k_t^2 - k_{\parallel}^2] \sin \left( k_l \frac{b}{2} \right) \cos(k_t z) \right] \right. \\ \left. + \hat{\mathbf{k}}_{\parallel} \cdot i k_t \left[ 2k_{\parallel}^2 \sin \left( k_t \frac{b}{2} \right) \sin(k_l z) + [k_t^2 - k_{\parallel}^2] \sin \left( k_l \frac{b}{2} \right) \sin(k_t z) \right] \right\} e^{i(\mathbf{k}_{\parallel} \cdot \mathbf{r} - \omega t)}, \quad (4c) \end{aligned}$$

where  $\hat{\mathbf{k}}_{\parallel}$  is the unit vector along the propagation direction and  $\mathbf{k}_{\parallel} = k_{\parallel} \cdot \hat{\mathbf{k}}_{\parallel}$ . In the  $s$  and  $a$  modes, the wave vector in the  $z$  direction takes two values,  $k_t$  and  $k_l$ , one corresponding to the transversal component of the mode, the other one corresponding to the longitudinal component. The constants  $N_h$ ,  $N_s$  and  $N_a$  are the normalization constants, which will be calculated in section 3.

The components of the wave vectors,  $k_t$ ,  $k_l$  and  $k_{\parallel}$  obey the transcendental equations [11]

$$\frac{\tan \left( \frac{d}{2} k_t \right)}{\tan \left( \frac{d}{2} k_l \right)} = - \frac{4k_l k_t k_{\parallel}^2}{[k_t^2 - k_{\parallel}^2]^2} \quad (5a)$$

and

$$\frac{\tan \left( \frac{d}{2} k_t \right)}{\tan \left( \frac{d}{2} k_l \right)} = - \frac{[k_t^2 - k_{\parallel}^2]^2}{4k_l k_t k_{\parallel}^2} \quad (5b)$$

for symmetric and antisymmetric waves, respectively.

All the solutions of the elastic equation (2) are given by equations (4a), (4b) and (4c), with  $m$  taking all the natural values,  $0, 1, \dots$ , whereas  $k_t$  and  $k_l$  are the solutions of equations (5a) and (5b).  $k_{\parallel}$  can be a complex number, but the complete, orthogonal set of phonon modes that we shall use in the quantization of the elastic field are those with  $k_{\parallel}$  running from 0 to  $\infty$ .

The paper is organized as follows. In section 2.1, we show that  $\tilde{L}$  is self-adjoint even when applied to the displacement field of an elastic body of arbitrary shape. Therefore, we can form a complete orthonormal set of its eigenfunctions.

The fact that the elastic modes (4) corresponding to different quantum numbers are orthogonal to each other is proved in section 2.2, based on the Hermiticity of  $\tilde{L}$ .

The normalization constants are calculated in section 3 and the formal procedure of quantizing the elastic field is presented in section 4.

We apply this formalism elsewhere to calculate the phonon scattering in amorphous thin plates (membranes) [14].

## 2. Orthogonality and completeness of the set of elastic eigenmodes

### 2.1. Self-adjointness of the operator $\tilde{L}$

We shall prove the self-adjointness of  $\tilde{L}$  on an arbitrary finite volume  $V$ . We assume that  $V$  has the smooth border  $\partial V$ . The operator  $\tilde{L}$  acts on the Hilbert space  $\mathcal{H}$  which consists of the vector functions defined on  $V$ , which are integrable in modulus square. The scalar product on  $\mathcal{H}$  is defined as usual,

$$\langle \mathbf{v} | \mathbf{u} \rangle = \int_V \mathbf{v}^\dagger(\mathbf{r}) \cdot \mathbf{u}(\mathbf{r}) \, d^3 \mathbf{r}, \quad (6)$$

and the norm is  $\|\mathbf{u}\| \equiv \langle \mathbf{u} | \mathbf{u} \rangle^{1/2}$ . The domain of  $\tilde{L}$ , denoted by  $\mathcal{D}(\tilde{L})$ , is formed of such functions  $\mathbf{u}(\mathbf{r}) \in \mathcal{H}$ , so that  $\tilde{L}\mathbf{u}(\mathbf{r})$  exists and it is contained in  $\mathcal{H}$ . Moreover, the functions from  $\mathcal{D}(\tilde{L})$  should obey the boundary conditions (3). Ezawa showed that  $\tilde{L}$  is Hermitian [10]. We will show now that  $L$  is also self-adjoint.

First, for a formal treatment, in this section we will understand the derivatives in a generalized sense. If  $f(\mathbf{r})$  is an arbitrary function on  $V$ , then  $\partial_i f$  is defined as the function that satisfies

$$\int_V g(\mathbf{r}) \partial_i f(\mathbf{r}) \, d^3 \mathbf{r} = \int_{\partial V} f(\mathbf{r}) g(\mathbf{r}) n_i \, d^2 \mathbf{r} - \int_V f(\mathbf{r}) \partial_i g(\mathbf{r}) \, d^3 \mathbf{r}, \quad (7)$$

for any function  $g(\mathbf{r})$  of class  $C^1(V)$  (i.e.  $g(\mathbf{r})$  is derivable, with continuous first derivatives on  $V$ ) and all the integrals on the right-hand side of equation (7) exist and are finite.

Returning to our operator, let us first note that the space  $\mathcal{D}(\tilde{L})$  includes the space of twice derivable functions, with continuous, integrable, second derivatives,  $C^2(V)$ , which is dense in  $\mathcal{H}$ . Therefore  $\mathcal{D}(\tilde{L})$  is also dense in  $\mathcal{H}$  and  $\tilde{L}$  is a *symmetric operator*. Then we define the adjoint operator  $\tilde{L}^\dagger$  and its domain  $\mathcal{D}(\tilde{L}^\dagger)$ . For this, let  $\mathbf{v}$  be a function in  $\mathcal{H}$ , so that  $\langle \mathbf{v} | \tilde{L}\mathbf{u} \rangle$  is a continuous linear functional in  $\mathbf{u} \in \mathcal{D}(\tilde{L})$ , i.e. there exists an  $M_v > 0$  for which

$$\langle \mathbf{v} | \tilde{L}\mathbf{u} \rangle \leq M_v \|\mathbf{u}\| \quad (8)$$

for any  $\mathbf{u} \in \mathcal{D}(\tilde{L})$ . By the Riesz–Fréchet theorem [12], there exists  $\mathbf{v}^* \in \mathcal{H}$  so that  $\langle \mathbf{v} | \tilde{L}\mathbf{u} \rangle = \langle \mathbf{v}^* | \mathbf{u} \rangle$  for any  $\mathbf{u} \in \mathcal{D}(\tilde{L})$ . The adjoint operator  $\tilde{L}^\dagger$  is defined by the relation  $\tilde{L}^\dagger \mathbf{v} = \mathbf{v}^*$ , for all  $\mathbf{v}$  that satisfy (8). The functions that satisfy (8) form the domain of  $\tilde{L}^\dagger$ , denoted by  $\mathcal{D}(\tilde{L}^\dagger)$ . Since  $\tilde{L}$  is Hermitian, if  $\mathbf{v}, \mathbf{u} \in \mathcal{D}(\tilde{L})$ , then  $\langle \mathbf{v} | \tilde{L}\mathbf{u} \rangle = \langle \tilde{L}\mathbf{v} | \mathbf{u} \rangle$  is a linear functional, so any function from  $\mathcal{D}(\tilde{L})$  is included in  $\mathcal{D}(\tilde{L}^\dagger)$ . Therefore we can write  $\mathcal{D}(\tilde{L}) \subset \mathcal{D}(\tilde{L}^\dagger)$ . To prove that  $\tilde{L}$  is self-adjoint, we have to show that also  $\mathcal{D}(\tilde{L}^\dagger) \subset \mathcal{D}(\tilde{L})$ , so in the end  $\mathcal{D}(\tilde{L}^\dagger) = \mathcal{D}(\tilde{L})$  and  $\tilde{L} = \tilde{L}^\dagger$ .

For this, let us take  $\mathbf{v} \in \mathcal{D}(\tilde{L}^\dagger)$  and  $\mathbf{u} \in \mathcal{D}(\tilde{L})$ . Integrating by parts we get

$$\begin{aligned} \rho \langle \mathbf{v} | \tilde{L}\mathbf{u} \rangle &= \int_V v_i^* c_{ijkl} \partial_j \partial_k u_l \, d^3 \mathbf{r} \\ &= - \int_{\partial V} (\partial_j v_i^*) c_{ijkl} n_k u_l \, d^2 \mathbf{r} + \int_V (\partial_k \partial_j v_i^*) c_{ijkl} u_l \, d^3 \mathbf{r} \\ &= - \int_{\partial V} (\partial_j v_i^*) c_{ijkl} n_k u_l \, d^2 \mathbf{r} + \rho \langle \tilde{L}\mathbf{v} | \mathbf{u} \rangle \equiv \rho \langle \tilde{L}^\dagger \mathbf{v} | \mathbf{u} \rangle, \end{aligned} \quad (9)$$

where we used  $c_{ijkl} n_j \partial_k u_l = 0$  on  $\partial V$  (equation (3)), and the simplified notation

$$\rho \langle \tilde{L}\mathbf{v} | \mathbf{u} \rangle \equiv \int_V (c_{ijkl} \partial_j \partial_k v_i^*) u_l \, d^3 \mathbf{r} = \int_V (c_{lkji} \partial_k \partial_j v_i^*) u_l \, d^3 \mathbf{r}, \quad (11)$$

although  $\mathbf{v}$  is not necessarily a function in  $\mathcal{D}(\tilde{L})$ . The last equality in equation (11) is obtained using  $c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$  [11] and permuting the partial derivatives. But equation (10) means that

$$\langle (\tilde{L} - \tilde{L}^\dagger)\mathbf{v} | \mathbf{u} \rangle = \rho^{-1} \int_{\partial V} \partial_j v_i^* n_k c_{ijkl} u_l \, d^2 \mathbf{r} \quad (12)$$

for any  $\mathbf{u} \in \mathcal{D}(\tilde{L})$ . If  $(\partial_j v_i^*) n_k c_{ijkl}$  is not identically zero on  $\partial V$ , then we can find  $\mathbf{u}$  so that the surface integral in (12) is different from zero. This implies that  $\langle (\tilde{L} - \tilde{L}^\dagger)\mathbf{v} | \mathbf{u} \rangle \neq 0$ , so  $\mathbf{v} \neq 0$  on a set of measure larger than zero in the interior of  $V$ , denoted as  $V^\circ$ . In such a case, we can find a nonempty compact set  $S \subset V^\circ$  and a function  $\mathbf{u}'$ , which is twice derivable, zero outside  $S$  and satisfies

$$\langle (\tilde{L} - \tilde{L}^\dagger)\mathbf{v} | \mathbf{u}' \rangle \neq 0. \quad (13)$$

Since  $\mathbf{u}'$  is zero outside  $S$  and  $S$  is a compact set in  $V^\circ$ , this means that both  $\mathbf{u}'$  and any of its derivatives are zero on  $\partial V$ ; therefore  $\mathbf{u}' \in \mathcal{D}(\tilde{L})$ . Now,  $\mathbf{u}'(\mathbf{r} \in \partial V) = 0$  implies that

$$\int_{\partial V} \partial_j v_i^* n_k c_{ijkl} (u_l + u'_l) \, d^2 \mathbf{r} = \int_{\partial V} \partial_j v_i^* n_k c_{ijkl} u_l \, d^2 \mathbf{r}, \quad (14)$$

which would mean that  $\langle (\tilde{L} - \tilde{L}^\dagger)\mathbf{v} | (\mathbf{u} + \mathbf{u}') \rangle = \langle (\tilde{L} - \tilde{L}^\dagger)\mathbf{v} | \mathbf{u} \rangle$ . Equation (13), on the other hand, implies that

$$\langle (\tilde{L} - \tilde{L}^\dagger)\mathbf{v} | (\mathbf{u} + \mathbf{u}') \rangle \neq \langle (\tilde{L} - \tilde{L}^\dagger)\mathbf{v} | \mathbf{u} \rangle. \quad (15)$$

Since equation (12) should be valid for any function in  $\mathcal{D}(\tilde{L})$ , including  $\mathbf{u}$  and  $\mathbf{u} + \mathbf{u}'$  and equation (14) is true by construction, this implies that (15) is a contradiction; therefore  $(\partial_j v_i^*) n_k c_{ijkl} = 0$  on  $\partial V$ . Moreover, by the definition of  $\mathcal{D}(\tilde{L}^\dagger)$  and the Riesz–Fréchet theorem,  $c_{lkji} \partial_k \partial_j v_i \equiv \rho \tilde{L}^\dagger \mathbf{v} \in \mathcal{H}$ . Therefore  $\mathbf{v} \in \mathcal{D}(\tilde{L})$ , so  $\mathcal{D}(\tilde{L}^\dagger) = \mathcal{D}(\tilde{L})$  and  $\tilde{L} = \tilde{L}^\dagger$ , i.e. the operator  $\tilde{L}$  is self-adjoint on an arbitrary, finite volume  $V$ . Therefore it has an infinite, discrete set of eigenvalues and its eigenfunctions form a complete set.

It is then straightforward to extend the above proof to the rectangular plate with the boundary conditions (3) on its surfaces and periodic boundary conditions on the edges. Then the operator  $\tilde{L}$  is self-adjoint also in this case and it has a discrete, infinite set of real eigenvalues and a complete set of eigenfunctions. To find from the wavefunctions of the form (4) a complete, orthonormal set, we shall use the operator  $\tilde{\mathbf{k}}_{\parallel} \equiv i(\partial_x + \partial_y)$ , which is also self-adjoint when acting on the rectangular plate with periodic boundary conditions at the edges. Since  $\tilde{L}$  and  $\tilde{\mathbf{k}}_{\parallel}$  commute and they are both self-adjoint operators, if we find *all* the eigenfunctions common to  $\tilde{L}$  and  $\tilde{\mathbf{k}}_{\parallel}$ , then we have a complete set. But these functions are given by equations (4), with real  $k_{\parallel}$ , and  $k_l$ ,  $k_l$  satisfying (5).

## 2.2. Orthogonality of the elastic eigenmodes

Now we study the orthogonality of the elastic eigenmodes of the plate. For this, we write the functions that appear in equations (4a)–(4c) in general as  $\mathbf{u}_{\mathbf{k}_{\parallel}, k_l, \sigma}(\mathbf{r}) \equiv \mathbf{u}_{k_l, \sigma}(z) e^{i\mathbf{k}_{\parallel} \cdot \mathbf{r}}$ , where we separated the  $x$  and  $y$  dependence of the fields from the  $z$  dependence and we disregarded the time dependence. By  $\sigma$  we denote the ‘polarization’  $h$ ,  $s$  or  $a$ . We shall not use  $k_l$  explicitly in the notations below, since it is determined implicitly by  $k_l$ ,  $k_{\parallel}$ , and equations (5);  $k_{\parallel}$  takes discrete, but very dense real values, fixed by the periodic boundary conditions at the edges of the plate. First we observe that

$$\langle \mathbf{u}_{\mathbf{k}_{\parallel}, k_l, \sigma} | \mathbf{u}_{\mathbf{k}'_{\parallel}, k'_l, \sigma'} \rangle = 2\pi \delta_{\mathbf{k}_{\parallel}, \mathbf{k}'_{\parallel}} \int_{-b/2}^{b/2} \mathbf{u}_{k_l, \sigma}^\dagger(z) \mathbf{u}_{k'_l, \sigma'}(z) \, dz, \quad (16)$$

so we are left to study the orthogonality of functions with the same  $\mathbf{k}_\parallel$ . For simplicity we choose  $\mathbf{k}_\parallel = \hat{\mathbf{x}} \cdot k_\parallel$ , so the  $h$  waves are polarized in the  $\hat{\mathbf{y}}$  direction and the  $s$  and  $a$  waves have displacement fields in the  $(xz)$  plane. Since the displacement fields of the  $h$  waves are perpendicular to the displacement fields of the  $s$  and  $a$  waves, any  $h$  wave is orthogonal to any  $s$  or  $a$  wave. Similarly, for the same  $\mathbf{k}_\parallel$ , the displacement fields of any of the  $s$  and  $a$  waves, although in the same plane, are orthogonal to each other due to their opposite symmetries. We conclude that

$$\int_{-b/2}^{b/2} \mathbf{u}_{k_t, \sigma}^\dagger(z) \mathbf{u}_{k_t', \sigma'}(z) dz = 0$$

for any  $k_t$  and  $k_t'$ , if  $\sigma \neq \sigma'$ . Therefore  $\langle \mathbf{u}_{\mathbf{k}_\parallel, k_t, \sigma} | \mathbf{u}_{\mathbf{k}_\parallel, k_t', \sigma'} \rangle \propto \delta_{\mathbf{k}_\parallel, \mathbf{k}_\parallel'} \delta_{\sigma, \sigma'}$ .

We are left to show that  $\langle \mathbf{u}_{\mathbf{k}_\parallel, k_t, \sigma} | \mathbf{u}_{\mathbf{k}_\parallel, k_t', \sigma} \rangle \propto \delta_{k_t, k_t'}$  which follows simply from the Hermiticity of the operator  $\tilde{L}$ . We calculate the matrix element

$$\begin{aligned} \langle \mathbf{u}_{\mathbf{k}_\parallel, k_t, \sigma} | \tilde{L} \mathbf{u}_{\mathbf{k}_\parallel, k_t', \sigma} \rangle &= -\omega_{\mathbf{k}_\parallel, k_t', \sigma}^2 \langle \mathbf{u}_{\mathbf{k}_\parallel, k_t, \sigma} | \mathbf{u}_{\mathbf{k}_\parallel, k_t', \sigma} \rangle \\ &= \langle \tilde{L} \mathbf{u}_{\mathbf{k}_\parallel, k_t, \sigma} | \mathbf{u}_{\mathbf{k}_\parallel, k_t', \sigma} \rangle = -\omega_{\mathbf{k}_\parallel, k_t, \sigma}^2 \langle \mathbf{u}_{\mathbf{k}_\parallel, k_t, \sigma} | \mathbf{u}_{\mathbf{k}_\parallel, k_t', \sigma} \rangle. \end{aligned}$$

But since for given  $\mathbf{k}_\parallel$  and  $\sigma$ , the eigenstates of  $\tilde{L}$  are not degenerate,  $\langle \mathbf{u}_{\mathbf{k}_\parallel, k_t, \sigma} | \mathbf{u}_{\mathbf{k}_\parallel, k_t', \sigma} \rangle = 0$ , unless  $k_t = k_t'$  and, of course,  $k_l = k_l'$ . This completes the proof and

$$\langle \mathbf{u}_{\mathbf{k}_\parallel, k_t, \sigma} | \mathbf{u}_{\mathbf{k}_\parallel, k_t', \sigma'} \rangle \propto \delta_{\sigma, \sigma'} \delta_{\mathbf{k}_\parallel, \mathbf{k}_\parallel'} \delta_{k_t, k_t'}. \quad (17)$$

### 3. Normalization of the elastic modes

If  $A = l \times l$  is the area of the plate, then its volume is  $V = Ad$ . The scalar product (17) should give

$$\langle \mathbf{u}_{\mathbf{k}_\parallel, k_t, \sigma} | \mathbf{u}_{\mathbf{k}_\parallel, k_t', \sigma'} \rangle = \delta_{\sigma, \sigma'} \delta_{\mathbf{k}_\parallel, \mathbf{k}_\parallel'} \delta_{k_t, k_t'}. \quad (18)$$

For horizontal shear waves, the normalization constant is simple to calculate. In this case, let us write  $\mathbf{u}_{\mathbf{k}_\parallel, k_t, h}$  simply as  $\mathbf{u}_{\mathbf{k}_\parallel, m, h}$  (see (4a)) and we obtain

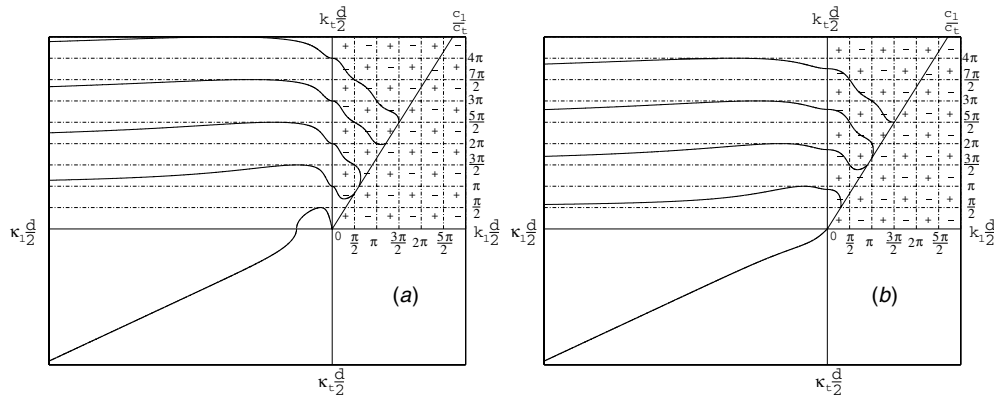
$$\|\mathbf{u}_{\mathbf{k}_\parallel, m, h}\|^2 = (N_{\mathbf{k}_\parallel, m, h})^2 \frac{V}{2}, \quad \text{for } m > 0, \quad (19a)$$

$$= (N_{\mathbf{k}_\parallel, m, h})^2 V, \quad \text{for } m = 0. \quad (19b)$$

So  $N_{\mathbf{k}_\parallel, 0, h} = V^{-1/2}$  and  $N_{\mathbf{k}_\parallel, m > 0, h} = (2/V)^{1/2}$ .

For symmetric Lamb modes, from equation (5b) we calculate  $k_l$  and  $k_t$  as functions of  $k_\parallel$ . The results are shown in figure 2(a). As expected, for each value of  $k_\parallel$ , the wave vectors  $k_l$  and  $k_t$  take only discrete values, but not as simple as the values corresponding to the  $h$  modes (4a). Each curve in figure 2(a) corresponds to a different branch of the dispersion relation,  $\omega_{k_\parallel, k_t(k_\parallel, m), s}$ , where  $m = 0, 1, \dots$ , denotes the branch number. Branches with bigger  $m$  are placed above branches with smaller  $m$ .

As  $k_\parallel$  increases,  $k_l(k_\parallel, m)$  decreases and, after reaching the value 0, turns imaginary. On the other hand, first  $k_t(k_\parallel, m)$  increases with  $k_\parallel$ , reaches a maximum value and then decreases monotonically as  $k_\parallel$  increases to infinity. Its decrease is bounded for all branches, except the lowest one, where, after reaching zero at some finite  $k_\parallel$ , it turns imaginary in the lower-left quadrant of figure 2(a). For the clarity of the calculations, the imaginary values of  $k_l$  and  $k_t$



**Figure 2.** The curves  $(k_l(k_{||}), k_t(k_{||}))$  for symmetric (a) and antisymmetric (b) Lamb modes. On each curve  $k_{||}$  varies from 0 to  $\infty$ , as we go from its right end to the left. The different curves correspond to different branches of the dispersion relation, i.e. to different values of  $k_l$  and  $k_t$  at  $k_{||} = 0$ . Both  $k_l$  and  $k_t$  may take real and imaginary values. The imaginary values of  $k_l$ , denoted by  $\kappa_l$  are to the right of the vertical axis and the imaginary values of  $k_t$ , denoted by  $\kappa_t$  are below the horizontal axis in both figures. As in the rest of the paper,  $d$  is the thickness of the plate. For these plots we used  $c_l/c_t = 1.66$ , corresponding to  $\text{SiN}_x$  plates. The + and - signs in the upper-right quadrants are only for orientation purposes.

are denoted as  $i\kappa_l$  and  $i\kappa_t$ , respectively, where  $\kappa_l$  and  $\kappa_t$  take real, positive values. Both  $\kappa_l$  and  $\kappa_t$  increase without limit as  $k_{||}$  increases to infinity.

We encounter a similar situation for the antisymmetric Lamb modes (see figure 2(b)). The only marked differences between the symmetric and antisymmetric modes are the following: the asymptotic values of  $(d/2) \cdot k_t(k_{||} \rightarrow \infty, m)$  are  $\pi, 2\pi, \dots$ , for symmetric modes and  $\pi/2, 3\pi/2, \dots$ , for antisymmetric modes; the maxima of  $(d/2) \cdot k_t(k_{||}, m)$  are  $(2m + 1)\pi/2$  for the symmetric modes and  $m\pi$  for the antisymmetric modes. Also, for the antisymmetric modes  $k_t(k_{||}, 0)$  (i.e. the lowest branch) takes only imaginary values.

Integrating  $|\mathbf{u}_s|^2$  over the volume of the plate and using the transcendental equation (5a), together with Snell's law,  $\omega^2 = c_t^2(k_t^2 + k_{||}^2) = c_l^2(k_l^2 + k_{||}^2)$ , we obtain the normalization constant for the symmetric modes in the *quadrant I* (upper right) of figure 2(a)

$$\begin{aligned}
 (N_s^I)^{-2} = A \left\{ 4k_t^2 k_{||}^2 \cos^2(k_t d/2) \left( (k_l^2 + k_{||}^2) \frac{d}{2} - (k_t^2 - k_{||}^2) \frac{\sin(k_l d)}{2k_l} \right) \right. \\
 + (k_t^2 - k_{||}^2)^2 \cos^2(k_l d/2) \left( (k_t^2 + k_{||}^2) \frac{d}{2} + (k_t^2 - k_{||}^2) \frac{\sin(k_t d)}{2k_t} \right) \\
 \left. + 4k_t k_{||}^2 (k_t^2 - k_{||}^2) \cos^2(k_l d/2) \sin(k_t d) \right\} \quad (20a)
 \end{aligned}$$

in the *quadrant II* (upper left),

$$\begin{aligned}
 (N_s^{II})^{-2} = A \left\{ 4k_t^2 k_{||}^2 \cos^2(k_t d/2) \left( (k_l^2 + k_{||}^2) \frac{\sinh(\kappa_l d)}{2\kappa_l} - (\kappa_l^2 - k_{||}^2) \frac{d}{2} \right) \right. \\
 + (k_t^2 - k_{||}^2)^2 \cosh^2(\kappa_l d/2) \left( (k_t^2 + k_{||}^2) \frac{d}{2} + (k_t^2 - k_{||}^2) \frac{\sin(k_t d)}{2k_t} \right) \\
 \left. + 4k_t k_{||}^2 (k_t^2 - k_{||}^2) \cosh^2(\kappa_l d/2) \sin(k_t d) \right\} \quad (20b)
 \end{aligned}$$



and in the *quadrant III* (lower left),

$$(N_s^{III})^{-2} = A \left\{ 4\kappa_t^2 k_{\parallel}^2 \cosh^2(\kappa_t d/2) \left( (\kappa_t^2 + k_{\parallel}^2) \frac{\sinh(\kappa_t d)}{2\kappa_t} - (\kappa_t^2 - k_{\parallel}^2) \frac{d}{2} \right) \right. \\ \left. + (\kappa_t^2 + k_{\parallel}^2)^2 \cosh^2(\kappa_t d/2) \left( (\kappa_t^2 + k_{\parallel}^2) \frac{\sinh(\kappa_t d)}{2\kappa_t} + (\kappa_t^2 - k_{\parallel}^2) \frac{d}{2} \right) \right. \\ \left. - 4\kappa_t k_{\parallel}^2 (\kappa_t^2 + k_{\parallel}^2) \cosh^2(\kappa_t d/2) \sinh(\kappa_t d) \right\}. \quad (20c)$$

Similarly, the normalization constants for antisymmetric modes in the three quadrants are

$$(N_a^I)^{-2} = A \left\{ 4k_t^2 k_{\parallel}^2 \sin^2(k_t d/2) \left( (k_t^2 + k_{\parallel}^2) \frac{d}{2} + (k_t^2 - k_{\parallel}^2) \frac{\sin(k_t d)}{2k_t} \right) + (k_t^2 - k_{\parallel}^2)^2 \sin^2(k_t d/2) \right. \\ \left. \times \left( (k_t^2 + k_{\parallel}^2) \frac{d}{2} - (k_t^2 - k_{\parallel}^2) \frac{\sin(k_t d)}{2k_t} \right) - 4k_t k_{\parallel}^2 (k_t^2 - k_{\parallel}^2) \sin^2(k_t d/2) \sin(k_t d) \right\} \quad (21a)$$

$$(N_a^{II})^{-2} = A \left\{ 4k_t^2 k_{\parallel}^2 \sin^2(k_t d/2) \left( (\kappa_t^2 + k_{\parallel}^2) \frac{\sinh(\kappa_t d)}{2\kappa_t} + (\kappa_t^2 - k_{\parallel}^2) \frac{d}{2} \right) \right. \\ \left. + (k_t^2 - k_{\parallel}^2)^2 \sinh^2(\kappa_t d/2) \left( (k_t^2 + k_{\parallel}^2) \frac{d}{2} - (k_t^2 - k_{\parallel}^2) \frac{\sin(k_t d)}{2k_t} \right) \right. \\ \left. - 4k_t k_{\parallel}^2 (k_t^2 - k_{\parallel}^2) \sinh^2(\kappa_t d/2) \sin(k_t d) \right\} \quad (21b)$$

and

$$(N_a^{III})^{-2} = A \left\{ 4\kappa_t^2 k_{\parallel}^2 \sinh^2(\kappa_t d/2) \left( (\kappa_t^2 + k_{\parallel}^2) \frac{\sinh(\kappa_t d)}{2\kappa_t} + (\kappa_t^2 - k_{\parallel}^2) \frac{d}{2} \right) \right. \\ \left. + (\kappa_t^2 + k_{\parallel}^2)^2 \sinh^2(\kappa_t d/2) \left( (\kappa_t^2 + k_{\parallel}^2) \frac{\sinh(\kappa_t d)}{2\kappa_t} - (\kappa_t^2 - k_{\parallel}^2) \frac{d}{2} \right) \right. \\ \left. - 4\kappa_t k_{\parallel}^2 (\kappa_t^2 + k_{\parallel}^2) \sinh^2(\kappa_t d/2) \sinh(\kappa_t d) \right\}. \quad (21c)$$

As one can note, in the *quadrants II* and *III* the trigonometric functions are replaced by hyperbolic functions because of the change  $k_l \rightarrow i\kappa_l$  in *quadrant II* and  $k_l \rightarrow i\kappa_l$ ,  $k_t \rightarrow i\kappa_t$  in *quadrant III*.

Although equations (20) and (21), with all their cases, are convenient because they are ready to use in practical calculations, we shall pack each set into one equation with the help of the complex wave vector components  $\bar{k}_t \equiv k_t + i\kappa_t$  and  $\bar{k}_l \equiv k_l + i\kappa_l$ ,

$$N_s^{-2} = A \left\{ 4|\bar{k}_t|^2 k_{\parallel}^2 |\cos(\bar{k}_t d/2)|^2 \left( (|\bar{k}_l|^2 + k_{\parallel}^2) \frac{\sinh(\kappa_l d)}{2\kappa_l} - (|\bar{k}_l|^2 - k_{\parallel}^2) \frac{\sin(k_l d)}{2k_l} \right) \right. \\ \left. + |\bar{k}_t^2 - k_{\parallel}^2|^2 |\cos(\bar{k}_t d/2)|^2 \left( (|\bar{k}_l|^2 + k_{\parallel}^2) \frac{\sinh(\kappa_l d)}{2\kappa_l} + (|\bar{k}_l|^2 - k_{\parallel}^2) \frac{\sin(k_l d)}{2k_l} \right) \right. \\ \left. - 4k_{\parallel}^2 |\cos(\bar{k}_t d/2)|^2 (\kappa_t (|\bar{k}_t|^2 + k_{\parallel}^2) \sinh(\kappa_t d) - k_t (|\bar{k}_t|^2 - k_{\parallel}^2) \sin(k_t d)) \right\} \quad (22a)$$

$$N_a^{-2} = A \left\{ 4|\bar{k}_t|^2 k_{\parallel}^2 |\sin(\bar{k}_t d/2)|^2 \left( (|\bar{k}_l|^2 + k_{\parallel}^2) \frac{\sinh(\kappa_l d)}{2\kappa_l} + (|\bar{k}_l|^2 - k_{\parallel}^2) \frac{\sin(k_l d)}{2k_l} \right) \right. \\ \left. + |\bar{k}_t^2 - k_{\parallel}^2|^2 |\sin(\bar{k}_t d/2)|^2 \left( (|\bar{k}_l|^2 + k_{\parallel}^2) \frac{\sinh(\kappa_l d)}{2\kappa_l} - (|\bar{k}_l|^2 - k_{\parallel}^2) \frac{\sin(k_l d)}{2k_l} \right) \right. \\ \left. - 4k_{\parallel}^2 |\sin(\bar{k}_t d/2)|^2 (\kappa_t (|\bar{k}_t|^2 + k_{\parallel}^2) \sinh(\kappa_t d) + k_t (|\bar{k}_t|^2 - k_{\parallel}^2) \sin(k_t d)) \right\}. \quad (22b)$$

Equations (22) are very general. For the phonon modes (i.e. for real  $k_{\parallel}$ ),  $\bar{k}_t$  and  $\bar{k}_l$  are either real or imaginary and the normalization constants (20) and (21) can be extracted from (22) by *taking the limit* of the redundant component of  $\bar{k}_t$  and  $\bar{k}_l$  going to zero. The normalization constants (19)–(21) are chosen so that  $\|\mathbf{u}_{\mathbf{k}_{\parallel},k_t,\sigma}\| = 1$ , for any  $\sigma$ ,  $\mathbf{k}_{\parallel}$  and  $k_t$ . In the following section the wavefunctions will be multiplied by still another constant, which will give the correct dimensions to the phonon field.

#### 4. Quantization of the elastic field

For the quantization of the elastic field we start from the classical Hamiltonian

$$U = \int_V \left( \frac{\rho \dot{\mathbf{u}}^2}{2} + \frac{S_{ij} c_{ijkl} S_{kl}}{2} \right), \quad (23)$$

where  $S_{ij}$  are the components of the strain field, which is the *symmetric gradient* of the displacement field,  $S_{ij} \equiv (\nabla_S \mathbf{u})_{ij} = (\partial_i u_j + \partial_j u_i)/2$ . The canonical variables are the field  $\mathbf{u}$  and the conjugate momentum  $\boldsymbol{\pi} = \rho \dot{\mathbf{u}}$  which satisfy the Hamilton equations

$$\dot{\mathbf{u}} = \frac{\delta U}{\delta \boldsymbol{\pi}}, \quad (24a)$$

$$\dot{\boldsymbol{\pi}} = -\frac{\delta U}{\delta \mathbf{u}}. \quad (24b)$$

Equation (24b) is nothing but the dynamic equation (1).

In the second quantization,  $\mathbf{u}$  and  $\boldsymbol{\pi}$  become the field operators  $\tilde{\mathbf{u}}$  and  $\tilde{\boldsymbol{\pi}}$ , respectively. If we denote by  $\tilde{b}_{\mathbf{k}_{\parallel},k_t,\sigma}^{\dagger}$  and  $\tilde{b}_{\mathbf{k}_{\parallel},k_t,\sigma}$ , the creation and annihilation operators of a phonon with quantum numbers  $\mathbf{k}_{\parallel}$ ,  $k_t$  and polarization  $\sigma$  (in the notation that we used before), then the *real* displacement and generalized momentum field operators,  $\tilde{\mathbf{u}}(\mathbf{r}) = \tilde{\mathbf{u}}^{\dagger}(\mathbf{r})$  and  $\tilde{\boldsymbol{\pi}}(\mathbf{r}) = \tilde{\boldsymbol{\pi}}^{\dagger}(\mathbf{r})$ , are

$$\tilde{\mathbf{u}}(\mathbf{r}) = \sum_{\mathbf{k}_{\parallel},k_t,\sigma} [\mathbf{f}_{\mathbf{k}_{\parallel},k_t,\sigma}(\mathbf{r}) \tilde{b}_{\mathbf{k}_{\parallel},k_t,\sigma} + \mathbf{f}_{\mathbf{k}_{\parallel},k_t,\sigma}^*(\mathbf{r}) \tilde{b}_{\mathbf{k}_{\parallel},k_t,\sigma}^{\dagger}] \quad (25a)$$

and

$$\begin{aligned} \tilde{\boldsymbol{\pi}}(\mathbf{r}) &= \rho \sum_{\mathbf{k}_{\parallel},k_t,\sigma} [\hat{\mathbf{f}}_{\mathbf{k}_{\parallel},k_t,\sigma}(\mathbf{r}) \tilde{b}_{\mathbf{k}_{\parallel},k_t,\sigma} + \hat{\mathbf{f}}_{\mathbf{k}_{\parallel},k_t,\sigma}^*(\mathbf{r}) \tilde{b}_{\mathbf{k}_{\parallel},k_t,\sigma}^{\dagger}] \\ &= -i\rho \sum_{\mathbf{k}_{\parallel},k_t,\sigma} \omega_{\mathbf{k}_{\parallel},k_t,\sigma} [\mathbf{f}_{\mathbf{k}_{\parallel},k_t,\sigma}(\mathbf{r}) \tilde{b}_{\mathbf{k}_{\parallel},k_t,\sigma} - \mathbf{f}_{\mathbf{k}_{\parallel},k_t,\sigma}(\mathbf{r}) \tilde{b}_{\mathbf{k}_{\parallel},k_t,\sigma}^{\dagger}], \end{aligned} \quad (25b)$$

where  $\mathbf{f}_{\mathbf{k}_{\parallel},k_t,\sigma}(\mathbf{r}) \equiv C \mathbf{u}_{\mathbf{k}_{\parallel},k_t,\sigma}(\mathbf{r})$  and  $C$  is a real constant which we shall determine from the commutation relations of the  $\tilde{b}$  operators. In equations (25) we do not take  $k_l$  as a summation variable, since this is either zero (for  $h$  fields), or is determined by  $\mathbf{k}_{\parallel}$  and  $k_t$  (via equation (5a) or (5b) for the symmetric and antisymmetric cases, respectively).

From equations (25) we can extract the operators  $\tilde{b}$  and  $\tilde{b}^{\dagger}$  in terms of  $\tilde{\mathbf{u}}$  and  $\tilde{\boldsymbol{\pi}}$ . In order to do this, let us first note from (4) that

$$[\mathbf{u}_{\mathbf{k}_{\parallel},k_t,h}(\mathbf{r})]^* = \mathbf{u}_{-\mathbf{k}_{\parallel},k_t,h}(\mathbf{r}), \quad (26a)$$

whereas

$$[\mathbf{u}_{\mathbf{k}_{\parallel},k_t,s}(\mathbf{r})]^* = \mathbf{u}_{-\mathbf{k}_{\parallel},k_t,s}(\mathbf{r}) \quad \text{or} \quad [\mathbf{u}_{\mathbf{k}_{\parallel},k_t,s}(\mathbf{r})]^* = -\mathbf{u}_{-\mathbf{k}_{\parallel},k_t,s}(\mathbf{r}), \quad (26b)$$

depending on whether  $k_t$  is real or imaginary, and

$$[\mathbf{u}_{\mathbf{k}_{\parallel},k_t,a}(\mathbf{r})]^* = \mathbf{u}_{-\mathbf{k}_{\parallel},k_t,a}(\mathbf{r}) \quad \text{or} \quad [\mathbf{u}_{\mathbf{k}_{\parallel},k_t,a}(\mathbf{r})]^* = -\mathbf{u}_{-\mathbf{k}_{\parallel},k_t,a}(\mathbf{r}), \quad (26c)$$

depending on whether  $k_l$  is real or imaginary. So

$$\int_V \mathbf{f}_{\mathbf{k}_{\parallel},k_t,\sigma}^T(\mathbf{r}) \cdot \mathbf{f}_{\mathbf{k}'_{\parallel},k'_t,\sigma'}(\mathbf{r}) d^3\mathbf{r} = a_{\sigma,k_t,k_l} C^2 \delta_{\sigma,\sigma'} \delta_{\mathbf{k}_{\parallel},-\mathbf{k}'_{\parallel}} \delta_{k_t,k'_t}, \quad (27)$$

where by  $\mathbf{f}^T$  we denote the transpose of the vector  $\mathbf{f}$  and  $a_{\sigma,k_t,k_l} = \pm 1$ , according to equations (26b) and (26c). Multiplying (25a) and (25b) by  $\mathbf{f}_{\mathbf{k}_{\parallel},k_t,\sigma}^\dagger(\mathbf{r})$  and integrating over  $V$ , we get

$$\int_V \mathbf{f}_{\mathbf{k}_{\parallel},k_t,\sigma}^\dagger(\mathbf{r}) \tilde{\mathbf{u}}(\mathbf{r}) d^3\mathbf{r} = C^2 [\tilde{b}_{\mathbf{k}_{\parallel},k_t,\sigma} + a_{\sigma,k_t,k_l} \tilde{b}_{-\mathbf{k}_{\parallel},k_t,\sigma}^\dagger], \quad (28a)$$

$$\int_V \mathbf{f}_{\mathbf{k}_{\parallel},k_t,\sigma}^\dagger(\mathbf{r}) \tilde{\pi}(\mathbf{r}) d^3\mathbf{r} = -i\rho\omega_{\mathbf{k}_{\parallel},k_t,\sigma} C^2 [\tilde{b}_{\mathbf{k}_{\parallel},k_t,\sigma} - a_{\sigma,k_t,k_l} \tilde{b}_{-\mathbf{k}_{\parallel},k_t,\sigma}^\dagger]. \quad (28b)$$

Solving the system we obtain

$$\tilde{b}_{\mathbf{k}_{\parallel},k_t,\sigma} = \frac{1}{2C^2} \left[ \int_V \mathbf{f}_{\mathbf{k}_{\parallel},k_t,\sigma}^\dagger(\mathbf{r}) \tilde{\mathbf{u}}(\mathbf{r}) d^3\mathbf{r} + \frac{i}{\rho\omega_{\mathbf{k}_{\parallel},k_t,\sigma}} \int_V \mathbf{f}_{\mathbf{k}_{\parallel},k_t,\sigma}^\dagger(\mathbf{r}) \tilde{\pi}(\mathbf{r}) d^3\mathbf{r} \right] \quad (29)$$

and

$$\tilde{b}_{\mathbf{k}_{\parallel},k_t,\sigma}^\dagger = \frac{1}{2C^2} \left[ \int_V \mathbf{f}_{\mathbf{k}_{\parallel},k_t,\sigma}^T(\mathbf{r}) \tilde{\mathbf{u}}(\mathbf{r}) d^3\mathbf{r} - \frac{i}{\rho\omega_{\mathbf{k}_{\parallel},k_t,\sigma}} \int_V \mathbf{f}_{\mathbf{k}_{\parallel},k_t,\sigma}^T(\mathbf{r}) \tilde{\pi}(\mathbf{r}) d^3\mathbf{r} \right]. \quad (30)$$

Using the canonical commutation relations  $[\tilde{\mathbf{u}}(\mathbf{r}), \tilde{\mathbf{u}}(\mathbf{r}')] = [\tilde{\pi}(\mathbf{r}), \tilde{\pi}(\mathbf{r}')] = 0$  and  $[\tilde{\mathbf{u}}(\mathbf{r}), \tilde{\pi}(\mathbf{r}')] = i\hbar\delta(\mathbf{r} - \mathbf{r}')$ , we obtain the commutation relations for the operators  $\tilde{b}$  and  $\tilde{b}^\dagger$ ,

$$[\tilde{b}_{\mathbf{k}_{\parallel},k_t,\sigma}, \tilde{b}_{\mathbf{k}'_{\parallel},k'_t,\sigma'}] = [\tilde{b}_{\mathbf{k}_{\parallel},k_t,\sigma}^\dagger, \tilde{b}_{\mathbf{k}'_{\parallel},k'_t,\sigma'}^\dagger] = 0$$

and

$$[\tilde{b}_{\mathbf{k}_{\parallel},k_t,\sigma}, \tilde{b}_{\mathbf{k}'_{\parallel},k'_t,\sigma'}^\dagger] = \delta_{\sigma,\sigma'} \delta_{\mathbf{k}_{\parallel},\mathbf{k}'_{\parallel}} \delta_{k_t,k'_t},$$

provided that

$$C = \sqrt{\frac{\hbar}{2\rho\omega_{\mathbf{k}_{\parallel},k_t,\sigma}}}. \quad (31)$$

Using equations (25), with the proper normalization of  $\mathbf{f}$ , we can write  $U$  (23) in the operator form

$$\begin{aligned} U &= \frac{\rho}{2} \int_V d^3\mathbf{r} \sum_{\mathbf{k}_{\parallel},k_t,\sigma} \sum_{\mathbf{k}'_{\parallel},k'_t,\sigma'} [\dot{\mathbf{f}}_{\mathbf{k}_{\parallel},k_t,\sigma}^\dagger(\mathbf{r}) \tilde{b}_{\mathbf{k}_{\parallel},k_t,\sigma}^\dagger + \dot{\mathbf{f}}_{\mathbf{k}_{\parallel},k_t,\sigma}(\mathbf{r}) \tilde{b}_{\mathbf{k}_{\parallel},k_t,\sigma}] \\ &\quad \times [\dot{\mathbf{f}}_{\mathbf{k}'_{\parallel},k'_t,\sigma'}(\mathbf{r}) \tilde{b}_{\mathbf{k}'_{\parallel},k'_t,\sigma'} + \dot{\mathbf{f}}_{\mathbf{k}'_{\parallel},k'_t,\sigma'}^*(\mathbf{r}) \tilde{b}_{\mathbf{k}'_{\parallel},k'_t,\sigma'}^\dagger] \\ &\quad + \frac{1}{2} \int_V d^3\mathbf{r} \sum_{\mathbf{k}_{\parallel},k_t,\sigma} \sum_{\mathbf{k}'_{\parallel},k'_t,\sigma'} [\partial_i [\mathbf{f}_{\mathbf{k}_{\parallel},k_t,\sigma}^*(\mathbf{r})]_j \tilde{b}_{\mathbf{k}_{\parallel},k_t,\sigma}^\dagger + \partial_i [\mathbf{f}_{\mathbf{k}_{\parallel},k_t,\sigma}(\mathbf{r})]_j \tilde{b}_{\mathbf{k}_{\parallel},k_t,\sigma}] \\ &\quad \times c_{ijkl} \cdot [\partial_k [\mathbf{f}_{\mathbf{k}_{\parallel},k_t,\sigma}(\mathbf{r})]_j \tilde{b}_{\mathbf{k}_{\parallel},k_t,\sigma} + \partial_k [\mathbf{f}_{\mathbf{k}_{\parallel},k_t,\sigma}^*(\mathbf{r})]_j \tilde{b}_{\mathbf{k}_{\parallel},k_t,\sigma}^\dagger] \\ &= \sum_{\mathbf{k}_{\parallel},k_t,\sigma} \hbar\omega_{\mathbf{k}_{\parallel},k_t,\sigma} [\tilde{b}_{\mathbf{k}_{\parallel},k_t,\sigma}^\dagger \tilde{b}_{\mathbf{k}_{\parallel},k_t,\sigma} + 1/2]. \end{aligned} \quad (32)$$

As expected, the Hamiltonian of the elastic body can be written as a sum of Hamiltonians of harmonic oscillators. These oscillators are the phonon modes of the plate.

We use this formalism elsewhere to describe the interaction of phonons with the disorder in amorphous materials [13, 14].

## 5. Conclusions

The vibrational modes of a thin plate (4) are well known from elasticity theory [11]. The purpose of the paper is to quantize the elastic field and for this we have to know if these modes, or part of them, form a complete set of orthogonal functions. But since the modes are the solutions of the eigenvalue–eigenvector problem of the operator  $\tilde{L}$  (2), we showed that they form a complete set by proving that  $\tilde{L}$  is self-adjoint.

Nevertheless, not all the functions of the form (4) are orthogonal to each other, so to build the complete orthogonal set of functions, we made use of a generic ‘momentum’ operator,  $\tilde{\mathbf{k}}_{\parallel} \equiv i(\partial_x + \partial_y)$ , which commutes with  $\tilde{L}$ . Since for a plate with infinite lateral extension or a finite rectangular plate with periodic boundary conditions at the edges the operator  $\tilde{\mathbf{k}}_{\parallel}$  is also self-adjoint,  $\tilde{L}$  and  $\tilde{\mathbf{k}}_{\parallel}$  admit a common, complete set of orthogonal eigenfunctions. The degenerate eigenvalues of  $\tilde{\mathbf{k}}_{\parallel}$  are, of course, the (real) wave vector components  $\mathbf{k}_{\parallel}$  which are parallel to the plate surfaces. Therefore, the complete set of eigenfunctions are those given by equations (4), with real  $\mathbf{k}_{\parallel}$ .

In section 2.2, based on the Hermiticity of the operator  $\tilde{L}$ , we showed that these functions (4) are indeed orthogonal to each other and in section 3 we calculated the normalization factors.

Having all these ingredients, in section 4 we presented the formal quantization procedure which is applied elsewhere [13, 14] to calculate the thermal properties of ultra-thin plates at low temperatures, and to deduce some of the observed features of the *standard tunneling model* in bulk amorphous materials.

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