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ANALYTICITY IN CERTAIN BANACH ALGEBRAS

BY

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1. **Introduction.** Consider a Riemann surface S. (All Riemann surfaces are surfaces without boundary and are assumed to be separable but not necessarily connected.) Consider also a set $\mathfrak{A}$ of analytic (that is, holomorphic) functions on S which are not simultaneously constant on any component of S. From the functions in $\mathfrak{A}$ it is possible to construct a wider class of functions analytic on S by the operations of addition, multiplication, and scalar multiplication. Further functions analytic on S are obtained by taking those functions which are uniform limits on each compact subset of S of functions already obtained. Thus from $\mathfrak{A}$ we pass to the set $\overline{\mathfrak{A}}$—the *holomorphic completion* of $\mathfrak{A}$.

In the sequel we only study *holomorphically complete* sets $\mathfrak{A}$ of analytic functions on a Riemann surface S. This means by definition that $1 \in \mathfrak{A}$, that the functions in $\mathfrak{A}$ are not all constant on any component of S, that $\mathfrak{A}$ is an algebra over the complex field with the natural algebraic operations, and that each function on S which can be uniformly approximated on each compact subset of S by functions in $\mathfrak{A}$ is in $\mathfrak{A}$. The set $\mathfrak{A}$ will be topologized by the topology of uniform convergence on compact subsets of S.

For such a holomorphically complete $\mathfrak{A}$ there are certain natural questions: Given a sequence of points in S having no cluster point, does there exist a function in $\mathfrak{A}$ having prescribed values at the given points? Or: When is it possible to approximate a function given on a compact subset of S uniformly by functions in $\mathfrak{A}$? and so forth. It is well known (see for example [1]) that the space X should be *holomorphically convex* (or at least *weakly holomorphically convex*) relative to the given algebra $\mathfrak{A}$ of analytic functions if such questions are to have satisfactory answers.

Definition 1. A Riemann surface S is *holomorphically convex* (respectively *weakly holomorphically convex*) relative to a holomorphically complete set $\mathfrak{A}$ of analytic functions on S if for each compact subset K of S the set (respectively each component of the set)

$$\tilde{K} = \{p \text{ in } S\colon |f(p)| \leqq \sup\{|f(q)| : q \in K\} \text{ for all } f \text{ in } \mathfrak{A}\}$$

is compact.

One of the purposes of this paper is to show that if $\mathfrak{A}$ is a holomorphically complete algebra of analytic functions on a Riemann surface S then S can be canonically *extended* to a Riemann surface S' and the functions in $\mathfrak{A}$ can be *extended* to S' to give an algebra $\mathfrak{A}'$ of analytic functions on S' with re-

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spect to which S' is weakly holomorphically convex. Thus the condition of weak holomorphic convexity is always realized on a suitable extension of the given surface S'. It might be thought that an analogous theorem would hold for a holomorphically complete algebra $\mathfrak{A}$ of analytic functions on a higher-dimensional complex analytic manifold S, but an example of Wermer [9] can easily be adapted to show that this is not the case. The following definition gives a precise meaning to the term extension just employed.

DEFINITION 2. Let the pair $(S, \mathfrak{A})$ consist of a Riemann surface S and a holomorphically complete algebra of analytic functions on S. An *extension* of $(S, \mathfrak{A})$ consists of a second such pair $(S', \mathfrak{A}')$, of an analytic map σ from S to S', and of a one-one map τ of $\mathfrak{A}$ *onto* $\mathfrak{A}'$ such that

$$(\tau(f))(\sigma(p)) = f(p) \tag{*}$$

for all f in $\mathfrak{A}$ and p in S.

Clearly σ need not be one-one since it is possible for σ to identify points of S which are identified by all functions in $\mathfrak{A}$.

One of our main results is then the following.

THEOREM 2. *Let the pair $(S, \mathfrak{A})$ consist of a Riemann surface S and a holomorphically complete algebra $\mathfrak{A}$ of analytic functions on S. Then $(S, \mathfrak{A})$ admits an extension $(S', \mathfrak{A}')$ such that*

(i) *For each compact subset K of S' there exists a compact subset K_0 of S with*

$$K \subset \tilde{L},$$

where $L = \sigma(K_0)$, and $\tilde{L}$ is formed relative to $\mathfrak{A}'$.

(ii) *To each continuous homomorphism ϕ of $\mathfrak{A}$ onto the complex numbers there exists p in S' with*

$$(\tau(f))(p) = \phi(f)$$

for all f in $\mathfrak{A}$.

(iii) *The set*

$$T = \{(p, q) : p \in S', q \in S', p \neq q, f(p) = f(q) \text{ for all } f \text{ in } \mathfrak{A}'\}$$

is a countable subset of $S' \times S'$ which has no cluster point in $S' \times S'$.

(iv) *For each compact subset K of S' the set $\tilde{K}$ is the union of a compact set L and all points p in S' for which there exists q in L with $(p, q) \in T$.*

Added in proof. From the argument used in proving Theorem 3 below it follows that the set L of (iv) can in fact be taken to be the union of K and all those components of $S' - K$ which are relatively compact subsets of S', so that in particular bdry $L \subset K$.

It follows from (iv) and the countability of T that S' is weakly holomorphically convex relative to $\mathfrak{A}'$.

Property (ii) of Theorem 2 contains the key to the construction of the extension $(S', \mathfrak{A}')$. The surface S' is constructed abstractly by considering the set of continuous homomorphisms of $\mathfrak{A}$ into the complex numbers and imposing the structure of a Riemann surface on this set. If a certain countable set of homomorphisms are counted more than once this can be done and gives the Riemann surface S'. The mapping σ from S to S' is then easily found, as is the mapping τ from $\mathfrak{A}$ onto a certain set $\mathfrak{A}'$ of analytic functions on S'. The pair $(S', \mathfrak{A}')$ and the maps σ and τ are then shown to be an extension of $(S, \mathfrak{A})$ having the properties of Theorem 2. Since a continuous homomorphism ϕ of $\mathfrak{A}$ into the complex numbers has the property that there exists a compact subset K of S with

$$|\phi(f)| \leqq \sup\{|f(p)| : p \in K\}$$

for all f in $\mathfrak{A}$, to get all continuous homomorphisms ϕ it is sufficient to consider compact subsets K of S' and homomorphisms ϕ satisfying the inequality.

Thus we come to a well-known problem in Banach algebras—the investigation of the set of continuous homomorphisms (called the spectrum) of an algebra of continuous complex-valued functions defined on a compact Hausdorff space K. Here continuous means continuous in the uniform norm for functions on K. The bulk of this paper is concerned with aspects of this problem, and the results obtained in this investigation are applied in the proof of Theorem 2. The particular type of Banach algebra which arises will be called a *uniform algebra*.

DEFINITION 3. A uniform algebra is a Banach algebra with unit whose norm and spectral norm coincide.

If $\mathfrak{A}$ is a uniform algebra with spectrum Y and Šilov boundary X, it is clear that $\mathfrak{A}$ can be considered as a closed subalgebra of either $C(X)$ or $C(Y)$, where $C(\Gamma)$, for a compact Hausdorff space Γ, is the uniformly-normed algebra of all continuous complex-valued functions on Γ. Conversely it is clear that if Γ is a compact Hausdorff space then every closed subalgebra of $C(\Gamma)$ which contains the function 1 is a uniform algebra.

Most of this paper is a systematic investigation of conditions which imply that certain open subsets of the spectrum of a uniform algebra can be given the structure of a Riemann surface on which the functions of the algebra all are analytic functions. The following is the principal result of this investigation.

THEOREM 1. *Let $\mathfrak{A}$ be a uniform algebra with Šilov boundary X and spectrum Y. Let $\mathfrak{A}$ contain a function g which has the following properties*:

(a) *The interior of $g(X)$ is void.*

(b) *Each point of $g(X)$ is the vertex of some nondegenerate triangle whose interior lies in $-g(X)$.*

(c) *For each z in $g(X)$ there are only a finite number of points p in X with $g(p)=z$.*

(d) *If w_1 and w_2 are points in $-g(X)$ there exists a Jordan arc γ joining w_1 and w_2 and intersecting $g(X)$ in a finite number of points $z_1, \cdots, z_\lambda$. Each point z_i has the property that there exists a smooth open Jordan arc $J_0 \subset g(X)$ which contains z_i, which is an open subset of $g(X)$, and which is the homeomorphic image under the mapping g of the subset $\{p: p \in X, g(p) \in J_0\}$ of X.*

Then there is a Riemann surface S and a continuous map λ of S onto $Y-X$ such that $f \circ \lambda$ is analytic on S for all f in $\mathfrak{A}$, such that each point in $Y-X$ except for those in a countable set is the image of exactly one point in S, and such that when $-g(X)$ has a finite number of components j each point in $Y-X$ is the image of at most $j-1$ points in S.

In condition (d) a "smooth" arc is one with a continuously turning tangent. To introduce another piece of notation, g^- will denote the function (or relation) inverse to a function g, the designation g^{-1} being reserved for the function $1/g$.

The investigations of this paper have their origin in ideas of Wermer [7; 8]. In particular special cases of Theorems 1 and 2 follow from Wermer's work. The present paper carries the theory further in certain directions than did Wermer's work and develops some of the material more systematically. In particular a definitive theorem (Theorem 2 above) about algebras of functions on Riemann surfaces is obtained. H. Royden has also extended Wermer's work, by methods different from those used here.

2. **Functions rational over $\mathfrak{A}$.** The motivations for the following definition are clear.

Definition 4. Let $\mathfrak{A}$ be a uniform algebra with spectrum Y, and let X be the Šilov boundary of $\mathfrak{A}$. Let h be a function in $C(X)$, and let $p_1, \cdots, p_n$ be points in Y, not necessarily distinct. Let G denote the set of all products of the form

$$g = g_1 g_2 \cdots g_n,$$

with $g_i \in \mathfrak{A}$ and $g_i(p_i) = 0$. Then h will be called a *rational* function over $\mathfrak{A}$ with *poles* $p_1, \cdots, p_n$ if

(i) $gh \in \mathfrak{A}$ for all g in G.

(ii) There exists g in G with $(gh)(p_i) \neq 0$, $1 \leqq i \leqq n$.

Lemma 1. *Let h be a rational function over the Banach algebra $\mathfrak{A}$ with poles $p_1, \cdots, p_n$. Let $q_1, \cdots, q_m$ be another set of poles for h. Then $m = n$ and the sequence $q_1, \cdots, q_m$ is a rearrangement of the sequence $p_1, \cdots, p_n$.*

Proof. Assume that there is a point p which occurs j times in the sequence $p_1, \cdots, p_n$ and $k < j$ times in the sequence $q_1, \cdots, q_m$. To prove the lemma, it will be enough to contradict this assumption. We may take it that $p_1 = p_2 = \cdots = p_j = p$, $q_1 = q_2 = \cdots = q_k = p$. Choose $g = g_1 \cdots g_n$ with $g_i \in \mathfrak{A}$, $g_i(p_i) = 0$, $gh \in \mathfrak{A}$, $(gh)(p) \neq 0$. For $k < i \leqq m$, choose f_i in $\mathfrak{A}$ with $f_i(q_i) = 0$, $f_i(p) = 1$. Thus the function

$$f = g_1 \cdots g_k f_{k+1} \cdots f_m$$

is in $\mathfrak{A}$, and $fh\in\mathfrak{A}$ because $q_1, \cdots, q_m$ is a set of poles for h. Thus the function $\alpha=fhg_{k+1}\cdots g_n$ is in $\mathfrak{A}$. We have $\alpha(p)=0$, since $g_{k+1}(p)=g_{k+1}(p_{k+1})=0$. But

$$\alpha = ghf_{k+1}\cdots f_m,$$

and the quantities $(gh)(p), f_{k+1}(p), \cdots, f_m(p)$ do not vanish. This contradiction shows that our assumption was false, thereby proving the lemma.

We are now justified in speaking of *the* poles $p_1, \cdots, p_n$ of a function h rational over $\mathfrak{A}$. If p is in the spectrum of $\mathfrak{A}$ and k is a non-negative integer such that p occurs k times in the sequence $p_1, \cdots, p_n$ we say that p is a pole of multiplicity k of h.

LEMMA 2. *Let h be a rational function over the uniform algebra $\mathfrak{A}$ and let the point p with multiplicity $n>0$ be the only pole of h. Then there exists g in $\mathfrak{A}$ such that gh is rational over $\mathfrak{A}$ and the point p with multiplicity $n-1$ is the only pole of gh.*

Proof. Choose $g_1, \cdots, g_n$ in $\mathfrak{A}$ with $\alpha=g_1\cdots g_nh\in\mathfrak{A}$, $g_i(p)=0$, $\alpha(p)\neq 0$. Let $g=g_1$. It is clear that g satisfies the required conditions.

LEMMA 3. *Let h_1 be a rational function over the uniform algebra $\mathfrak{A}$ such that the point p with multiplicity $n\geqq 0$ is the only pole of h_1. Let the function h_2 in $C(X)$, where X is the Šilov boundary of $\mathfrak{A}$, have the property that $gh_2\in\mathfrak{A}$ for each g in G, where G is the set of all $g=g_1\cdots g_n$ with $g_i\in\mathfrak{A}$ and $g_i(p)=0$. Then there exist elements f_1 and f_2 in $\mathfrak{A}$ with $h_2=f_1h_1+f_2$.*

Proof. We proceed by induction on n. The theorem is clearly true if $n=0$, for then both h_1 and h_2 are in $\mathfrak{A}$. Assume now that the lemma is true for all integers up to and including $n-1$. By hypothesis, there exists g in G with $gh_1=\alpha\in\mathfrak{A}$ and $\alpha(p)\neq 0$, say $\alpha(p)=1$. If we let $\beta=1-\alpha$, then $\beta\in\mathfrak{A}$, $\beta(p)=0$, and $1=gh_1+\beta$. Multiplying this equality by h_2, we have

$$h_2 = gh_1h_2 + \beta h_2 = (gh_2)h_1 + \beta h_2 = \delta h_1 + \beta h_2,$$

where $\delta\in\mathfrak{A}$. By the previous lemma, there exists γ in $\mathfrak{A}$ with $\gamma(p)=0$ such that γh_1 is rational over $\mathfrak{A}$ and has the point p with multiplicity $n-1$ as its only pole. By the hypothesis of the induction, we therefore have

$$\beta h_2 = \gamma_0(\gamma h_1) + \gamma_1,$$

where γ_0 and γ_1 are in $\mathfrak{A}$. Therefore

$$h_2 = \delta h_1 + \beta h_2 = (\delta + \gamma_0\gamma)h_1 + \gamma_1.$$

If we write $f_1=\delta+\gamma_0\gamma$ and $f_2=\gamma_1$, this proves the lemma.

LEMMA 4. *Let h be a rational function over the uniform algebra $\mathfrak{A}$, with poles at the distinct points $p_1, \cdots, p_n$ of multiplicities $k_1, \cdots, k_n$ respectively. Then there exist functions $\alpha_1, \cdots, \alpha_n$ in $\mathfrak{A}$ with $\alpha_1+\cdots+\alpha_n=1$ such that for each i the function $\alpha_ih=h_i$ is rational over $\mathfrak{A}$ with a single pole of multiplicity k_i at p_i.*

Proof. For $1 \leqq i \leqq n$ let G_i consist of all products $g_1 g_2 \cdots g_{k_i}$ with $g_j \in \mathfrak{A}$ for $1 \leqq j \leqq k_i$ and $g_j(p_i) = 0$. For $1 \leqq i \leqq n$ let F_i consist of all products of the form $f_1 \cdots f_{i-1} f_{i+1} \cdots f_n$ with $f_j \in G_j$. Let $\tilde{F}_i$ be the ideal which F_i generates in $\mathfrak{A}$. Since the ideals $\tilde{F}_i$ are simultaneously contained in no maximal ideal of $\mathfrak{A}$, $\tilde{F}_1 + \cdots + \tilde{F}_n$ is an ideal of $\mathfrak{A}$ which is contained in no maximal ideal. There therefore exist elements α_i in $\tilde{F}_i$ with $\alpha_1 + \cdots + \alpha_n = 1$. Let $h_i = \alpha_i h$. Clearly $h_i f_i \in \mathfrak{A}$ for each f_i in G_i. To complete the proof that h_i has a pole of order k_i at p_i and no other pole it is sufficient to show that there exists f_i in G_i with $(h_i f_i)(p_i) \neq 0$. By hypothesis and Definition 4 there exist elements f_i in G_i with $(fh)(p_i) \neq 0$ for all i, where $f = f_1 \cdots f_n$. We shall show that $(h_i f_i)(p_i) \neq 0$, thereby completing the proof. Since clearly $\alpha_j(p_i) = 0$ for $j \neq i$, we have $\alpha_i(p_i) = 1$. Therefore

$$(fh_i)(p_i) = (fh)(p_i)\alpha_i(p_i) \neq 0.$$

Since

$$(fh_i)(p_i) = f_1(p_i) \cdots f_{i-1}(p_i) f_{i+1}(p_i) \cdots f_n(p_i)(f_i h_i)(p_i),$$

it follows that $(f_i h_i)(p_i) \neq 0$. Thus h_i has a pole of order k_i at p_i, as was to be proved.

LEMMA 5. *Let h be a rational function over the uniform algebra $\mathfrak{A}$ with poles at $p_1, \cdots, p_n$. There exists a unique continuous function $\hat{h}$ on $Y - \{p_1, \cdots, p_n\}$, where Y is the spectrum of $\mathfrak{A}$, such that if f is any element of $\mathfrak{A}$ for which $fh \in \mathfrak{A}$ then*

$$(*) \qquad (fh)(p) = f(p)\hat{h}(p) \qquad \text{for all } p \text{ in } Y - \{p_1, \cdots, p_n\}.$$

The function $\hat{h}$ becomes infinite at the points p_i. Thus if we define $\hat{h}$ to be infinity at the points p_i then $\hat{h}$ is a continuous function from Y to the Riemann sphere.

Proof. Consider a point p in $Y - \{p_1, \cdots, p_n\}$. Choose $g = g_1 \cdots g_n$ with $g_i \in \mathfrak{A}$, $g_i(p_i) = 0$, $g_i(p) \neq 0$. Define $\hat{h}(q) = (gh)(q)[g(q)]^{-1}$ for all q in Y for which $g(q) \neq 0$. Now if f is any function in $\mathfrak{A}$ such that $fh \in \mathfrak{A}$, and if q is a point in Y with $g(q) \neq 0$, then $f(q)(gh)(q) = (fgh)(q) = (fh)(q)g(q)$, so that

$$(*) \qquad (fh)(q) = f(q)\hat{h}(q)$$

as was required. This equation can be written $\hat{h}(q) = (fh)(q)[f(q)]^{-1}$, which shows that $\hat{h}$ is independent of the choice of g. Since $\hat{h}$ is clearly continuous, it remains only to prove the last statement of the lemma. To this end, choose g as above with $(gh)(p_i) \neq 0$, $1 \leqq i \leqq n$. Since $g(q)\hat{h}(q) = (gh)(q)$, we have $g(q) \neq 0$ for all q sufficiently near p_i. The equations $\hat{h}(q) = (gh)(q)[g(q)]^{-1}$ and $g(p_i) = 0$ then show that $|\hat{h}(q)| \to \infty$ as $q \to p_i$, as was to be proved.

To simplify notation we shall write simply $h(p)$ for $\hat{h}(p)$ in the situation of Lemma 5. By a rational function h over a uniform algebra $\mathfrak{A}$ with a simple pole at p we shall mean a rational function h over $\mathfrak{A}$ which has $\{p\}$ as its complete set of poles.

DEFINITION 5. Let h be a rational function over the uniform algebra $\mathfrak{A}$ with a simple pole at the point p. We define the transformation T_h from $\mathfrak{A}$ to $\mathfrak{A}$ by

$$T_h f = h(f - f(p)).$$

It is clear that $\|T_h\| \leqq 2\|h\|$, where $\|h\|$ will now be defined.

DEFINITION 6. Let h be a rational function over the uniform algebra $\mathfrak{A}$. We define

$$\|h\| = \sup\{ \mid h(x) \mid : x \in X\},$$

where X is the Šilov boundary of $\mathfrak{A}$, and $[h] = (2\|h\|)^{-1}$.

DEFINITION 7. If h is a rational function over a uniform algebra $\mathfrak{A}$ with a simple pole at a point p, D_h will denote the set $\{z : |z| < [h]\}$ in the complex plane. If $f \in \mathfrak{A}$ the expansion of f in powers of h^{-1} will denote the power series $a_0 + a_1 z + \cdots$, where $a_n = f_n(p)$ with $f_n = (T_h)^n f$.

LEMMA 6. *Under the hypothesis of Definition 7, the expansion of f in powers of h^{-1} converges for all z in D_h.*

Proof. We have

$$| a_n | \leqq \|T_h\|^n \|f\| \leqq (2\|h\|)^n \|f\|,$$

and so the series converges for $|z| < [h]$.

DEFINITION 8. Under the hypothesis of Definition 7, the value of the sum of the power series at the point z in D_h will be denoted by $P(f, h, z)$.

DEFINITION 9. If h is a rational function over a uniform algebra $\mathfrak{A}$ with a simple pole p, if z is any point in D_h, and if x is any point in the Šilov boundary X of $\mathfrak{A}$, then we define the quantity $u(h, z)$ in $C(X)$ by the equation

$$u(h, z, x) = h(x)(1 - zh(x))^{-1},$$

where $u(h, z, x)$ denotes the value of $u(h, z)$ at x. (It should be noted that $1 - zh(x) \neq 0$ for all z in D_h since $|z| < [h]$.)

LEMMA 7. *Let h be a rational function over the uniform algebra $\mathfrak{A}$ which has a simple pole p. Let f be an element of $\mathfrak{A}$. Then $\sigma(z) = [f - P(f, h, z)]u(h, z)$ defines an analytic mapping σ from D_h into $\mathfrak{A}$.*

Proof. We have seen that $P(f, h, \cdot)$ is an analytic complex-valued function

$$P(f, h, z) = \sum_{n=0}^{\infty} a_n z^n,$$

where $a_n = f_n(p)$ and $f_n = (T_h)^n f$. It is easily seen by induction that

$$f_n = h^n(f - a_0) - \sum_{k=1}^{n-1} h^k a_{n-k}$$

for $n \geqq 1$. We also have $|z| < [h] = (2\|h\|)^{-1}$, which implies that $\|zh\| < 1$ for all z in D_h. Thus the mapping $z \to u(h, z)$ is an analytic mapping from D_h into $C(X)$ and has the power series expansion

$$u(h, z) = \sum_{n=0}^{\infty} z^n h^{n+1}.$$

It follows that σ is an analytic mapping from D_h into $C(X)$ with the power series expansion

$$\sigma(z) = \left(f - a_0 - \sum_{n=1}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} z^n h^{n+1}\right)$$

$$= h(f - a_0) + \sum_{n=1}^{\infty} f_{n+1} z^n = \sum_{n=0}^{\infty} f_{n+1} z^n,$$

with f_n as above. Since $f_n \in \mathfrak{A}$ for all values of n, we see that $\sigma(z) \in \mathfrak{A}$, as was to be proved.

LEMMA 8. *Under the hypothesis of Definitions 7 and 8 the mapping*

$$\lambda_h(z): f \to P(f, h, z)$$

is a homomorphism of $\mathfrak{A}$ into the complex numbers, for each z in D_h, and therefore is a point in the spectrum of $\mathfrak{A}$. The mapping $\lambda_h: z \to \lambda_h(z)$ is a continuous mapping of D_h into the spectrum of $\mathfrak{A}$.

Proof. It is only necessary to prove that this map is multiplicative, the other properties of a homomorphism being obvious. Fix z in D_h and write $u = u(h, z)$. Since $|z| < (2\|h\|)^{-1}$, we have $\|zh\| < 1/2$, so that

$$\|u\| = \|h(1 - zh)^{-1}\| < \|h\|(1 - 1/2)^{-1} = 2\|h\|.$$

Thus $\|zu\| < 1$. It follows that $1 + zu$ has an inverse in $C(X)$. Therefore $h = u(1+zu)^{-1}$. Now if u were in $\mathfrak{A}$ we should have $(1+zu)^{-1} \in \mathfrak{A}$, since $\|zu\| < 1$, and therefore $h \in \mathfrak{A}$. Since h has a pole, it is not in $\mathfrak{A}$, and therefore u is not in $\mathfrak{A}$.

Consider functions f and g in $\mathfrak{A}$. By Lemma 7, we see that the functions

$$g(f - P(f, h, z))u \quad \text{and} \quad P(f, h, z)(g - P(g, h, z))u$$

are in $\mathfrak{A}$. It follows that their sum

$$[fg - P(f, h, z)P(g, h, z)]u$$

is in $\mathfrak{A}$. Since also $(fg - P(fg, h, z))u$ is in $\mathfrak{A}$, we see that

$$[P(fg, h, z) - P(f, h, z)P(g, h, z)]u$$

is in $\mathfrak{A}$. Since u itself is not in $\mathfrak{A}$, this implies $P(fg, h, z) - P(f, h, z)P(g, h, z) = 0$. Thus $\lambda_h(z)$ is a homomorphism of $\mathfrak{A}$ into the complex numbers and therefore is a point in the spectrum of $\mathfrak{A}$ (see [5, p. 68]). The fact that λ_h is continuous follows from the fact that for each f in $\mathfrak{A}$ the composite map $f \circ \lambda_h = P(f, h, \cdot)$ is continuous.

DEFINITION 10. Let h be a rational function with a simple pole p over a uniform algebra $\mathfrak{A}$. For each z in D_h the quantity $\lambda_h(z)$ is the point in the spectrum Y of $\mathfrak{A}$ defined by

$$f(\lambda_h(z)) = P(f, h, z)$$

for all f in $\mathfrak{A}$. E_h will denote the subset of Y of all $\lambda_h(z)$ for z in D_h. The mapping $z \to \lambda_h(z)$ of D_h onto E_h will be denoted by λ_h.

LEMMA 9. *Let h be a rational function with a simple pole p over the uniform algebra $\mathfrak{A}$. Then for each z in D_h, $u(h, z)$ is a rational function over $\mathfrak{A}$ with a simple pole at $\lambda_h(z)$. For all z and t in D_h with $t \neq 0$ and $t \neq z$ respectively we have*

$$h(\lambda_h(t)) = t^{-1}, \qquad u(h, z, \lambda_h(t)) = (t - z)^{-1}.$$

For any q in the spectrum Y of $\mathfrak{A}$ with $q \neq \lambda_h(z)$ and $q \neq p$ we have

$$u(h, z, q)[1 - zh(q)] = h(q).$$

Proof. Let f be any element in $\mathfrak{A}$ such that $f(p) \neq 0$ and $f_1(p) \neq 0$, where $f_1 = h(f - f(p))$. Then $P(f, h, z)$ has the expansion $a_0 + a_1 z + \cdots$ with $a_1 = f_1(p) \neq 0$. Thus the set S of points z in D_h with $P(f, h, z) = a_0$ is isolated. Since $P(f, h, z) = f(\lambda_h(z))$ and $a_0 = f(p)$, it follows that $\lambda_h(z) \neq p$ for all z in $D_h - S$. Now the function σ defined by $\sigma(z) = [f - P(f, h, z)]u(h, z)$ has been seen in Lemma 7 to be an analytic map from D_h to $\mathfrak{A}$, and therefore it has an expansion

$$\sigma(z) = \sum_{n=0}^{\infty} \alpha_n z^n$$

converging on D_h, with $\alpha_n \in \mathfrak{A}$. For each x in the spectrum Y of $\mathfrak{A}$, let

$$\sigma(z, x) = (\sigma(z))(x) = \sum_{n=0}^{\infty} \alpha_n(x) z^n.$$

Thus if we define $w(z) \equiv \sigma(z, \lambda_h(z))$ it follows that

$$w(z) = \sum_{n=0}^{\infty} \alpha_n(\lambda_h(z)) z^n.$$

Since $\alpha_n \in \mathfrak{A}$, we see that $\alpha_n \circ \lambda_h = P(\alpha_n, h, \cdot)$ is analytic on D_h and has absolute values there which do not exceed $\|\alpha_n\|$. It follows that w is an analytic function on D_h. Since $\sigma(0) = (f - f(p))h = f_1$, we have

$$w(0) = \sigma(0, p) = f_1(p) \neq 0,$$

so that the set T of all z in D_h with $w(z) = 0$ is isolated.

Consider now any point z in $D_h - S - T$. From the defining equations for σ and $u(h, z)$ we obtain

$$[f - f(\lambda_h(z))]h = \sigma(z)(1 - zh).$$

By Lemma 5, since $\lambda_h(z) \neq p$, we may evaluate both sides of this equality at the point $\lambda_h(z)$ in Y, obtaining

$$0 = w(z)(1 - zh(\lambda_h(z))).$$

Since z is not in T, this gives $h(\lambda_h(z)) = z^{-1}$. Since h is continuous on Y, by Lemma 5, it follows that $h(\lambda_h(t)) = t^{-1}$ for all $t \neq 0$ in D_h.

Since by Lemma 7 we see that $u(h, z)(f - f(\lambda_h(z))) \in \mathfrak{A}$ for all f in $\mathfrak{A}$, to show that $u(h, z)$ is rational over $\mathfrak{A}$ with a simple pole at $\lambda_h(z)$, it is sufficient to show that there exists f in $\mathfrak{A}$ such that $u(h, z)(f - f(\lambda_h(z)))$ does not vanish at $\lambda_h(z)$. To this end, we consider an element f in $\mathfrak{A}$ with the property that $uf \in \mathfrak{A}$, where $u = u(h, z)$. By the definition of u, we have

$$(1 - zh)uf = hf.$$

so that

$$uf = h(f + zuf).$$

For t in D_h and $t \neq 0$, this implies by Lemma 5 that

$$(uf)(\lambda_h(t)) = h(\lambda_h(t))[f(\lambda_h(t)) + z(uf)(\lambda_h(t))].$$

Since $h(\lambda_h(t)) = t^{-1}$ this gives

$$(*) \qquad (uf)(\lambda_h(t)) = (t - z)^{-1} f(\lambda_h(t))$$

if $t \neq 0$ and $t \neq z$. By continuity, (*) is valid whenever t and z are in D_h and $t \neq z$.

Now let f be any element in $\mathfrak{A}$ with $f(\lambda_h(z)) = 0$, $f(p) = f(\lambda_h(0)) \neq 0$. Let $f_n = u^n f$. Assume that $f_n \in \mathfrak{A}$ for all n and that $f_n(\lambda_h(z)) = 0$ for all n. It follows by induction from the formula (*) that $f_n(\lambda_h(t)) = (t - z)^{-n} f(\lambda_h(t))$, for all $t \neq z$. Thus $f(\lambda_h(\cdot)) = f \circ \lambda_h$ is an analytic function on D_h which is not identically zero such that the function $(\cdot - z)^{-n} f(\lambda_h(\cdot))$ is analytic for each n. This contradiction shows that our assumption was false. Thus there exists a positive integer n such that $f_{n-1} \in \mathfrak{A}$ and $f_{n-1}(\lambda_h(z)) = 0$ and either $f_n \notin \mathfrak{A}$ or $f_n \in \mathfrak{A}$ and $f_n(\lambda_h(z)) \neq 0$. We must have $f_n \in \mathfrak{A}$, since $f_{n-1} \in \mathfrak{A}$ and $f_{n-1}(\lambda_h(z)) = 0$. Thus $f_n(\lambda_h(z)) = (uf_{n-1})(\lambda_h(z)) \neq 0$. This is just what was needed to show that u is rational over $\mathfrak{A}$ with a simple pole at $\lambda_h(z)$. By the formula (*), we have $u(h, z, \lambda_h(t)) = (t - z)^{-1}$.

Now if p is any point in Y with $q \neq \lambda_h(z)$ and $q \neq p$, choose f in $\mathfrak{A}$ with $f(q) = 1$, $f(\lambda_h(z)) = 0$. By the above, we have $uf = h(f + zuf)$. Evaluating at q, we have

$$u(q) = u(q)f(q) = (uf)(q) = h(q)(1 + z(uf)(q))$$
$$= h(q)(1 + zu(q)),$$

so that

$$u(q)(1 - zh(q)) = h(q),$$

as was to be proved.

LEMMA 10. *Let h be a rational function over the uniform algebra $\mathfrak{A}$ with a simple pole p. The mapping λ_h is a homeomorphism of D_h onto E_h and E_h is an open set in the spectrum Y of $\mathfrak{A}$.*

Proof. The mapping λ_h is one-to-one because we have seen that $h(\lambda_h(z)) = z^{-1}$ for all z in D_h. We shall show that $\lambda_h(U)$ is open in Y for all open subsets U of D_h. Since λ_h is continuous and one-one, it will follow that λ_h is a homeomorphism. By taking $U = D_h$, it will follow that E_h is open.

Assume then that $\lambda_h(U)$ is not open, and let z in U be chosen so that $\lambda_h(z)$ is not in the interior of $\lambda_h(U)$. Since h is continuous on Y and since $[h(\lambda_h(z))]^{-1} = z$ is in D_h, it follows that $t = [h(q)]^{-1}$ will be in D_h whenever the point q in $Y - \lambda_h(U)$ is near enough to $\lambda_h(z)$. Choose such a point q. Since $q \notin \lambda_h(U)$ we have $q \neq p$ and $q \neq \lambda_h(t)$. By Lemma 9 it follows that

$$t^{-1} = h(q) = u(h, t, q)[1 - th(q)]$$
$$= u(h, t, q)[1 - tt^{-1}] = 0,$$

a contradiction. Thus $\lambda_h(U)$ is open, as was to be proved.

DEFINITION 11 AND LEMMA 11. Let $\mathfrak{A}$ be a uniform algebra and $\mathfrak{A}_0$ the set of functions rational over $\mathfrak{A}$, so that $\mathfrak{A} \subset \mathfrak{A}_0$. We define $\Lambda = \Lambda(\mathfrak{A})$, called the *analytic part* of Y, to be the set of all points p in the spectrum Y of $\mathfrak{A}$ such that there exists h in $\mathfrak{A}_0$ with a simple pole at p. Thus the poles of any function in $\mathfrak{A}_0$ lie in Λ. The set Λ is open in Y and can be given uniquely the structure of a Riemann surface in such a way that all functions h in $\mathfrak{A}_0$ are analytic on Λ except for a finite number of poles which with their multiplicities coincide with the poles and multiplicities of h when considered as a function rational over $\mathfrak{A}$. A point p in Λ is said to be a *zero of order k* of a function h in $\mathfrak{A}_0$ if the analytic function h on Λ has a zero of order k at p. For each p in Λ there exists g in $\mathfrak{A}$ with a zero of order 1 at p. If h is in $\mathfrak{A}_0$, if the points $p_1, \cdots, p_n$ in Λ with multiplicities $k_1, \cdots, k_n$ respectively are the poles of h, and if $h_1, \cdots, h_n$ are elements of $\mathfrak{A}_0$ having simple poles at $p_1, \cdots, p_n$ respectively then h can be written in the form

$$(*) \qquad h = \sum_{i=1}^{n} \sum_{j=1}^{k_i} a_{ij}(h_i)^j + f,$$

where the a_{ij} are constants and f is in $\mathfrak{A}$. The set $\mathfrak{A}_0$ is a subalgebra of $C(X)$, where X is the Šilov boundary of $\mathfrak{A}$. If h_1 and h_2 are in $\mathfrak{A}_0$ and if the point p in Y is a pole of neither then

$$(h_1h_2)(p) = h_1(p)h_2(p)$$

and

$$(a_1h_1 + a_2h_2)(p) = a_1h_1(p) + a_2h_2(p)$$

for all scalars a_1 and a_2.

Proof of Lemma 11. Lemma 10 establishes a local coordinate system at each point of Λ. Since the functions in $\mathfrak{A}$ separate points on Λ and are analytic in each such local coordinate system, we see that overlapping coordinate systems are analytically related. Thus Λ can uniquely be given the structure of a Riemann surface on which the functions in $\mathfrak{A}$ are analytic. If $h\in\mathfrak{A}_0$ and $p\in\Lambda$ is not a pole of h, then there exists g in $\mathfrak{A}$ with $g(p)=1$ and $gh=f\in\mathfrak{A}$. Thus $g(q)h(q)=f(q)$ for all q near p, so that h is analytic at p. If on the other hand p is a pole of h of order n, let h_1 have a simple pole p. Such an h_1 exists by Lemmas 2 and 4. Thus $p\in\Lambda$. There exists g in $\mathfrak{A}$ with $g(p)=0$ and $f_1(p)\neq 0$, where $f_1=gh_1$. Since by Lemma 10 the function h_1—as a meromorphic function on Λ—has a simple pole at p, we see that g has a simple zero at p. Since the element g^nh of $\mathfrak{A}_0$ is analytic at p, we see that as a meromorphic function on Λ, h has a pole at p of order at most n. On the other hand, there exists $g_1, \cdots, g_n$ in $\mathfrak{A}$ with $g_i(p)=0$ and $g_1 \cdots g_nh=f\in\mathfrak{A}_0$, $f(p)\neq 0$. This shows that as a meromorphic function on Λ, h has a pole at p of order at least n. Thus, as a meromorphic function on Λ, the order of the pole of h at p is exactly n. It follows that the poles of h and their orders are the same whether h is considered to be a meromorphic function on Λ or as a function rational over $\mathfrak{A}$. We have incidentally constructed a function, g above, with a zero of order 1 at any point p in Λ. The fact that Λ is open in Y follows from Lemma 10.

We now prove the representation (*) for an arbitrary h in $\mathfrak{A}_0$. To this end, notice that it is sufficient to consider the case in which h has only one pole, since by Lemma 4 an arbitrary h can be written as a linear combination of such h. We assume therefore that h has the single pole p of multiplicity k, and that h_1 is any function in $\mathfrak{A}_0$ with a simple pole at p. The representation (*) which we wish to obtain now reduces to

$$h = \sum_{j=1}^{k} a_j(h_1)^j + f, \tag{**}$$

with $f\in\mathfrak{A}$. Since h and h_1 are meromorphic on Λ with poles of orders k and 1 respectively at p, we can find constants $a_1, \cdots, a_j$ such that the function

$$h - \sum_{j=1}^{k} a_j(h_1)^j$$

is regular at p, where h and h_1 are considered as meromorphic functions on Λ.

If we let g be the function $\sum_{j=1}^{k} a_j(h_1)^j$ in $C(X)$, it follows from Lemma 3 that there exist f_1 and f_2 in $\mathfrak{A}$ with $g=f_1h+f_2$. For all j it is clear that $(h_1)^j$ is in $\mathfrak{A}_0$ with a pole of order j at p, and that $(h_1)^j(q)=(h_1(q))^j$ for all q in $Y-\{p\}$. If β in $\mathfrak{A}$ has a simple zero at p and $\alpha=\beta^k$ we see that $\alpha h_1^j \in \mathfrak{A}$ for $j\leqq k$, $(\alpha h_1^j)(p)=0$ for $j<k$, $(\alpha h_1)^k(p)\neq 0$. Since $a_k\neq 0$, it follows that $\alpha g\in\mathfrak{A}$ and $(\alpha g)(p)\neq 0$. For all q near p we have $(\alpha h)(q)=\alpha(q)h(q)$ and

$$(\alpha h_1^j)(q) = \alpha(q)(h_1^j)(q) = \alpha(q)(h_1(q))^j.$$

Thus

$$(\alpha h)(q) - (\alpha g)(q) = \alpha(q)\left[h(q) - \sum_{j=1}^{k} a_j(h_1(q))^j\right].$$

Since the function

$$h - \sum_{j=1}^{k} a_j h_1^j,$$

where h and h_1 are considered as meromorphic functions on Λ, is regular at p, it follows that $\alpha h-\alpha g$ has a zero of order at least k at p. Since

$$\alpha g = f_1\alpha h + f_2\alpha$$

it follows that

$$\begin{aligned}(\alpha g)(1-f_1) &= \alpha g - f_1\alpha g\\ &= \alpha g - f_1\alpha h + f_1(\alpha h - \alpha g) = f_2\alpha + f_1(\alpha h - \alpha g)\end{aligned}$$

has a zero of order at least k at p. But $(\alpha g)(p)\neq 0$ so that $1-f_1$ therefore has a zero of order at least k at p. Therefore $(1-f_1)h_1$ is in $\mathfrak{A}$ and has a zero of order at least $k-1$ at p. Therefore $(1-f_1)h_1^2$ is in $\mathfrak{A}$ and has a zero of order at least $k-2$ at $\mathfrak{A}$. Continuing this argument we see finally that $(1-f_1)h_1^k$ is in $\mathfrak{A}$. Since by Lemma 3 h has the form

$$h = \gamma_1(h_1)^k + \gamma_2$$

with γ_1 and γ_2 in $\mathfrak{A}$, it follows that $(1-f_1)h\in\mathfrak{A}$. Therefore

$$g = f_1h + f_2 = h - (1-f_1)h + f_2 = h - f,$$

where $f\in\mathfrak{A}$. But this is just (**).

It remains to show that $\mathfrak{A}_0$ is an algebra. Let f and g be in $\mathfrak{A}_0$. Let the points $p_1, \cdots, p_n$ in Λ include the poles of f and g, and let h_i, $1\leqq i\leqq n$, be in $\mathfrak{A}_0$ and have a simple pole at p_i. Thus both f and g have representations of the form (*). It follows that $f+g$ has a representation of the form (*). It therefore suffices to prove that the element h of $C(X)$ defined by (*) is in $\mathfrak{A}_0$, for arbitrary constants a_{ij} and an arbitrary f in $\mathfrak{A}$. We may assume that $a_{ik_i}\neq 0$ for each i. For each i, choose α_i in $\mathfrak{A}$ with a simple zero at p_i and with $\alpha_i(p_j)\neq 0$

for $i \neq j$. Set $\beta_i = (\alpha_i)^{k_i}$ and $\beta = \beta_1 \cdots \beta_n$. Thus $\beta_i(h_i)^j \in \mathfrak{A}$ for $j \leqq k_i$. Now $(\beta(h_i)^j)(p_i)$ is 0 if $j < k_i$ and not zero if $j = k_i$. Also $(\beta(h_m)^j)(p_i) = 0$ if $j \leqq k_m$ and $m \neq i$. It follows that βh is in $\mathfrak{A}$ and $(\beta h)(p_i) \neq 0$ for all i. Thus h is rational over $\mathfrak{A}$ with poles $p_1, \cdots, p_n$ having respective multiplicities $k_1, \cdots, k_n$. It follows that $\mathfrak{A}_0$ is closed under addition.

To see that $\mathfrak{A}_0$ is closed under multiplication, it is sufficient to consider elements f and g in $\mathfrak{A}_0$ each of which has at most one pole, since by Lemma 4 any element in $\mathfrak{A}_0$ is a linear combination of such elements, and since we have already seen $\mathfrak{A}_0$ to be closed under addition. First let f have a pole of multiplicity n at the point p, and let g have a pole of multiplicity m at the same point p, where $m > 0$, $n > 0$. We show that in this case fg has a pole of multiplicity $n+m$ at p. It is clear that $h = g_1 \cdots g_{n+m} fg \in \mathfrak{A}_0$ whenever $g_i(p) = 0$ for $1 \leqq i \leqq n+m$, since we can write

$$h = g_1 \cdots g_n f \, g_{n+1} \cdots g_{n+m} g.$$

It is also clear that we can choose the g_i to give $h(p) \neq 0$, since we can choose them to give

$$(g_1 \cdots g_n f)(p) \neq 0, \qquad (g_{n+1} \cdots g_{n+m} g)(p) \neq 0.$$

This shows that $fg \in \mathfrak{A}$. We now consider the case for $f \in \mathfrak{A}$, g has a pole of multiplicity $m > 0$ at the point q. By the decomposition (**), it is sufficient to consider functions g of the form $(h)^j$, where h has a simple pole at q. If r is the order of the zero of f at q, it is clear from successive multiplications by h that $h^i f$ is in $\mathfrak{A}$ for $i \leqq r$ and has a zero of order $r - i$ at q. If $j \leqq r$, this shows that $fg \in \mathfrak{A}$. If $j > r$, we see that $\alpha = h^r f$ is in $\mathfrak{A}$ and does not vanish at q, and that $fg = \alpha h^{j-r}$. From this it is clear that fg is in $\mathfrak{A}_0$ with a pole of order $j - r$ at q.

There remains the case in which f has a pole of order $n > 0$ at p and g has a pole of order $m > 0$ at q, with $p \neq q$. Let h_1 with $h_1(q) \neq 0$ have a simple pole at p and h_2 with $h_2(p) \neq 0$ have a simple pole at q. By the representation (**), it is sufficient to assume that either $f = (h_1)^j$ for $j > 0$ or $f \in \mathfrak{A}$, and $g = (h_2)^k$ for $k > 0$ or $g \in \mathfrak{A}$. Since we have already settled the cases $f \in \mathfrak{A}$ or $g \in \mathfrak{A}$, we assume $f = (h_1)^j$ and $g = (h_2)^k$.

It is then clear that

$$\alpha = f_1 \cdots f_j \, g_1 \cdots g_k f g \in \mathfrak{A}$$

whenever $f_i \in \mathfrak{A}$, $f_i(p) = 0$, $g_i \in \mathfrak{A}$, $g_i(q) = 0$. Also, if we choose each f_i to have a simple zero at p and not to vanish at q, and each g_i to have a simple zero at q and not to vanish at p, we see that $\alpha(p) \neq 0$, $\alpha(q) \neq 0$. Thus fg is in $\mathfrak{A}_0$. This completes the proof that $\mathfrak{A}_0$ is an algebra.

Now consider h_1 and h_2 in $\mathfrak{A}_0$ and p in Y which is a pole of neither h_1 nor h_2. If g in $\mathfrak{A}$ vanishes to sufficiently large order at the poles of h_1 and h_2 then $gh_1 \in \mathfrak{A}$, $gh_2 \in \mathfrak{A}$, and $gh_1h_2 \in \mathfrak{A}$. Choose such a g with $g(p) \neq 0$. We then have

$$g(p)(h_1h_2)(p) = (gh_1h_2)(p) = (gh_1)(p)h_2(p) = g(p)h_1(p)h_2(p),$$

so that $(h_1h_2)(p) = h_1(p)h_2(p)$. The proof that $(h_1+h_2)(p) = h_1(p)+h_2(p)$ is similar.

LEMMA 12. *Let $\mathfrak{A}$ be a uniform algebra with spectrum Y whose analytic part is Λ. Let U be an open subset of Λ. Let ϕ be a bounded linear functional on $C(Y-U)$ which vanishes on $\mathfrak{A}$. Then there exists a unique analytic differential $d\omega_\phi$ on U such that*

$$[*] \qquad \phi(h) = (2\pi i)^{-1} \int_C h d\omega_\phi,$$

if h is any rational function over $\mathfrak{A}$ whose poles all lie in U, if C is the union of a finite set of disjoint simple, closed rectifiable curves lying in U, and if C bounds a relatively compact open set $V \subset U$ which contains the poles of h.

Proof. Let p be any point in U and choose h in $\mathfrak{A}_0$ with a simple pole at p. Thus h^{-1} is analytic at p, so that $d(h^{-1})$, considered at p, is in the space of differentials at p (see Chevalley [2] for this notion). We define the form $d\omega_\phi$ to have the value $\phi(h)d(h^{-1})$ at p. To see that this does not depend on the choice of h, consider a second function g in $\mathfrak{A}_0$ with a simple pole at p. By Lemma 11, there exists a constant λ such that $g-\lambda h \in \mathfrak{A}$. Therefore $\phi(g) = \lambda\phi(h)$. Viewing g and h as meromorphic functions on Λ we see that $g-\lambda h$ is regular at p. Since g^{-1} and h^{-1} are regular at p and vanish there we see that

$$g^{-1}h^{-1}(g - \lambda h) = h^{-1} - \lambda g^{-1}$$

has a zero of order at least 2 at p. Therefore we have $d(h^{-1}) = \lambda d(g^{-1})$ at p. We therefore have

$$\phi(g)d(g^{-1}) = \phi(h)d(h^{-1})$$

at p, so that $d\omega_\phi$ is uniquely defined.

To see that $d\omega_\phi$ is an analytic differential, notice that the function $u(h, z)$ of Definition 9 has a simple pole at $\lambda_h(z)$ for each z in D_h. If we consider h, which is defined on $Y-\{p\}$, to be an element of $C(Y-U)$ then the mapping

$$z \to u(h, z) = h(1 - zh)^{-1}$$

is an analytic mapping from D_h^0 to $C(Y-U)$, where

$$D_h^0 = \{z: |z| < r\}$$

with r chosen so small that $|zh(p)| < 1$ for all z in D_h^0 and all p in $Y-U$. Thus

$$z \to \phi(u(h, z))$$

is an analytic function on D_h^0, so that

$$q \to \phi[u(h, \gamma_h(q))]$$

is an analytic function on E_h^0, where $E_h^0=\lambda_h(D_h^0)$ and γ_h denotes the mapping of E_h onto D_h which is inverse to λ_h. If h and $u(h, z)$ are considered as meromorphic functions on Λ, it follows from Lemma 10 that

$$u(h, z)(1 - zh) = h.$$

Therefore

$$[u(h, z)]^{-1} = h^{-1} - z$$

so that

$$d[u(h, z)]^{-1} = dh^{-1}.$$

Thus we see that the differential $d\omega_\phi$ is given on E_h^0 by

$$\begin{aligned}[d\omega_\phi]_q &= \phi[(u(h, \gamma_h(q))][d[u(h, \gamma_h(q))]^{-1}]_q \\ &= \phi[u(h, \gamma_h(q))]dh^{-1}(q),\end{aligned}$$

and is therefore an analytic differential.

To prove the formula [*], we avail ourselves of the representation (*) of Lemma 11. Thus in proving [*] it suffices to consider functions h^n, where h has a simple pole in V, and functions f in $\mathfrak{A}$. Now if $f\in\mathfrak{A}$, both sides of [*] vanish, the left side by the hypothesis on ϕ and the right side because $fd\omega_\phi$ is analytic on $V\cup C$.

Thus we consider h with a simple pole at a point p in V. Since $h^n d\omega_\phi$ is analytic on $V\cup C$ except at p, we may replace the contour C by any simple contour about p. Thus in proving [*] we may choose C to be a simple closed rectifiable curve lying in E_h^0 and surrounding the point p. Using the representation obtained above for $d\omega_\phi$ in E_h^0, we now compute, letting B be the curve in D_h^0 corresponding to the curve C in E_h^0,

$$\begin{aligned}\int_C h^n d\omega_\phi &= \int_C h^n(q)\phi(u[h, \gamma_h(q)])dh^{-1}(q) \\ &= \int_B z^{-n}\phi(u(h, z))dz = \phi\left[\int_B z^{-n}u(h, z)dz\right] \\ &= \phi\left[\int_B z^{-n}h(1 - zh)^{-1}dz\right] = \phi\left[\sum_{k=0}^{\infty}\int_B z^{k-n}h^{k+1}dz\right] \\ &= \phi[2\pi i h^n] = 2\pi i\phi(h^n).\end{aligned}$$

This proves [*] and thereby completes the proof of Lemma 12, since the fact that $d\omega_\phi$ is unique clearly follows from [*].

LEMMA 13. *Let $\mathfrak{A}$ be a uniform algebra with spectrum Y whose analytic part is Λ and whose Šilov boundary is X. Let U be an open set in Λ, and let B be the boundary of U. Let $\mathfrak{B}_1$ consist of all rational functions over $\mathfrak{A}$ whose poles lie in*

U, and let $\mathfrak{B}$ be the closure in $C(Y-U)$ of $\mathfrak{B}_1$. Then $Y-U$ is the spectrum of $\mathfrak{B}$ and the Šilov boundary of $\mathfrak{B}$ is a subset of $X\cup B$.

Proof. Consider any element λ in the spectrum of $\mathfrak{B}$. The restriction of λ to $\mathfrak{A}$ is some point p in Y, so that

$$\lambda(f) = f(p)$$

for all f in $\mathfrak{A}$. Assume that $p\in U$. Choose h in $\mathfrak{A}_0$ with a simple pole at p and f in $\mathfrak{A}$ with $f(p)=0$, $(fh)(p)\neq 0$. Then

$$0 \neq (fh)(p) = \lambda(fh) = \lambda(f)\lambda(h) = 0\cdot\lambda(h) = 0.$$

This contradiction proves that $p\in Y-U$. For any h in $\mathfrak{B}_1$ choose g in $\mathfrak{A}$ with $g(p)\neq 0$ and $gh\in\mathfrak{A}$. Then

$$g(p)\lambda(h) = \lambda(g)\lambda(h) = \lambda(gh) = (gh)(p) = g(p)h(p),$$

so that $\lambda(h)=h(p)$. Since this is true for all h in $\mathfrak{B}_1$ it is true for all h in $\mathfrak{B}$. Thus $Y-U$ is the spectrum of $\mathfrak{B}$.

We now show that the Šilov boundary of $\mathfrak{B}$ is a subset of $X\cup B$. Consider a point p in $Y-U-(X\cup B)$. Since X is the Šilov boundary of $\mathfrak{A}$ there exists a bounded linear functional ϕ_0 on $C(X)$ such that

$$\phi_0(f) = f(p)$$

for all f in $\mathfrak{A}$. Define the bounded linear functional ϕ on $C(X\cup\{p\})$ by

$$\phi(f) = \phi_0(f) - f(p).$$

Thus $\phi(f)=0$ for all f in $\mathfrak{A}$. Let V be any relatively compact subset of U whose boundary C consists of a finite number of disjoint rectifiable simple closed curves. Let $\mathfrak{B}_V$ consist of all functions in $\mathfrak{B}_1$ whose poles lie in V. From [*] of Lemma 12 it follows that

$$\phi(h) = (2\pi i)^{-1}\int_C h\,d\omega_\phi$$

for all h in $\mathfrak{B}_V$. If we define the bounded linear functional ϕ_1 on $C(C)$ by

$$\phi_1(f) = (2\pi i)^{-1}\int_C f\,d\omega_\phi,$$

it follows that $\phi_2=\phi_0-\phi_1$ is a bounded linear functional on $C(X\cup C)$ and that $\phi_2(h)=h(p)$ for all h in $\mathfrak{B}_V$. Thus, for each positive integer n,

$$|h(p)|^n = |\phi_2(h^n)| \leqq \|\phi_2\|[\sup\{|h(q)| : q\in X\cup C\}]^n$$

for all h in $\mathfrak{B}_V$. By taking roots and letting $n\to\infty$ it follows that

$$|h(p)| \leqq \sup\{|h(q)| : q\in X\cup C\}.$$

Now an arbitrary element h of $\mathfrak{B}_1$ will be in $\mathfrak{B}_V$ if V is a large enough subset of U, and the inequality just derived will obtain. Letting V converge to U it follows that

$$|h(p)| \leqq \sup\{|h(q)| : q \in X \cup B\}$$

for all h in $\mathfrak{B}_1$. This inequality therefore holds for all h in $\mathfrak{B}$, so that $X \cup B$ contains the Šilov boundary of $\mathfrak{B}$.

LEMMA 14. *Let $\mathfrak{A}$ be a uniform algebra with spectrum Y whose analytic part is Λ. Let X be the Šilov boundary of $\mathfrak{A}$. Let g be a rational function over $\mathfrak{A}$ such that g vanishes at only a finite set $p_1, \cdots, p_n$ of points in Y, all of which lie in Λ. Then g^{-1} is a rational function over $\mathfrak{A}$.*

Proof. Let $k_1, \cdots, k_n$ be the orders to which g vanishes at $p_1, \cdots, p_n$ respectively. Let U be an open set in Λ containing the poles and zeros of g such that the boundary C of U consists of a finite set of disjoint rectifiable Jordan arcs lying in Λ. Let $\mathfrak{B}_1$ consist of all rational functions over $\mathfrak{A}$ whose poles lie in U. Let $\mathfrak{B}$ be the closure of $\mathfrak{B}_1$ in $C(Y-U)$. By Lemma 13, $Y-U$ is the spectrum of $\mathfrak{B}$.

Now since $g \in \mathfrak{B}$ and g does not vanish on $Y-U$, it follows that $g^{-1} \in \mathfrak{B}$.

Let ϕ be any bounded linear functional on $C(X)$ which vanishes on $\mathfrak{A}$. By Lemma 12, we have

$$\phi(h) = (2\pi i)^{-1} \int_C h \, d\omega_\phi$$

for all h in $\mathfrak{B}_1$, and therefore for all h in $\mathfrak{B}$. Now let f be any function in $\mathfrak{A}$ which vanishes at $p_1, \cdots, p_n$ to orders at least $k_1, \cdots, k_n$. We shall show that $fg^{-1} \in \mathfrak{A}$. This will help prove that g^{-1} is rational over $\mathfrak{A}$ with poles at $p_1, \cdots, p_n$ of multiplicities $k_1, \cdots, k_n$. To see that $fg^{-1} \in \mathfrak{A}$, notice that $fg^{-1} \in \mathfrak{B}$, so that

$$\phi(fg^{-1}) = (2\pi i)^{-1} \int_C fg^{-1} d\omega_\phi = 0$$

since fg^{-1} is regular on Λ. Since ϕ is an arbitrary bounded linear function on $C(X)$ which annihilates $\mathfrak{A}$, it follows that $fg^{-1} \in \mathfrak{A}$. To complete the proof that g^{-1} is rational over $\mathfrak{A}$, choose $g_i \in \mathfrak{A}$, $1 \leqq i \leqq n$, such that g_i has a simple zero at p_i and $g_i(p_j) \neq 0$ for $i \neq j$. Write $f = (g_1)^{k_1} \cdots (g_n)^{k_n}$. By the above we have $fg^{-1} \in \mathfrak{A}$. Since $(fg^{-1})g = f$ and g have zeros of the same order at $p_1, \cdots, p_n$, it follows that $(fg^{-1})(p_i) \neq 0$ for $1 \leqq i \leqq n$. This completes the proof that g^{-1} is rational over $\mathfrak{A}$.

3. Conditions for analyticity of the spectrum. In this section we derive conditions which imply that certain points in the spectrum of a uniform algebra belong to the analytic part of the spectrum. Somewhat more exact conditions could be given, by refining the techniques employed here, but the added generality which would be obtained does not seem to justify the attendant complication of the proofs.

LEMMA 15. *Let $\mathfrak{A}$ be a uniform algebra with Šilov boundary X and spectrum Y with analytic part Λ. Let g be a function in $\mathfrak{A}$ which vanishes at a finite set $p_1, \cdots, p_n$ of points in Y, all of which lie in Λ and are simple zeros of g. Let $A = \inf\{|g(x)| : x \in X\}$. Let f in $\mathfrak{A}$ have the properties $f(p_1)=1$ and $f(p_i)=0$ for $2 \leqq i \leqq n$. Then there exists a neighborhood F_1 of p_1 in Λ which g maps homeomorphically onto $\{z: |z| < A(32\|f\|^3)^{-1}\}$.*

Proof. By Lemmas 11 and 14, we see that $h = f^2 g^{-1}$ is rational over $\mathfrak{A}$ with a simple pole at p_1. Let D be the set

$$\{z: |z| \leqq (4\|f\|)^{-1}[h]\}.$$

Since $\|f\| \geqq |f(p_1)| = 1$, we see that

$$D \subset D_h = \{z: |z| \leqq [h]\},$$

so that $E = \lambda_h(D)$ is a subset of E_h. By Lemma 9, $h^{-1} \circ \lambda_h$ is the identity map on $D \subset D_h$. Since $\|g^{-1}\| \leqq A^{-1}$, we have $\|h\| \leqq A^{-1}\|f\|^2$ so that $[h] = (2\|h\|)^{-1} \geqq A(2\|f\|^2)^{-1}$. Now $|f(q)-1| \leqq 2\|f\|$ for q in E_h, and $f(p_1)-1=0$. Since h^{-1} maps E_h homeomorphically onto D_h, it follows by Schwarz's lemma that

$$|f(q) - 1| \leqq 2\|f\| \, |h^{-1}(q)| \, [h]^{-1}$$

for all q in E_h. In particular, for q in E this gives

$$|f(q) - 1| \leqq 2\|f\|(4\|f\|)^{-1}[h][h]^{-1} = 1/2,$$

so that $|f(q)| \geqq 1/2$. Thus for q in the boundary of E we have

$$\begin{aligned} |g(q)| &= |f(q)|^2 |h(q)|^{-1} \geqq 1/4\,(4\|f\|)^{-1}[h] \\ &\geqq 1/4(4\|f\|)^{-1}A(2\|f\|^2)^{-1} = A(32\|f\|^3)^{-1}. \end{aligned}$$

Assume for the moment that p_1 is the only point in E at which g vanishes. Then this last inequality when combined with Rouche's theorem tells us that each value of z with $|z| < A(32\|f\|^3)^{-1}$ is assumed by g exactly once on the set E. If we set

$$F_1 = \{q: q \in E, \ |g(q)| < A(32\|f\|^3)^{-1}\},$$

the set F_1 has the required properties.

It only remains to see that p_1 is the only point in E at which g vanishes, i.e., that none of the points p_i, $2 \leqq i \leqq n$, is in E. If such a p_i were in E, we would have $h(p_i) = f(p_i)(fg^{-1})(p_i) = 0$, contradicting the fact that h does not vanish on E_h. This completes the proof of Lemma 15.

DEFINITION 12. Let $\mathfrak{A}$ be a uniform algebra with Šilov boundary X and with spectrum Y whose analytic part is Λ. Let g be a function in $\mathfrak{A}$. A point z in $-g(X)$ will be called *g-regular of multiplicity n* if $g^{-}(\{z\})$ consists of n

points $p_1, \cdots, p_n$ of Y, all of which lie in Λ and at each of which $g-z$ has a simple zero. A component U of $-g(X)$ will be called *g-regular of multiplicity n* if all points in U with the exception of an isolated set in U are g-regular of multiplicity n.

DEFINITION 13. A point p in the spectrum Y of a uniform algebra $\mathfrak{A}$ will be called *one-dimensional of multiplicity n* if there exists a connected neighborhood U of p such that

(i) $U-\{p\}$ consists of n components $U_1, \cdots, U_n$ each of which is a subset of Λ.

(ii) For $1 \leqq i \leqq n$ there exists a homeomorphism σ_i of $U_i \cup \{p\}$ onto $\{z: |z| < 1\}$ which is analytic on U_i.

LEMMA 16. *Let V be a bounded open set in the complex plane with boundary B. Let N be a relatively open subset of B and Δ an analytic function in V such that*

$$\lim_{z \to t} \Delta(z) = 0$$

for each t in N. Then N is an isolated set (the relative topology of N is discrete).

Proof. Let t_0 be any point in N and let L be some neighborhood

$$L = \{z: |z - t_0| < \epsilon\}$$

of t_0 with the property that $L \cap B \subset N$. Define the function Δ_0 on L by $\Delta_0(z) = \Delta(z)$ if $z \in V$ and $\Delta_0(z) = 0$ otherwise. Thus Δ_0 is continuous on L and analytic at those points where it does not vanish. By a theorem of Radó (see [3]), Δ_0 is analytic on L. Since Δ_0 vanishes on $L \cap N$, the point t_0 is isolated in $L \cap N$ and therefore isolated in N. This completes the proof.

LEMMA 17. *If $\mathfrak{A}$ is a uniform algebra with spectrum Y and Šilov boundary X and if g is a function in $\mathfrak{A}$, then any component U of $-g(X)$ which contains a g-regular point z_0 of multiplicity n is g-regular of multiplicity n. If z is any point in U then there are at most n points p in Y with $g(p) = z$, each of which is one-dimensional. If there are exactly n such p then they all lie in Λ and are simple zeros of $g-z$.*

Proof. Let f be any function in $\mathfrak{A}$ with $\|f\| \leqq 1/2$ which has distinct values at the points p in Λ with $g(p) = z_0$. Let U_0 consist of all points in U which are g-regular of multiplicity n. For each z in U_0 let $p_z^1, \cdots, p_z^n$ be the points in Λ where g takes the value z, and define the function Δ on U_0 by

$$\Delta(z) = \prod_{1 \leqq i < j \leqq n} (f(p_z^i) - f(p_z^j))^2.$$

Let V be the set of all z in U_0 with $\Delta(z) \neq 0$, so that $z_0 \in V$. For each z in U_0 define the functions f_z^i in $\mathfrak{A}$, $1 \leqq i \leqq n$, by

$$\overset{i}{f_z}(p) = \prod_{j \neq i} (f(p) - f(\overset{j}{p_z})).$$

Since $\|f\| \leqq 1/2$ we have $|\Delta(z)| \leqq 1$, $\|f_z^i\| \leqq 1$. Also,

$$|\Delta(z)| = \prod_{i=1}^{n} |\Delta_i(z)|,$$

where

$$\Delta_i(z) = \overset{i}{f_z}(\overset{i}{p_z}).$$

We therefore see that $|\Delta_i(z)| \geqq |\Delta(z)|$ for $1 \leqq i \leqq n$ and z in U_0. For z in V we define

$$\overset{i}{g_z} = (\Delta_i(z))^{-1} \overset{i}{f_z}.$$

It follows that $g_z^i \in \mathfrak{A}$, that $\|g_z^i\| \leqq |\Delta_i(z)|^{-1} \leqq |\Delta(z)|^{-1}$, that $g_z^i(p_z^i) = 1$, and that $g_z^i(p_z^j) = 0$ for $j \neq i$. It follows from Lemma 15 that there exists a neighborhood of p_z^i in Λ which $g - z$ maps homeomorphically onto

$$\{t: |t| < A_z(32\|\overset{i}{g_z}\|^3)^{-1}\},$$

where A_z is the distance of z to $g(X)$. Thus g maps some neighborhood F_z^i of p_z^i homeomorphically onto

$$D_z = \{t: |t - z| < K_z\}$$

where

$$K_z = 1/32 A_z |\Delta(z)|^3.$$

For $i \neq j$ there thus exists a unique analytic homeomorphism σ of F_z^i onto F_z^j which identifies points at which g has equal values. Now if F_z^i and F_z^j had a common point q, it would follow that $\sigma(q) = q$. Thus $\Omega = F_z^i \cap F_z^j$ would be nonvoid and σ is the identity map on Ω. Clearly Ω is open in F_z^i because both F_z^i and F_z^j are open. Since Ω is the fixed set of σ on F_z^i it is also clear that Ω is closed in F_z^i. Since F_z^i is connected it follows that $F_z^i = \Omega$ if Ω is nonvoid. From this it would follow that $\sigma(p) = p$ for all p in F_z^i, so that $p_z^j = \sigma(p_z^i) = p_z^i$, a contradiction. Therefore the sets $F_z^1, \cdots, F_z^n$ are disjoint for each z in V.

If $z \in V$ and $t \in D_z$, it follows that $g - t$ has exactly one simple zero in each of the sets F_z^i, which we denote by $p_t^1, \cdots, p_t^n$. If we can show that $g - t$ vanishes at no other point of Y, it will follow that $t \in U_0$. To see this, let H be the set of all t in D_z such that $g(p) = t$ for some p in $Y - F_z^1 - \cdots - F_z^n$. Clearly H is a closed subset of D_z and $z \notin H$. To see that H is open in D_z or that $D_z - H$ is closed in D_z, let t in D_z be in the closure of $D_z - H$, so that there exists u in $D_z - H$ arbitrarily near to t. By Lemma 14, $(g - u)^{-1}$ is in $\mathfrak{A}_0$, so that $(h-h(p_u^1)) \cdots (h-h(p_u^n))(g-u)^{-1}$ is in $\mathfrak{A}$, for each h in $\mathfrak{A}$. Letting u converge to t, we see that

$$(h - h(p_t^1)) \cdots (h - h(p_t^n))(g - t)^{-1} = h_0$$

is in $\mathfrak{A}$. Thus for any p in Y with $g(p) = t$ we have

$$(h(p) - h(p_t^1)) \cdots (h(p) - h(p_t^n)) = (g(p) - t)h_0(p) = 0.$$

Since h was any element in $\mathfrak{A}$, it follows that p is one of the points p_t^i. Thus $t \in D_z - H$. Since H is both open and closed, and since $z \notin H$, we see that H is void. Therefore $D_z \subset U_0$. Since $\Delta(z) \neq 0$ and the p_t^i depend continuously on t for t in D_z, we see that $\Delta(t) \neq 0$ for all t sufficiently near z. Thus V is an open subset of U.

Now Δ is an analytic function on V, because for $1 \leqq i \leqq n$ the mapping $t \rightarrow p_t^i$ is an analytic function from D_z to Λ. We see by the above formula for K_z that every boundary point of V is either a point of $g(X)$ or a point at which Δ converges to 0. Let B be the boundary of V. It follows from Lemma 16 that $N = B - g(X)$ is an isolated set. Therefore $U - V$ is an isolated subset of U. Therefore U is a g-regular component of $-g(X)$ of multiplicity n. If $p \in Y$ and $g(p) \in V$, then $p \in \Lambda$ so that p is a one-dimensional point of Y of multiplicity 1. If, on the other hand, $g(p) \in U - V$, let W be a neighborhood of $z = g(p)$ such that $W - \{z\} \subset V$. Thus $g^{-}(W - \{z\})$ (where g^{-} is the relation inverse to g) is an n-sheeted Riemann surface S over $W - \{z\}$ and the functions in $\mathfrak{A}$ are all analytic and bounded on S. Thus S can be completed to a Riemann surface S_0 over W, with possible branch points at z, on which the functions in $\mathfrak{A}$ can be extended to be analytic. Thus S_0 has a natural mapping into Y, and it is clear that every point in the image T_0 of $S_0 - S$ in Y is one-dimensional. It is also clear that there are at most n such points. It remains to prove that $p \in T_0$. To see this, we use the same type of proof that was used above to show that $g - t$ vanishes only at $p_t^1, \cdots, p_t^n$. Thus we consider arbitrary functions $h_1, \cdots, h_n$ in $\mathfrak{A}$, so that

$$(h_1 - h_1(p_u^1)) \cdots (h_n - h_n(p_u^n))(g - u)^{-1}$$

is in $\mathfrak{A}$ for each u in $W - \{z\}$. Letting u converge to z we see that

$$(h_1 - h_1(p^1)) \cdots (h_n - h_n(p^n))(g - z)^{-1} = h_0$$

is in $\mathfrak{A}$, where $p^1, \cdots, p^n$ are certain points (not necessarily distinct) in T_0 with $g(p^i) = z$. We therefore have

$$(h_1(p) - h_1(p^1)) \cdots (h_n(p) - h_n(p^n)) = (g(p) - z)h_0(p) = 0.$$

Since h_i is any element of $\mathfrak{A}$, p is one of the points p^i, as was to be proved.

It remains to show that $(g - z)^{-1}$ is rational over $\mathfrak{A}$ with poles $p^1, \cdots, p^n$ whenever $p^1, \cdots, p^n$ are distinct points in Y with $g(p^i) = z \in U$, $1 \leqq i \leqq n$.

Since we have just seen that the function h_0 is in $\mathfrak{A}$ for all choices of the h_i, this will follow from the following lemma.

LEMMA 18. *Let g be a function in a uniform algebra $\mathfrak{A}$ and let $p_1, \cdots, p_n$ be points in the spectrum Y of $\mathfrak{A}$ such that $h_0 = g^{-1}h_1 \cdots h_n$ is in $\mathfrak{A}$ whenever the functions h_i are in $\mathfrak{A}$ and $h_i(p_i) = g(p_i) = 0$ for $1 \leqq i \leqq n$. Then g^{-1} is rational over $\mathfrak{A}$ with poles $p_1, \cdots, p_n$.*

Proof. Let

$$h_i = tg + f_i,$$

where $f_i \in \mathfrak{A}$, $f_i(p_i) = 0$, $f_i(p_j) = 1$ for $j \neq i$ and where t will be chosen. It is clear that $h_0(p_i)$ is a polynomial F_i of degree $\leqq n$ in t and that the coefficient of t in F_i is 1. Thus t may be chosen so that $F_i(t) \neq 0$ for all i, and thus $h_0(p_i) \neq 0$ for all i. It follows from Definition 4 that g^{-1} is rational over $\mathfrak{A}$ with poles $p_1, \cdots, p_n$.

We next investigate the nature of points p in $Y - X$ with $g(p) \in g(X)$.

LEMMA 19. *Let $\mathfrak{A}$ be a uniform algebra with Šilov boundary X and spectrum Y. Let g be a function in $\mathfrak{A}$ and U be a g-regular component of $-g(X)$ of multiplicity n. Let the point z_0 in $g(X)$ be the vertex of a nondegenerate triangle whose interior lies in U. Let there exist only a finite number of points q in X with $g(q) = z_0$. Then there exist at most n points p_0 in $Y - X$ with $g(p_0) = z_0$, and each of these points is a one-dimensional point of $Y - X$.*

Proof. It is no loss of generality to assume that $z_0 = 0$ and that the segment $(0, 1]$ of the real axis lies interior to the triangle in question. There therefore exists a constant $K > 0$ such that

$$\operatorname{dist}(x, g(X)) \geqq Kx$$

for $0 < x \leqq 1$.

Let p_0 be any point in $Y - X$ such that $g(p_0) = z_0 = 0$. The idea of the proof will be to perturb g to a function g_0 such that $g_0(p_0)$ will lie in a g_0-regular component of $-g_0(X)$, thereby showing that p_0 is one-dimensional. The perturbing function h will be any function in $\mathfrak{A}$ such that $h(p_0) = 1$ and $h(q) = 0$ whenever $q \in X$ and $g(q) = 0$. Such a function h exists because the number of such points q is finite.

Since h vanishes on the set $g^{-}(0) \cap X$, there exists x in $(0, 1]$ such that $|h(p)| < K$ whenever $p \in X$ and $|g(p)| \leqq (1 + K/2)x$. Write $g_0 = g + Mh$, where $M = \min\{x/2, (2\|h\|)^{-1}Kx\}$. Let p be any point in Y with $g_0(p) = x$. Then

$$|g(p) - x| \leqq M|h(x)| \leqq M\|h\| \leqq \frac{1}{2} Kx < \operatorname{dist}(x, g(X)).$$

It follows that $g(p) \in U$, so that p is a one-dimensional point, and some neighborhood of p with p deleted lies in Λ. It also follows that there are only a finite number of such p, since an accumulation point of $(g_0)^-(x)$ would be a point in $(g_0)^-(x)$ which could not be one-dimensional. Thus $(g_0)^-(x)$ is finite and consists of points each of which has a deleted neighborhood lying in Λ. It follows that y is a g_0-regular point of $-g_0(X)$ for all y sufficiently near to x. Choose such a y with $M<y\leqq x$, and let V be the component of $-g_0(X)$ which contains y. By Lemma 17, V is g_0-regular. We shall show that the interval $[M, x]$ belongs to V. It will follow that p_0 is one-dimensional, since $g_0(p_0)=g(p_0)+Mh(p_0)=0+M=M\in V$ and V is g_0-regular.

To see that $[M, x]\subset V$, it is clearly enough to show $z\notin g_0(X)$ for z in $[M, x]$. Therefore consider z with $M\leqq z<x$, and let p be any point in X. To show that $g_0(p)\neq z$, there are two cases to consider. First consider the case $|g(p)|>(1+K/2)x$. Then

$$| g_0(p) | \geqq | g(p) | - M\|h\| > \left(1+\frac{1}{2}K\right)x - \frac{1}{2}Kx = x > z$$

so that $g_0(p)\neq z$. Next consider the case $|g(p)|\leqq(1+K/2)x$, so that $|h(p)|<K$. Then

$$\begin{aligned}\operatorname{dist}(g_0(p), g(X)) &\leqq | g_0(p) - g(p) | = M | h(p) | < MK \leqq zK \\ &\leqq \operatorname{dist}(z, g(X)),\end{aligned}$$

so that $g_0(p)\neq z$ in this case also.

Thus we have shown that every p_0 in $Y-X$ with $g(p_0)=0$ is a one-dimensional point of Y which has a deleted neighborhood consisting of points in Λ. Thus for each neighborhood N of p_0 we see that $g(N)$ is a neighborhood of 0. It follows that there are at most n such points p_0, since otherwise all points t in the complex plane which are sufficiently near to 0 would be the images under g of more than n points in Y, and we know that this is not the case for t in U. This completes the proof of Lemma 19.

It remains to give conditions which make a component of $-g(X)$ g-regular. In doing this we essentially follow Wermer [1], although the details are different. The idea is to start from the unbounded component of $-g(X)$, which is obviously g-regular, and to proceed step by step, showing that a component of $-g(X)$ which is close enough to a g-regular component is itself g-regular. The crucial lemma is the following, which is derived following Wermer.

Lemma 20. *Let $\mathfrak{A}$ be a uniform algebra with Šilov boundary X and spectrum Y. Let g be a function in $\mathfrak{A}$ and U and V components of $-g(X)$, such that there exists z in V with $(g-z)^{-1}\in\mathfrak{A}$ for some z in V (so that V is g-regular of multiplicity* 0). *Let there exist an open Jordan arc J_1 which is an open subset of $g(X)$ such that $J=g^-(J_1)\cap X$ is mapped homeomorphically by g onto J_1 and such that U and V are the components of $-g(X)$ which border on J_1. Then U is g-regular of multiplicity* 0 *or* 1.

Proof. By replacing the function g in $\mathfrak{A}$ by the function $(g-z)^{-1}$ in $\mathfrak{A}$ we reduce to the case in which V is the unbounded component of $-g(X)$. After replacing the arc J_1 by a slightly smaller arc—if necessary—we can find a simple closed curve Γ in the complex plane with interior Φ such that $g(J) = J_1 \subset \Gamma$ and $g(X) \subset \Gamma \cup \Phi$. Let ϕ be a conformal map of Φ onto $\{w: |w| < 1\}$, so that ϕ can be extended to a homeomorphism of $\Gamma \cup \Phi$ onto

$$D = \{w: |w| \leqq 1\}.$$

By a theorem of Mergelyan [6] we see that ϕ is a uniform limit on $\Gamma \cup \Phi$ of polynomials, so that $g_0 = \phi \circ g$ is in $\mathfrak{A}$.

Let z_0 be any point in U. Write $w_0 = \phi(z_0)$ and $U_0 = \phi(U)$. Let ψ be the inverse mapping to ϕ, so that $g = \psi \circ g_0$. Now if $g_0 - w_0$ vanishes at a unique point p_0 in Y then clearly $g - z_0$ vanishes at the unique point p_0 in Y. If p_0 lies in Λ and $g_0 - w_0$ vanishes to multiplicity 1 at p_0 then, since $g = \psi \circ g_0$, the function $g - z_0$ also vanishes to multiplicity 1 at p_0. It follows that to show z_0 is g-regular of multiplicity 1 it is sufficient to show that w_0 is g_0-regular of multiplicity 1. The same statement holds of multiplicity 0. Thus to show that z_0 is g-regular of multiplicity 0 or 1 (and thereby prove the lemma) it is sufficient to show that w_0 is g_0-regular of multiplicity 0 or 1. There are two cases to consider, depending on whether $(g_0 - w_0)^{-1}$ is in $\mathfrak{A}$. If it is then w_0 is g_0-regular of multiplicity 0. Thus we have left the case $(g_0 - w_0)^{-1} \notin \mathfrak{A}$. Under this assumption there exists a finite, complex-valued Baire measure μ on X with

$$\int (g_0 - w_0)^{-1} d\mu \neq 0$$

and

$$\int f d\mu = 0$$

for all f in $\mathfrak{A}$. Now

$$0 \neq \int (g_0 - w_0)^{-1} d\mu = \int (w - w_0)^{-1} d\nu(w),$$

where $\nu = g_0(\mu)$. If we let J_0 be the arc $g_0(J) = \phi(J_1)$ of the boundary of D, and $X_0 = g_0(X)$, we see that J_0 is open in X_0 and that g_0 maps $J = g_0^-(J_0)$ homeomorphically onto J_0. Also U_0 and the unbounded component of $-X_0$ are the components of $-X_0$ which adjoin J_0. Clearly ν is a measure on X_0.

For each f in $\mathfrak{A}$ the measure $f\mu$ on X defined by

$$(f\mu)(S) = \int_S f d\mu$$

for all Baire sets S will be orthogonal to $\mathfrak{A}$. In particular $f\mu$ is orthogonal to all polynomials in g_0 so that the measure

$$\nu_f = g_0(f\mu)$$

on X_0 is orthogonal to all polynomials. Clearly $\nu_1 = \nu$. For each f in $\mathfrak{A}$ define the analytic function $\tilde{f}$ on U_0 by

$$\tilde{f}(w) = \int (t-w)^{-1} d\nu_f(t) = \int (g_0 - w)^{-1} f d\mu.$$

We have

$$\tilde{1}(w_0) = \int (w - w_0)^{-1} d\nu(w) \neq 0,$$

so that $\tilde{1}$ does not vanish identically on U_0. Thus the set T of zeros of $\tilde{1}$ is an isolated subset of U_0. Since ν_f is orthogonal to all polynomials,

$$\int (t-w)^{-1} d\nu_f(t) = 0$$

for all w in $-D$. Since J_0 separates $-D$ from U_0, it follows (see for example Wermer [7, p. 49]) that $\tilde{f}$ has nontangential boundary values $\tilde{f}(t_0)$ at almost all points t_0 in J_0, given by

$$\tilde{f}(t_0) = 2\pi i \left[\frac{d\nu_f(t)}{dt}\right]_{t=t_0},$$

where $d\nu_f(t)/dt$ is the Radon-Nikodym derivative of the measure ν_f on J_0 with respect to the measure dt on J_0. If we let h be the map of J_0 onto J which is inverse to g_0 then the restriction of ν_f to J_0 has the representation

$$\nu_f = g_0(f\mu) = (f \circ h) g_0(\mu) = (f \circ h) \cdot \nu$$

for all f in $\mathfrak{A}$. It follows that

$$\tilde{f}(t_0) = (f \circ h) \cdot \tilde{1}(t_0)$$

for almost all t_0 in J_0. It follows that for arbitrary f_1 and f_2 in $\mathfrak{A}$ the function $\alpha = \tilde{f}_1 \tilde{f}_2 - [f_1 f_2]^{\sim} \cdot \tilde{1}$ on U_0 has nontangential boundary values which vanish almost everywhere on J_0. Therefore α vanishes identically on U_0. If T denotes the isolated subset of U_0 on which $\tilde{1}$ vanishes, for each f in $\mathfrak{A}$ define the function $\bar{f}$ on $U_0 - T$ by

$$\bar{f} = \tilde{f}/\tilde{1}.$$

Thus $\bar{f}$ is analytic in $U_0 - T$ and $\bar{f}_1 \bar{f}_2 = [f_1 f_2]^{-}$ for all f_1 and f_2 in $\mathfrak{A}$. For each w in $U_0 - T$ it follows that the map $f \to \bar{f}(w)$ is a homomorphism of $\mathfrak{A}$ into the complex numbers and therefore defines a point in the spectrum Y of $\mathfrak{A}$. Thus we see that $|\bar{f}(w)| \leq \|f\|$ for all w in $U_0 - T$. It follows that $\bar{f}$ can be extended to all of U_0 and has nontangential boundary values $f\circ h$ at almost all points of J_0.

To show that w_0 is g_0-regular of multiplicity 1, let p be the point in Y defined by

$$f(p) = \tilde{f}(w_0)$$

for all f in $\mathfrak{A}$. It is enough to show that $(g_0-w_0)^{-1}$ is rational over $\mathfrak{A}$ with a simple pole at p. To do this, consider f in $\mathfrak{A}$ with $f(p)=\tilde{f}(w_0)=0$. Then

$$\begin{aligned}\int f(g_0 - w_0)^{-1}d\mu &= \int (g_0 - w_0)^{-1}d(f\mu)\\ &= \int (w - w_0)^{-1}d\nu_f(w) = \tilde{f}(w_0) = \tilde{f}(w_0)\cdot\tilde{1}(w_0)\\ &= f(p)\tilde{1}(w_0) = 0.\end{aligned}$$

Now μ can be any measure on X which is orthogonal to $\mathfrak{A}$ and which is not orthogonal to $(g-z_0)^{-1}$. Since such a μ exists, every measure σ on X which is orthogonal to $\mathfrak{A}$ can be written as the difference of two such μ. Thus

$$\int f(g_0 - w_0)^{-1}d\sigma = 0$$

for all f in $\mathfrak{A}$ with $f(p)=0$ and all such σ. The function $f_1=f(g_0-w_0)^{-1}$ is therefore in $\mathfrak{A}$. If $f=(g_0-w_0)$ then $f_1(p)=1\neq 0$. Thus $(g_0-w_0)^{-1}$ is rational over $\mathfrak{A}$ with a simple pole at p. It follows that g_0-w_0 vanishes on Y at the unique point p in Λ which is a simple zero of g_0-w_0. Thus w_0 is g_0-regular of multiplicity 1, as was to be proved.

LEMMA 21. *Let $\mathfrak{A}$ be a uniform algebra with spectrum Y and Šilov boundary X. Let Λ be the analytic part of Y. Let g be a function in $\mathfrak{A}$ and U and V be components of $-g(X)$. Let J_0 be a smooth simple open Jordan arc which is an open subset of $g(X)$ such that the set $J=g^{-}(J_0)\cap X$ is mapped homeomorphically by g onto J_0. Let U and V be the components of $-g(X)$ which adjoin J_0. Let V be g-regular of multiplicity n. Then U is g-regular of multiplicity n, $n+1$, or $n-1$.*

Proof. By the smoothness of J_0, all points in J_0 are vertices of nondegenerate triangles whose interiors lie in V. Consider f in $\mathfrak{A}$ and form the function Δ on V defined at any g-regular point z of V by

$$\Delta(z) = \prod_{1\le i<j\le n} (f(p_z^i) - f(p_z^j))^2,$$

where $p_z^1, \cdots, p_z^n$ are the points in $g^{-}(\{z\})$. The definition of Δ is completed by defining it to be 0 at other points of V, so that Δ is an analytic function on

V. If f is chosen to have distinct values at the points $p_z^1, \cdots, p_z^n$ for some particular z then Δ will not vanish identically on V. Let f be so chosen. Then the set Γ of points z_0 in J_0 such that $\Delta(z)$ does not converge to 0 as $z \to z_0$ is dense in J_0. For each z_0 in Γ there exist at least n distinct points q in Y with $g(q) = z_0$. Now for any z_0 in J_0 there exists exactly one q in X with $g(q) = z_0$ and by Lemma 19 there are at most n such q in $Y - X$. Thus there are at most $n+1$ distinct points q in Y with $g(q) = z_0$, for all z_0 in J_0. Let z_0 be chosen to be a point in J_0 for which the number k of such points q is a maximum. Thus $k \leqq n+1$. On the other hand, $k \geqq n$ because for z_0 in Γ there exist n such points q. The rest of the proof of Lemma 21 divides into the consideration of two cases. Case 1 will be the case $k = n+1$ and Case 2 the case $k = n$.

We consider first Case 1. Since there is exactly 1 point q in X with $g(q) = z_0$, there are exactly n points $q_1, \cdots, q_n$ in $Y - X$ with $g(q_i) = z_0$. By Lemma 19, each q_i is a one-dimensional point of Y and therefore has a deleted neighborhood which lies in Λ. Thus if we replace z_0 by a sufficiently near point of J_0 we may asume that $q_i \in \Lambda$, $1 \leqq i \leqq n$, and that $g - z_0$ has a simple zero at each of the points q_i. Thus there exist disjoint neighborhoods $W_1, \cdots, W_n$ in Λ of $q_1, \cdots, q_n$ respectively whose closures lie in Λ each of which g maps homeomorphically onto a neighborhood T of z_0. We may choose T so that $U \cap T$ and $V \cap T$ are connected. Write

$$W = W_1 \cup \cdots \cup W_n$$

and $B = \text{bdry } W$. Let $\mathfrak{B}_1$ be all rational functions over $\mathfrak{A}$ whose poles lie in W. Let $\mathfrak{B}$ be the closure of $\mathfrak{B}_1$ in the space $C(Y - W)$. We see by Lemma 13 that the Šilov boundary X_0 of $\mathfrak{B}$ is a subset of $X \cup B$. Thus $g(X_0) \subset (\text{bdry } T) \cup g(X)$. It follows that there are unique components U_0 and V_0 of $-g(X_0)$ with $U_0 \supset T \cap U$ and $V_0 \supset T \cap V$. Since V is g-regular for $\mathfrak{A}$ of multiplicity n, and since $T \subset g(W_i)$ for each i, we see that $g^{-}(T \cap V) \subset W_i$, so that $T \cap V \subset -g(T - W)$. Since $T - W$ is the spectrum of $\mathfrak{B}$ it follows that V_0 is g-regular of multiplicity 0 for the algebra $\mathfrak{B}$. If U_0 and V_0 are the same component of $-g(X_0)$ then U_0 is g-regular for the algebra $\mathfrak{B}$ of multiplicity 0. Otherwise (by Lemma 20) U_0 is g-regular for the algebra $\mathfrak{B}$ of multiplicity either 0 or 1. Thus in either case U_0 is g-regular for the algebra $\mathfrak{B}$ of multiplicity 0 or 1. In case U_0 is g-regular for $\mathfrak{B}$ of multiplicity 0 then for each z in U_0 $g - z$ does not vanish on $Y - W$ so that z is g-regular for $\mathfrak{A}$ of multiplicity n. Thus in this case U is g-regular of multiplicity n. In case U_0 is g-regular of multiplicity 1 for $\mathfrak{B}$ let z be any point in $T \cap U$ and let p_0 be the point in $Y - W$ with $g(p_0) = z$. Thus $g - z$ vanishes on Y at exactly the points $p_0, p_1, \cdots, p_n$, where p_i for $1 \leqq i \leqq n$ is that point in W_i with $g(p_i) = z$. To show that U is a g-regular component of multiplicity $n+1$ for the algebra $\mathfrak{A}$ it suffices to show that $(g - z)^{-1}$ is rational over $\mathfrak{A}$ with poles $p_0, p_1, \cdots, p_n$. To this end consider f in $\mathfrak{A}$ vanishing at $p_0, \cdots, p_n$. Since z is a g-regular point for $\mathfrak{B}$ of multiplicity 1 and $f(p_0) = 0$ we see that $f(g - z)^{-1} \in \mathfrak{B}$. Thus if ϕ is a bounded linear functional on $C(X)$ which vanishes on $\mathfrak{A}$ we have

$$\phi[f(g-z)^{-1}] = \int_B f(g-z)^{-1} d\omega_\phi = 0$$

since $f(g-z)^{-1}$ is analytic in W and on the boundary B of W. Since this is true for all ϕ we have $f(g-z)^{-1} \in \mathfrak{A}$. By Lemma 18 it follows that $(g-z)^{-1}$ is rational over $\mathfrak{A}$ with poles $p_0, \cdots, p_n$ and therefore that U is g-regular for $\mathfrak{A}$ of multiplicity $n+1$. Thus we see that in Case 1, U is g-regular of multiplicity n or $n+1$.

It remains to consider Case 2, so that there are $n-1$ points, say $q_1, \cdots, q_{n-1}$ in $Y-X$ with $g(q_i)=z_0$. Since by Lemma 19 each q_i has a deleted neighborhood which lies in Λ, we may assume—by replacing z_0 by a nearby point of J_0 if necessary—that each q_i belongs to Λ and is a simple zero of $g-z_0$. Choose disjoint neighborhoods $W_1, \cdots, W_{n-1}$ in Λ of $q_1, \cdots, q_{n-1}$ respectively which g maps homeomorphically onto a neighborhood T of z_0 such that the $\overline{W}_i$ are disjoint subsets of Λ and such that $T \cap U$ and $T \cap V$ are connected. If T is chosen small enough then $T = (T \cap U) \cup (T \cap V) \cup (T \cap J_0)$. Write

$$\Omega = W_1 \cup \cdots \cup W_{n-1} \cup g^{-}(T \cap V).$$

We first show that $\Omega \subset \Lambda$. To do this it is sufficient to show that

$$H = g^{-}(T \cap V) - W_1 - \cdots - W_{n-1} \subset \Lambda.$$

Consider p_0 in H, so that $z = g(p_0)$ is in V. Thus there exist p_1 in $W_1, \cdots, p_{n-1}$ in W_{n-1} with $g(p_i)=z$. Thus $p_0, \cdots, p_{n-1}$ are distinct points in $g^{-}(\{z\})$. Since $z \in V$ and V is g-regular of multiplicity n these points are all of $g^{-}(\{z\})$. It follows from Lemma 17 that $p_0 \in \Lambda$. Therefore $\Omega \subset \Lambda$.

Now let $\mathfrak{B}_1$ be the set of all functions rational over $\mathfrak{A}$ whose poles lie in Ω and let $\mathfrak{B}$ be the closure of $\mathfrak{B}_1$ in the space $C(Y-\Omega)$. Thus the Šilov boundary X_0 of $\mathfrak{B}$ is a subset of $X \cup \text{bdry } \Omega$. Since $U \cap T$ and $V \cap T$ are connected these sets are therefore contained respectively in components U_0 and V_0 of $-g(X_0)$. The component V_0 of $-g(X_0)$ is g-regular of multiplicity 0 relative to the algebra $\mathfrak{B}$ since $g-z$ does not vanish on $Y-\Omega$ whenever $z \in V \cap T$. By Lemma 20, U_0 is g-regular of multiplicity 0 or 1 for the algebra $\mathfrak{B}$. Now if U_0 is g-regular for $\mathfrak{B}$ of multiplicity 0 then $g-z$ does not vanish on $Y-\Omega$ for z in $U \cap T \subset U_0$, so that for such z the zeros of $g-z$ are in $W_1 \cup \cdots \cup W_{n-1}$. Therefore z is a g-regular point of $-g(X)$ of multiplicity $n-1$. Thus we need only consider the case in which U_0 is g-regular for $\mathfrak{B}$ of multiplicity 1. In this case for each z in $T \cap U \subset U_0$ there is exactly one point $\lambda(z)$ in $Y-\Omega$ with $g(\lambda(z))=z$, and the map $z \to \lambda(z)$ is a homeomorphism of $T \cap U$ onto an open subset of $\Lambda(\mathfrak{B})$, where $\Lambda(\mathfrak{B}) \subset Y-\Omega$ is the analytic set for the algebra $\mathfrak{B}$. We now extend the function λ from $T \cap U$ to the entire set $T = (T \cap U) \cup (T \cap V) \cup (T \cap J_0)$. For each z in $T \cap J_0$ let $\lambda(z)$ be the point in X with $g(\lambda(z))=z$, so that λ is a homeomorphism of $T \cap J_0$ onto a subset of J. For each z in $T \cap V$ let $\lambda(z)$ be that point in H with $g(\lambda(z))=z$. There exists one such point $\lambda(z)$ because otherwise $g-z$ would vanish on Y only at the points $p_1, \cdots, p_{n-1}$

in Λ, at which points $g-z$ has simple zeros, contradicting the fact that V is g-regular of multiplicity n. On the other hand there is at most one such point $\lambda(z)$ in H, again because V is g-regular of multiplicity n. Thus $\lambda(z)$ is uniquely defined on $T\cap V$. From Lemma 17 it follows that $\lambda(z)\in\Lambda$ for all z in $T\cap V$. Clearly λ is an analytic homeomorphism of $T\cap V$ onto an open subset of Λ. Thus we have defined a map λ from T to Y. Since λ is continuous on each of the sets $T\cap U$, $T\cap V$ and $T\cap J_0$, to show that λ is continuous on T it is only necessary to show that λ is continuous at points of $T\cap J_0$. If this were not so there would exist z in $T\cap J_0$ and a sequence $\{z_i\}$ in T converging to z with $\lambda(z_i)$ converging to a point $q\neq\lambda(z)$ in Y. Since $\lambda(z_i)\in Y-W_1-\cdots-W_{n-1}$ we have $q\in Y-W_1-\cdots-W_{n-1}$. Let u_i, $1\leqq i\leqq n-1$, be the point in W_i for which $g(u_i)=z$. Thus $u_1,\cdots,u_{n-1}$, q, $\lambda(z)$ are distinct points in Y mapping by g onto z. Since $z\in J_0$ this contradicts the fact that $k=n$ in Case 2. This contradiction shows that λ is a continuous map of T into Y. Now for each f in $\mathfrak{A}$ the function $f\circ\lambda$ is continuous on T and analytic on $T-J_0$. Since J_0 is smooth, $f\circ\lambda$ is analytic on T. Thus $f\circ\lambda$ has no strong maximum interior to T so that f has no strong maximum on the set $\lambda(T)$. Thus $\lambda(J_0)\subset J$ is an open subset of X such that every f in $\mathfrak{A}$ assumes its maximum on $X-\lambda(J_0)$. This contradicts the fact that X is the Šilov boundary of $\mathfrak{A}$. This contradiction shows that U_0 can not be a g-regular component of $-g(X_0)$ of multiplicity 1 for the algebra $\mathfrak{B}$. This was the last remaining case so that the proof of Lemma 21 is complete.

4. **Proofs of Theorems 1 and 2.**

Proof of Theorem 1. Let U be any component of $-g(X)$ and let w_2 be any point of U. Let w_1 be any point of the unbounded component of $-g(X)$. Let γ be a Jordan arc which joins w_1 to w_2 and fulfills conditions (d) of the statement of Theorem 1. If $-g(X)$ has a finite number j of components then clearly γ can be chosen to intersect $g(X)$ in at most $j-1$ points. Thus $\gamma-g(X)$ consists of a finite number of components $\gamma_1,\cdots,\gamma_k$, which we order according to the direction along γ from w_1 to w_2. If $-g(X)$ has a finite number j of components then $k\leqq j$. Now each γ_i belongs to some component U_i of $-g(X)$. Clearly $w_1\in U_1$ and $w_2\in U_k$, so that U_1 is the unbounded component of $-g(X)$ and $U_k=U$. We thereby see by applying Lemma 21 $k-1$ times that $U=U_k$ is a g-regular component of $-g(X)$ of multiplicity at most $k-1$. By Lemma 19, each p in $Y-X$ for which $g(p)$ is the vertex of some nondegenerate triangle whose interior lies in U is a one-dimensional point of $Y-X$, and at most $k-1$ such points lie over a given point in $g(Y)$. Thus $Y-X$ is the union of Λ and the set T of one-dimensional points of $Y-X$ which are not in Λ. Clearly T has no cluster point in $Y-X$ and so is an isolated set. To each p in T choose a deleted neighborhood U of p in Λ such that U is a finitely-sheeted covering space by the map g of $g(U)=\{z: 0<|z-g(p)|<r\}$. Thus U is a finite Riemann surface over $g(U)$. Therefore U can be completed to a finite Riemann surface V over $g(U)\cup\{g(p)\}$. Let $p_1,\cdots,p_m$ be those points

in V which cover $g(p)$. Consider the set S consisting of Λ and the points $p_1, \cdots, p_m$ for all p in T. This set can be given as follows the structure of a Riemann surface. At each p in Λ, S has the structure of Λ, and at each of the points p_i, S has the structure of V. Clearly S is a Riemann surface which satisfies the conditions of Theorem 1.

Proof of Theorem 2. Since S is separable it has only a countable number of components. Since the functions in $\mathfrak{A}$ are constant on no component of S, by a standard construction there exists g in $\mathfrak{A}$ which is constant on no component of S. Let

$$K_1 \subset K_2 \subset \cdots \subset K_n \subset \cdots$$

be an increasing sequence of compact subsets of S whose union is S. By induction we choose open sets $U_i \subset S$ with the following properties:

(1) $\overline{U}_{i-1} \cup K_i \subset U_i$ and $\overline{U}_i$ is compact.

(2) The boundary γ_i of U_i is the union of a finite number of disjoint smooth closed Jordan curves.

(3) dg vanishes nowhere on γ_i.

(4) $g(p_1) = g(p_2)$ for at most a finite set of pairs (p_1, p_2) of distinct points of γ_i.

(5) $g(\gamma_i) \cap g(\gamma_{i-1})$ is a finite set.

(6) $g(\gamma_i) \cap g(\gamma_{i-1}) \cap g(\gamma_{i-2})$ is void.

Assume that $U_1, \cdots, U_{i-1}$ have been chosen. Since $\overline{U}_{i-1} \cup K_i$ is compact, there exist U_i and γ_i satisfying (1) and (2). Since g is nonconstant on each component of S, the curves γ_i can be moved slightly, if necessary, so that (3), (4), and (5) are satisfied. Since by the induction hypothesis $g(\gamma_{i-1}) \cap g(\gamma_{i-2})$ is finite, we may choose γ_i so that (6) is also satisfied.

Having chosen the sets U_i and γ_i for all i, we let $\mathfrak{A}_i$ be the closure of $\mathfrak{A}$ in $C(\overline{U}_i) = C(U_i \cup \gamma_i)$. Let Y_i be the spectrum and X_i the Šilov boundary of $\mathfrak{A}_i$. Let π_i be the natural map of $\overline{U}_i$ into Y_i. Since every function in $\mathfrak{A}_i$ assumes its maximum for the set $\overline{U}_i$ on γ_i, it is clear that

$$X_i \subset \pi_i(\gamma_i).$$

Because of this and the properties (3) and (4) above we see that the algebra $\mathfrak{A}_i$ and the function g in $\mathfrak{A}_i$ satisfy all conditions of Theorem 1. Let Λ_i be the analytic part of Y_i and let $T_i = Y_i - X_i - \Lambda_i$, so that T_i is a countable isolated set. Let S_i be the Riemann surface corresponding to the algebra $\mathfrak{A}_i$ constructed in the proof of Theorem 1. Let λ_i be the map of S_i onto $Y_i - X_i$.

Since $\overline{U}_i \subset \overline{U}_j$ for $i < j$, there is a natural homeomorphism ϕ_{ji} of Y_i into Y_j such that

$$f(\phi_{ji}(p)) = f(p)$$

for all p in Y_i and all f in $\mathfrak{A}$, where f on the left of this equation is considered as a function in $\mathfrak{A}_j$ and f on the right is considered as a function in $\mathfrak{A}_i$. Clearly

$$\phi_{kj} \circ \phi_{ji} = \phi_{ki}$$

for $i<j<k$.

If $s_i \in S_i$ and $s_j \in S_j$ write $s_i \equiv s_j$ in case there exists a neighborhood V_i of s_i in S_i, a neighborhood V_j of s_j in S_j, and a homeomorphism β of V_i onto V_j such that

$$(*) \qquad f \circ \lambda_i = f \circ \lambda_j \circ \beta \qquad \text{on } V_i$$

for all f in $\mathfrak{A}$. Since $f \circ \lambda_i$ and $f \circ \lambda_j$ are analytic on S_i and S_j respectively, it follows that β is necessarily an analytic homeomorphism of V_i onto V_j. It is clear that $s_i \equiv s_j$ implies that $\beta(s) \equiv s$ for all s in V_i. If $i=j$ and $s_i \neq s_j$, so that $S_i = S_j$, it follows that $s \equiv \beta(s)$ for some s in S_i with $\beta(s)$ in S_i, $s \neq \beta(s)$, $\lambda_i(s) \notin T_i$, $\lambda_i(\beta(s)) \notin T_i$. This contradicts the equation (*) because there exists f in $\mathfrak{A}$ assuming distinct values at the distinct points $\lambda_i(s)$ and $\lambda_i(\beta(s))$ of Y_i. Therefore no two distinct points in S_i are equivalent. It is clear that $\equiv$ is an equivalence relation on the set $\bigcup_i S_i$, where the S_i are taken to be disjoint. Let S' be the set of all equivalence classes of $\bigcup_i S_i$. We thus have a natural one-one map σ_i of S_i onto a subset S_i' of S', where S_i' consists of those equivalence classes which contain elements of S_i. It is clear that for each f in $\mathfrak{A}$ there exists a unique function f' on S' with

$$f'(\sigma_i(s_i)) = f(\lambda_i(s_i))$$

for all s_i in S_i, $1 \leqq i < \infty$. Define

$$\mathfrak{A}' = \{f' : f \in \mathfrak{A}\}$$

so that $\mathfrak{A}'$ is an algebra of functions on S'. Let τ be the mapping $f \rightarrow f'$ from $\mathfrak{A}$ onto $\mathfrak{A}'$.

We topologize S' by defining $W \subset S'$ to be open if $\sigma_i^-(W)$ is open in S_i for all i. Clearly this gives a topology on S' and the functions in $\mathfrak{A}'$ are all continuous in this topology. The maps σ_i are also clearly continuous. Consider an open set $W_i \subset S_i$. We shall show that $W = \sigma_i(W_i)$ is open in S'. To this end we must show that $\sigma_j^-(W) = W_j$ is open for all j. Now if $s_j \in W_j$ then $s_j \equiv s_i$, where $s_i = \sigma_i^-(\sigma_j(s_j))$ is in W_i. Thus there exists a neighborhood $V_i \subset W_i$ of s_i, a neighborhood V_j of s_j, and a homeomorphism β of V_i onto V_j such that $\beta(s) \equiv s$ for all s in V_i. Therefore $\sigma_j(\beta(s)) = \sigma_i(s)$ so that $\beta(s) \in W_j$. Thus $V_j \subset W_j$. It follows that W_j is open for each j so that W is open in S'. Thus σ_i is a homeomorphism of S_i onto the open subset S_i' of S'. It follows that $\{S_i'\}$ is a covering of S' by open sets, each of which is homeomorphic to a Riemann surface S_i by a given map σ_i. Thus to give S' the structure of a Riemann surface it is sufficient to show that the map $\sigma_j^- \circ \sigma_i$ of $\sigma_i^-(S_i' \cap S_j')$ onto $\sigma_j^-(S_i' \cap S_j')$ is analytic for all i and j. Let p_i be any point in $\sigma_i^-(S_i' \cap S_j')$, so that

$$p_j = \sigma_j^-(\sigma_i(p_i)) \in \sigma_j^-(S_i' \cap S_j')$$

and $p_i \equiv p_j$. There therefore exist neighborhoods V_i and V_j of p_i and p_j respectively and an analytic homeomorphism β of V_i onto V_j satisfying (*). As above, $\sigma_i(s) = \sigma_j(\beta(s))$ for all s in V_i. Thus on V_i, $\beta = \sigma_j^{-} \circ \sigma_i$, so that $\sigma_j^{-} \circ \sigma_i$ is analytic at the point p_i. Thus $\sigma_j^{-} \circ \sigma_i$ is analytic on $\sigma_i^{-}(S_i' \cap S_j')$. It follows that S' can uniquely be given the structure of a Riemann surface in such a way that the maps σ_i are all analytic. As a consequence the functions f' in $\mathfrak{A}'$ are all analytic on S'.

Now let ϕ be a continuous homomorphism of $\mathfrak{A}$ onto the complex numbers. Thus there exists a compact subset K of S with

$$|\phi(f)| \leqq \sup\{|f(p)| : p \in K\}$$

for all f in $\mathfrak{A}$. Since the U_i cover S there exists n with $K \subset U_n$. Since $g(\gamma_n) \cap g(\gamma_{n+1}) \cap g(\gamma_{n+2})$ is void we may choose m with $\phi(g) \notin g(\gamma_m)$, where $m = n$, $n+1$, or $n+2$. Thus $K \subset U_m$. Therefore

$$|\phi(f)| \leqq \sup\{|f(p)| : p \in \overline{U}_m\}$$

for all f in $\mathfrak{A}$. There therefore exists q_0 in Y_m with $\phi(f) = f(q_0)$ for all f in $\mathfrak{A}$. Since $g(q_0) = \phi(g) \notin g(X_m)$ it follows that $q_0 \in Y_m - X_m$. Let p_m be any point in S_m with $\lambda_m(p_m) = q_0$. Write $p = \sigma_m(p_m)$. It follows that $p \in S'$ and

$$f'(p) = f'(\sigma_m(p_m)) = f(\lambda_m(p_m)) = f(q_0) = \phi(f)$$

for all f in $\mathfrak{A}$. This proves (2) of Theorem 2.

Now let p be any point in S. As above there exists i with $p \in U_i$, $g(p) \notin g(X_i)$. Thus $\pi_i(p) \notin X_i$ so that $\pi_i(p) \in Y_i - X_i$. Let V be a neighborhood of p in U_i with $\pi_i(V) \subset Y_i - X_i$ and $g(q) \neq g(p)$ for all q in $V - \{p\}$. Thus $\pi_i(q) \neq \pi_i(p)$ for all such q. Since the points of $T_i = Y_i - X_i - \Lambda_i$ are isolated in $Y_i - X_i$, we may choose V so small that $\pi_i(V - \{p\}) \subset \Lambda_i$. Now $f_i \circ \pi_i = f$ for all f in $\mathfrak{A}$, where we have subscripted f on the left to show that it is considered as a function on Y_i. Since f_i is analytic on Λ_i and f is analytic on V and since f can be chosen to have a simple zero at any point in Λ_i, the map π_i of $V - \{p\}$ into Λ_i is analytic. Therefore the map $\lambda_i^{-} \circ \pi_i$ of $V - \{p\}$ into S_i is analytic. Since $\lambda_i^{-}(\pi_i(q))$ must converge as $q \to p$ to one of the points t in S_i for which $\lambda_i(t) = \pi_i(p)$, it follows that $\lambda_i^{-} \circ \pi_i$ has a unique extension to an analytic map α_0 from V into S_i. Write $\alpha = \sigma_i \circ \alpha_0$. It is clear that α is an analytic map from V into S' such that

$$(**) \qquad f' \circ \alpha = f \circ \lambda_i \circ \alpha_0 = f \circ \pi_i = f \qquad \text{on } V$$

for all f in $\mathfrak{A}$.

Thus each p in S has a neighborhood V which admits an analytic map α into S' satisfying (**). Assume that some open set V in S admits two analytic maps α_1 and α_2 into S' both satisfying (**). We show that $\alpha_1 = \alpha_2$. Assume otherwise. There therefore exists p in V with $\alpha_1(p) \neq \alpha_2(p)$ and $d\alpha_1(p) \neq 0$, $d\alpha_2(p) \neq 0$. Thus there exists a neighborhood V_0 of p which α_1 and α_2

respectively map homeomorphically onto disjoint open sets V_1 and V_2 in S'. Since $\{S_i^-\}$ is an open cover of S' we may assume $V_1 \subset S_i'$, $V_2 \subset S_j'$ for certain i and j. Thus $\beta_0 = \alpha_2 \circ \alpha_1^-$ maps V_1 homeomorphically onto V_2. Thus $\beta = \sigma_j^- \circ \beta_0 \circ \sigma_i$ maps an open set in S_i homeomorphically onto an open set in S_j. For each f in $\mathfrak{A}$ we have

$$f \circ \lambda_j \circ \beta = f' \circ \sigma_j \circ \beta = f' \circ \beta_0 \circ \sigma_i$$
$$= f \circ \alpha_1^- \circ \sigma_i = f' \circ \sigma_i = f \circ \lambda_i.$$

Therefore $\beta(s) \equiv s$ for all s in $\sigma_i^-(V_1)$. It follows that $\beta_0(s') = s'$ for all s' in V_1. This contradicts the fact that V_1 and V_2 are disjoint, proving that $\alpha_1 = \alpha_2$. Thus we may define a map σ of S into S' by defining $\sigma(p) = \alpha(p)$ for each p in S, where α is an analytic map of some neighborhood V of p into S' which satisfies (**). Since α is unique the map σ is well-defined. Clearly σ is an analytic map from S into S' such that $f' \circ \sigma = f$ for all f in $\mathfrak{A}$, or

$$(\tau(f))(\sigma(p)) = f(p)$$

for all p in S and f in $\mathfrak{A}$. This is just (*) of Definition 2. Thus to show that $(\mathfrak{A}', S')$ and the mappings σ, τ define an extension of $(\mathfrak{A}, S)$ it only remains to prove that $\mathfrak{A}'$ is holomorphically complete. Clearly $\mathfrak{A}'$ is an algebra. Since g is not constant on any component of Λ_i, $1 \leqq i < \infty$, $g \circ \lambda_i$ is not constant on any component of S'.

Thus it remains to show that $\mathfrak{A}'$ is closed in the topology of uniform convergence on compact subsets of S'. Consider therefore a sequence $\{f_i'\}$ of elements in $\mathfrak{A}'$ converging uniformly on compact subsets of S' to a function F on S'. The sequence $\{f_i\}$ then converges uniformly on compact subsets of S to $F \circ \sigma$. Thus $f = F \circ \sigma \in \mathfrak{A}$. It follows that $F - f'$ vanishes on $\sigma(S)$. Once condition (1) of Theorem 2 is verified it will follow from this that $F - f'$, which is a uniform limit on compact subsets of S' of elements in $\mathfrak{A}'$, vanishes on all of S'. Thus $F = f'$ will be in $\mathfrak{A}'$, as was to be proved.

It only remains to verify conditions (1), (3), and (4) of Theorem 2. To verify (1) consider a compact subset K of S'. Since $\{S_i'\}$ is an open cover of S' there exist $S_{i_1}', \cdots, S_{i_n}'$ which cover K. Let $k = 1 + \sup\{i_1, \cdots, i_n\}$. Since $\sigma(\gamma_k)$ is a compact subset of S', it is enough to show that $S_i' \subset \tilde{\sigma}(\gamma_k)$ for all $i < k$. (Here $\tilde{\sigma}(\gamma_k) = \tilde{G}$, where $G = \sigma(\gamma_k)$.) Since

$$\sigma(\gamma_i) \subset \tilde{\sigma}(\overline{U}_k) \subset \tilde{\sigma}(\gamma_k)$$

for $i < k$, it is sufficient to show that

$$S_i' \subset \tilde{\sigma}(\gamma_i)$$

for all i. Consider p_0 in S_i' and write $p = \lambda_i(\sigma_i^-(p_0))$ so that $p \in Y_i$ and $f(p) = f'(p_0)$ for all f in $\mathfrak{A}$. Thus

$$\begin{aligned} |f'(p_0)| &= |f(p)| \leqq \sup\{|f(q)| : q \in X_i\} \\ &\leqq \sup\{|f(q)| : q \in \gamma_i\} \\ &= \sup\{|f'(q)| : q \in \sigma(\gamma_i)\}, \end{aligned}$$

so that $p_0 \in \bar{\sigma}(\gamma_i)$. This proves (1) of Theorem 2.

We turn to the proof of (3). Assume that (3) is false so that T intersects some compact subset of $S' \times S'$ in an infinite set. Then there exists a sequence $\{(p_n', q_n')\}$ of distinct elements of T converging to an element (p', q') in $S' \times S'$. We may assume that the p_n are distinct. Choose S_i' and S_j' with $p' \in S_i'$, $q' \in S_j'$. We may assume that $p_n' \in S_i'$ and $q_n' \in S_j'$ for all n. Let $p_n = \sigma_i^-(p_n')$, $q_n = \sigma_j^-(q_n')$, $p = \sigma_i^-(p')$, $q = \sigma_j^-(q')$, so that $\{p_n\}$ converges to p in S_i and $\{q_n\}$ converges to q in S_j. Choose k with $k > i$, $k > j$, and $g'(p') \notin g(X_k)$. Now λ_i is a homeomorphism of some neighborhood V in S_i of p into Y_i. Thus $\phi_{ki} \circ \lambda_i$ gives a homeomorphism of V into Y_k. Since λ_j is a homeomorphism of some neighborhood W of q into Y_j, $\phi_{kj} \circ \lambda_j$ is a homeomorphism of W into Y_k. We may assume that $p_n \in V$ and $q_n \in W$ for all n. For each f in $\mathfrak{A}$ we have

$$f \circ \phi_{ki} \circ \lambda_i = f \circ \lambda_i = f' \circ \sigma_i \qquad \text{on } V.$$

Similarly,

$$f \circ \phi_{kj} \circ \lambda_j = f \circ \lambda_j = f' \circ \sigma_j \qquad \text{on } W.$$

In particular,

$$\begin{aligned} f(\phi_{ki}[\lambda_i(p_n)]) &= f'(\sigma_i(p_n)) = f'(p_n') \\ &= f'(q_n') = f'(\sigma_j(q_n)) = f(\phi_{kj}[\lambda_j(q_n)]) \end{aligned}$$

for all f in $\mathfrak{A}$. Thus $\phi_{ki}(\lambda_i(p_n))$ and $\phi_{kj}(\lambda_j(q_n))$ are the same point y_n in Y_k. Now $g(\phi_{ki}[\lambda_i(p)]) = g(\lambda_i(p)) = g'(p') \notin g(X_k)$ so that $\phi_{ki}(\lambda_i(p)) \in Y_k - X_k$ for all n sufficiently large. Since T_k is isolated in $Y_k - X_k$ it follows that $y_n \in Y_k - X_k - T_k = \Lambda_k$ for all n sufficiently large. Fix such a value of n. There exists a neighborhood V_n of p_n in S_i which $\phi_{ki} \circ \lambda_i$ maps homeomorphically into Λ_k. Since S_i and Λ_k are Riemann surfaces it follows from invariance of domain (see [4, p. 95]) that $\phi_{ki} \circ \lambda_i$ maps V_n homeomorphically onto an open set in Λ_k containing y_n. Similarly $\phi_{kj} \circ \lambda_j$ maps some neighborhood W_n of q_n homeomorphically onto an open set in Λ_k containing y_n. We may assume that $\phi_{ki}(\lambda_i(V_n)) = \phi_{kj}(\lambda_j(W_n))$. Thus

$$\beta = (\phi_{kj} \circ \lambda_j)^- \circ (\phi_{ki} \circ \lambda_i)$$

is a homeomorphism of V_n onto W_n. Now

$$f \circ \lambda_j \circ \beta = f \circ \bar{\phi}_{kj} \circ \phi_{ki} \circ \lambda_i = f \circ \phi_{ki} \circ \lambda_i = f \circ \lambda_i$$

on V_n for all f in $\mathfrak{A}$. It follows that

$$p_n \equiv \beta(p_n) = (\phi_{kj} \circ \lambda_j)^{-} y_n = q_n.$$

Thus $\sigma_i(p_n) = \sigma_j(q_n)$, or $p_n' = q_n'$. This contradicts the fact that $(p_n', q_n') \in T$ and thereby establishes the truth of (3) of Theorem 2.

It remains to prove (4) of Theorem 2. We first show that the closure of S_i' is a compact subset of S' for each i. Since the set

$$R_i' = \sigma_i(R_i) \quad \text{where} \quad R_i = \lambda_i^{-}(\Lambda_i)$$

is dense in S_i', it is enough to show that the closure of R_i' is compact. Let $\{p_n'\}$ be a sequence of points of R_i'. Let $p_n = \lambda_i(\sigma_i^{-}(p_n'))$, so that $p_n \in \Lambda_i$. We may assume, by passing to a subsequence if necessary, that $\{p_n\}$ converges to a point p in Y_i. Choose $m > i$ with $g(p) \notin g(\gamma_m)$. Then $\phi_{mi}(p) \notin X_m$. We have the following diagram

$$S_i \xrightarrow{\lambda_i} Y_i \xrightarrow{\phi_{mi}} Y_m \xleftarrow{\lambda_m} S_m,$$

and $p_n \to p$ in Y_i as $n \to \infty$. Since ϕ_{mi} is continuous, $\phi_{mi}(p_n) \to \phi_{mi}(p)$ as $n \to \infty$, so that $\phi_{mi}(p_n) \in \Lambda_m$ for all n sufficiently large, say for all n. Thus for each n there exists a unique point t_n in S_m with $\lambda_m(t_n) = \phi_{mi}(p_n)$. Let $q_1, \cdots, q_k$ be those points in S_m with $\lambda_m(q_j) = \phi_{mi}(p)$, $1 \leqq j \leqq k$. Thus to each open set V in S_m containing the points $q_1, \cdots, q_k$ corresponds a neighborhood V_0 of $\phi_{mi}(p)$ in Y_m with $\lambda_m^{-}(V_0) \subset V$. Thus $t_n \in V$ for all n sufficiently large. We may therefore assume, by passing to a subsequence if necessary, that t_n converges to one of the points $q_1, \cdots, q_k$, call it t. Let W be a neighborhood of t in S_m mapped homeomorphically by λ_m into Y_m. Take W so small that $\lambda_m(W - \{t\}) \subset \Lambda_m$, so that λ_m is a homeomorphism of $W - \{t\}$ onto an open set in Y_m. We may assume that $t_n \in W$ for all n so that $\phi_{mi}(p_n) = \lambda_m(t_n) \in \lambda_m(W)$. Now $p_n \in \Lambda_i$ and $\phi_{mi}(p_n) \in \Lambda_m$. Since ϕ_{mi} maps a neighborhood of p_n homeomorphically into Λ_m, we may assume by invariance of domain that ϕ_{mi} maps a neighborhood V_n in Λ_i of p_n homeomorphically onto a neighborhood of $\phi_{mi}(p_n)$ in Λ_m. We may assume that $\phi_{mi}(V_n) \subset \lambda_m(W - \{p\})$. Thus $\lambda_m^{-} \circ \phi_{mi}$ maps V_n homeomorphically onto a neighborhood W_n of t_n in S_n and λ_i^{-} maps V_n homeomorphically onto a neighborhood W^n of $\lambda_i^{-}(p_n)$ in S_i. Therefore the map

$$\beta = \lambda_m^{-} \circ \phi_{mi} \circ \lambda_i \qquad \text{on } W^n$$

maps W^n homeomorphically onto W_n. If f is any function in $\mathfrak{A}$ then

$$f \circ \lambda_i = f \circ \phi_{mi} \circ \lambda_i = f \circ \lambda_m \circ \beta \qquad \text{on } W^n.$$

Therefore

$$\lambda_i^{-}(p_n) \equiv \beta(\lambda_i^{-}(p_n)) = t_n,$$

so that

$$p_n' = \sigma_i(\lambda_i^{-}(p_n)) = \sigma_m(t_n).$$

Therefore $\{p_n'\}$ converges to the point $\sigma_m(t)$ in S'. Thus there is a convergent subsequence of $\{p_n'\}$. Therefore the closure of S_i' is compact.

Consider any compact subset K of S'. We have seen above that $K \subset \tilde{\sigma}(\gamma_i)$ for some i. Therefore $\tilde{K} \subset \tilde{\sigma}(\gamma_i)$. The set

$$H = \overline{S}_i' \cup \sigma(\gamma_i)$$

is a subset of $\tilde{\sigma}(\gamma_i)$ because it was shown above that $S_i' \subset \tilde{\sigma}(\gamma_i)$. Since $\overline{S}_i'$ is compact H is compact. The set

$$L = \tilde{K} \cap H$$

is also compact. Clearly $\tilde{K}$ contains all points p such that $(p, q) \in T$ for some q in L. We shall complete the proof by showing that conversely if $p \in \tilde{K} - L$ then $(p, q) \in T$ for some q in L. Consider p in $\tilde{K} - L$. Thus

$$p \in \tilde{\sigma}(\gamma_i) - H.$$

It follows that

$$|f'(p)| \leq \sup\{|f(q)| : q \in \gamma_i\}$$

for all f in $\mathfrak{A}$. There therefore exists q_0 in Y_i with $f'(p) = f(q_0)$ for all f in $\mathfrak{A}$. Thus either $q_0 \in X_i$ or $q_0 = \lambda_i(q_1)$ for some q_1 in S_i. In the first case let q_1 be a point in γ_i with $\pi_i(q_1) = q_0$, so that in the first case $q = \sigma(q_1) \in H$ and

$$f'(q) = f(q_1) = f(q_0) = f'(p)$$

for all f in $\mathfrak{A}$. Thus $(p, q) \in T$. In the second case let $q = \sigma_i(q_1)$. Thus $q \in S_i' \subset H$ and $f'(q) = f(\lambda_i(q_1)) = f(q_0) = f'(p)$ for all f in $\mathfrak{A}$. Thus $(p, q) \in T$. Thus in either case there exists q in H such that $(p, q) \in T$. Also $q \in \tilde{K}$ because $f'(q) = f'(p)$ for all f in $\mathfrak{A}$. Therefore $q \in \tilde{K} \cap H = L$. This completes the proof of Theorem 2.

We end with a result which completely describes the uniform closure of an algebra of analytic functions on a compact subset of a Riemann surface.

THEOREM 3. *Let K be a compact subset of a Riemann surface S. Let $\mathfrak{A}$ be a holomorphically complete algebra of analytic functions on S. Let $\mathfrak{B}$ be the uniform closure of $\mathfrak{A}$ on K. Let Y be the spectrum of $\mathfrak{B}$. Let $(S', \mathfrak{A}')$ be the extension of $(S, \mathfrak{A})$ described in Theorem 2, and σ and τ the maps there described. Let M be the union of $L = \sigma(K)$ and all of those components of $S' - L$ which are relatively compact subsets of S'. Then*

(a) *$\mathfrak{B}$ is isomorphic to the uniform closure of $\mathfrak{A}'$ on M, via the maps σ and τ.*

(b) *For each ϕ in Y there exists p in M with $\phi(f) = f(p)$ for all f in $\mathfrak{B}$, where $\mathfrak{B}$ is considered as a subalgebra of $C(M)$.*

(c) *The linear space $\mathfrak{B}$ is of finite codimension in the space $\mathfrak{B}_0$ of all continuous functions on M which are analytic at interior points of M.*

Proof. From Theorem 2 it is clear that for each f in $\mathfrak{A}$ the uniform norm of f on K and the uniform norm of $\tau(f)$ on L are equal. From this (a) follows readily.

Now by Theorem 2 there exists a compact set $D\subset\tilde{M}$ such that for each ϕ in Y there exists p in D with $\phi(f)=f(p)$ for all f in $\mathfrak{A}'$. By (iv) of Theorem 2 and the principal theorem of [1] it follows that $\mathfrak{B}'$ is of finite codimension d in $\mathfrak{B}_1$, where $\mathfrak{B}'$ is the uniform closure of $\mathfrak{A}'$ on D and where $\mathfrak{B}_1$ is the set of all continuous functions on D which are analytic at interior points of D. Assume that $D-M$ is not a finite set. Thus there exist distinct points $p_1, \cdots, p_{d+1}$ in $D-M$. Since $D\subset\tilde{M}$, for each i there exists a finite measure μ_i on M such that $\delta_i-\mu_i\perp\mathfrak{A}'$, where δ_i is the point mass at p_i. Now considered as linear functionals on $\mathfrak{B}_1$ these measures $\delta_i-\mu_i$ are all linearly independent because by Runge's theorem there exists f_i in $\mathfrak{B}_1$ which has the value 1 at p_i and 0 at the other p's and is arbitrarily small on M. But since these $d+1$ measures annihilate $\mathfrak{B}'$ we have a contradiction. Thus $D-M$ is finite. Thus if ϕ in Y does not have property (b) above then ϕ corresponds to a point p in $D-M$. Since p is isolated in D an easy argument shows that ϕ is isolated in Y. By a theorem of Šilov it follows that ϕ is in the Šilov boundary of $\mathfrak{B}$. This contradiction shows that (b) is valid for all ϕ in Y. Finally (c) follows from Theorem 2 and the principal theorem of [1].

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