A Mean Value Theorem

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Several theorems go by this name. The present note adds to the assortment an unusual variant (Theorem 1), which involves the shape of the underlying region in an interesting way.

We work in Euclidean spaces, although Lemma 2 and the second inequality of Lemma 3 carry over to general Riemannian manifolds. ∇ and $|\ |$ denote gradient and norm with respect to the standard inner product $<,>$, and ∂ stands for boundary. All our functions are real-valued. A *gradient curve* of a function f is an integral curve of ∇f.

Theorem 1. *Let f be a C^1-function on a closed ball B. Then there exists $b \in B$ at which $|\nabla f(b)| \cdot \mathrm{diam}(B) = \max f - \min f$.*

The proof is obtained via Lemmas 2 and 3.

Lemma 2. *Let f be a C^1-function without critical points on a compact region B. Then every gradient curve of f begins and ends on ∂B.*

Proof: Say a gradient curve $\gamma(s)$ is defined for s from s_- to s_+. We have

$$\begin{aligned}\lim_{s \to s_+} f(\gamma(s)) - \lim_{s \to s_-} f(\gamma(s)) &= \int_\gamma \langle \nabla f, d\gamma \rangle \\ &= \int_\gamma |\nabla f|\ |d\gamma| \quad \text{because } \gamma \text{ is tangent to } \nabla f \quad (*) \\ &\geq \min|\nabla f| \cdot \mathrm{length}(\gamma).\end{aligned}$$

On compact B, f is bounded, so if f has no critical points ($\min|\nabla f| > 0$), $(*)$ shows that length(γ) is finite and $\gamma(s_\pm)$ exist. Unless both $\gamma(s_-)$ and $\gamma(s_+)$ lie on ∂B, γ can be extended beyond s_- or s_+ by the existence theorem for solutions of differential equations, contradicting the choice of $s_\pm$. ∎

Remark. In Lemma 2, compactness is indispensable: think of the height function on an infinite vertical cylinder.

Lemma 3. *Let f be a C^1-function on a closed ball B. Then*

$$\min|\nabla f| \leq \frac{\max f - \min f}{\mathrm{diam}(B)} \leq \max|\nabla f|.$$

Proof: First inequality: If f has critical points on B, then $\min|\nabla f| = 0$. Otherwise consider the gradient curve γ *through the center of* B. γ reaches ∂B by Lemma 2, so that length$(\gamma) \geq$ diam(B); combine this with $(*)$ to get

$$\max f - \min f \geq \min|\nabla f| \cdot \mathrm{diam}(B).$$

Second inequality: Let l be a line segment that joins a minimum and a maximum of f. Since length$(l) \leq$ diam(B),

$$\max f - \min f = \int_l \langle \nabla f, dl \rangle \leq \int_l |\nabla f|\, |dl| \leq \max |\nabla f| \cdot \operatorname{diam}(B). \qquad \blacksquare$$

Remark. In Lemma 3, the first inequality is true only on a ball: $f(x, y) = x$ is a counterexample on $[0, 1] \times [0, 1]$. The second inequality holds on any convex region. Both become equalities for affine functions on balls.

Theorem 1 is now immediate:

Proof of Theorem 1 Apply Lemma 3 and the intermediate value theorem to $|\nabla f|$. ∎

I do not know how close the mean value property of Theorem 1 comes to characterizing balls. However, Theorem 1 does admit a partial converse. To state it, we need a definition.

The *width* $w_B(\boldsymbol{e})$ of a compact region B in the direction of a unit vector $\boldsymbol{e}$ is defined as follows. 'Sandwich' B by a pair of parallel planes perpendicular to $\boldsymbol{e}$; $w_B(\boldsymbol{e})$ is the distance between these planes:

$$w_B(\boldsymbol{e}) = \max_{\boldsymbol{r} \in B} \langle \boldsymbol{e}, \boldsymbol{r} \rangle - \min_{\boldsymbol{r} \in B} \langle \boldsymbol{e}, \boldsymbol{r} \rangle.$$

B has *constant width* if $w_B(\boldsymbol{e})$ has the same value for all directions $\boldsymbol{e}$. A ball has constant width, but there are shapes of constant width that are not balls (e.g., Reuleaux's tetrahedron).

Aside. Why are lids on manholes round? Answer: because a lid whose rim is *not* a curve of constant width can fall into the hole if (un)suitably rotated. Of course, the lid and the hole need not be circular; any shape of constant width would be safe.

Return to the partial converse to Theorem 1.

Theorem 4. *Let B be a compact region such that for every linear function f on it, there exists $b \in B$ at which $|\nabla f(b)| \cdot \operatorname{diam}(B) = \max f - \min f$. Then B has constant width.*

Proof: Suppose B has maximal width in the direction of $\boldsymbol{e}_+$, minimal width in the direction of $\boldsymbol{e}_-$, and $w_B(\boldsymbol{e}_-) < w_B(\boldsymbol{e}_+)$. Then the linear function $f(\boldsymbol{r}) = \langle \boldsymbol{e}_-, \boldsymbol{r} \rangle$ violates the assumed property of f, as $|\nabla f| = 1$, diam$(B) = w_B(\boldsymbol{e}_+)$, $\max f - \min f = w_B(\boldsymbol{e}_-)$. ∎

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