Mini black holes and the relic gravitational waves spectrum

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In this paper we explore the impact of an era—right after reheating—dominated by mini black holes and radiation on the spectrum of relic gravitational waves. This era may lower the spectrum by several orders of magnitude.

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I. INTRODUCTION

Gravitational wave detection is a very topical subject and a great effort is underway in that direction. There are five cryogenic resonant-bar detectors of gravitational waves in operation, sensitive ground-based laser interferometers will be operational soon [Laser Interferometer Gravitational wave observation (LIGO), GEO600, and VIRGO] [1–3]; meanwhile TAMA300 is collecting data at their level of sensitivity and the space-based Laser Interferometer Space Antenna (LISA) is planned to be launched around the year 2011 [4]. Plans for advanced ground-based interferometers are also being developed.

The most serious hurdle in the detection arises from the very weak interaction of gravitational waves with matter and other fields. Only at energies of the order of the Planck scale ($\sim 10^{19}$ GeV) will gravitational interaction become as strong as the electromagnetic one. Paradoxically this fact makes their detection so attractive, as it would give us information about the epoch at which gravitational waves were decoupled from the cosmological dynamics, i.e., the Planck time.

Gravitational waves can be produced by local sources, such as coalescing stellar-mass black holes, compact binary stars, and supernovae explosions, or can have a cosmic origin, such as the decay of cosmic strings and the amplification of zero-point fluctuations due to the expansion of the Universe. Waves of the latter type are usually called relic gravitational waves (RGWs) and should form an isotropic stochastic background somehow similar to the cosmic microwave background radiation (CMBR), but without a thermal distribution [5]. As is well known, RGWs are a unavoidable consequence of general relativity and quantum field theory in curved space-time.

From the point of view of cosmologists, obtaining data of RGWs spectrum will be extremely interesting as it would make it possible to reconstruct the scale factor of the Universe. Different cosmic stages of expansion with different equations of estate for the matter content will produce a well determined power spectrum of RGWs. Even without data of the spectrum of RGWs, we can still get useful information by analyzing their influence in well-known processes such as primordial nucleosynthesis and the anisotropy they would induce in the CMBR. It is possible to test the validity of cosmological models (or to restrict them) by comparing the theoretical spectrum of RGWs produced with the maximum values allowed by the CMBR anisotropy data and primordial nucleosynthesis.

The RGWs production has been usually linked to inflation, though inflation is not the only process leading to a RGWs spectrum [5,6]. By contrast the recently proposed scenario of Khoury *et al.* [7] does not lead to a spectrum of RGWs. The spectrum in an expanding universe with a de Sitter inflationary era followed by a radiation dominated era and finally by a dust dominated era was calculated by several authors [8,10,11] and a general expression for the creation of RGWs in a multistage model was derived by Maia [11]. Other cosmological models can be used to obtain in each case a spectrum of RGWs, e.g., an initial inflationary stage different from the de Sitter one [12], a model with an intermediate additional era of stiff matter between a power law inflation and the radiation era [13].

The aim of this paper is to calculate the power spectrum of RGWs in a spatially flat Friedmann-Robertson-Walker universe that begins with de Sitter era, followed by an era dominated by a mixture of mini black holes (MBHs) and radiation, then a radiation dominated era (after the MBHs evaporated), and finally a dust dominated era. We will also see what constraints on the free parameters of the cosmological model can be drawn from the CMBR.

As it turns out, the era dominated by a mixture of MBHs and radiation leads to a significantly reduced power spectrum of RGWs at present time. This may have implications in case LISA does not detect a spectrum.

Except occasionally, we will use units for which $\hbar = c = k_B = 1$.

II. RGW SPECTRUM IN AN EXPANDING UNIVERSE

A. Basics of RGWs amplification

We consider a flat FRW universe with line element

$$ds^2 = -dt^2 + a(t)^2[dr^2 + r^2 d\Omega^2]$$
$$= a(\eta)^2[-d\eta^2 + dr^2 + r^2 d\Omega^2],$$

where t and η are, respectively, the cosmic and conformal time $[a(\eta)d\eta = dt]$.

By slightly perturbing the metric ($\bar{g}_{ij} = g_{ij} + h_{ij}, |h_{ij}| \ll |g_{ij}|, i,j = 0,1,2,3$) the perturbed Einstein equations follow.

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To first order, the transverse-traceless tensor solution which represents sourceless weak GWs can be expressed as [14,15]

$$h_{\alpha\beta}(\eta,\mathbf{x})=\int h^{(k)}_{\alpha\beta}(\eta,\mathbf{x})d^3k,$$

$$h^{(k)}_{\alpha\beta}(\eta,\mathbf{x})=\frac{\mu(\eta)}{a(\eta)}G_{\alpha\beta}(\mathbf{k},\mathbf{x}), \tag{1}$$

where greek indices run from 1 to 3, and $\mathbf{k}$ is the comoving wave vector. The functions $G_{\alpha\beta}(\mathbf{k},\mathbf{x})$ and $\mu(\eta)$ satisfy the equations

$$G^{\beta;\gamma}_{\alpha;\gamma}=-k^2G^{\beta}_{\alpha},\quad G^{\beta}_{\alpha;\beta}=G^{\alpha}_{\alpha}=0, \tag{2}$$

$$\mu''(\eta)+\left(k^2-\frac{a''(\eta)}{a(\eta)}\right)\mu(\eta)=0, \tag{3}$$

where the prime indicates derivative with respect to conformal time and $k=|\mathbf{k}|$ is the constant wave number related to the physical wavelength and frequency by $k=2\pi a/\lambda=2\pi af=a\ \omega$.

The functions G^{β}_{α} are combinations of $\exp(\pm i\mathbf{k}\cdot\mathbf{x})$ which contain the two possible polarizations of the wave, compatible with conditions (2).

Equation (3) can be interpreted as an oscillator parametrically excited by the term a''/a. When $k^2\gg a''/a$, i.e., for high-frequency waves, expression (3) becomes the equation of a harmonic oscillator whose solution is a free wave. The amplitude of $h^{(k)}_{\alpha\beta}(\eta,\mathbf{x})$ will decrease adiabatically as a^{-1} in an expanding universe. In the opposite regime, when $k^2\ll a''/a$, the solution to Eq. (3) is a lineal combination of $\mu_1\propto a(\eta)$ and $\mu_2\propto a(\eta)\int d\eta\ a^{-2}$. In an expanding universe μ_1 grows faster than μ_2 and will soon dominate. Accordingly, the amplitude of $h^{(k)}_{\alpha\beta}(\eta,\mathbf{x})$ will remain constant so long as the condition $k^2\ll a''/a$ is satisfied. When it is no longer satisfied, the wave will have an amplitude greater than it would have in the adiabatic behavior. This phenomenon is known as "superadiabatic" or "parametric amplification" of gravitational waves [8,15].

For power law expansion $a\propto\eta^l$ ($l=-1,1,2$ for inflationary, radiation dominated, and dust dominated universes, respectively) the general solution to Eq. (3) is

$$\mu(\eta)=(k\eta)^{1/2}[K_1J_{l-1/2}(k\eta)+K_2J_{-(l-1/2)}(k\eta)],$$

where $J_{l-1/2}(k\eta)$, $J_{-(l-1/2)}(k\eta)$ are Bessel functions of the first kind and $K_{1,2}$ are integration constants.

B. Spectrum

Here we succinctly recall the work of Grishchuk [8] to obtain the RGW spectrum in a universe that begins with a de Sitter stage, followed by a radiation dominated era and a dust era till the present time. Transitions between successive eras are assumed instantaneous. The scale factor is

$$a(\eta)=\begin{cases}-\dfrac{1}{H_1\eta} & (-\infty<\eta<\eta_1<0)\\[2mm] \dfrac{1}{H_1\eta_1^2}(\eta-2\eta_1) & (\eta_1<\eta<\eta_2)\\[2mm] \dfrac{1}{4H_1\eta_1^2}\dfrac{(\eta+\eta_2-4\eta_1)^2}{\eta_2-2\eta_1} & (\eta_2<\eta<\eta_0),\end{cases} \tag{4}$$

where the subindices 1,2 correspond to the sudden transitions from inflation to radiation era and from radiation to dust era, H_1 represents the Hubble factor at the end of the inflationary era and the subindex 0 indicates the present time.

The exact solution to Eq. (3) for each era specifies to

$$\mu_I(\eta)=C_I\left[\cos(k\eta+\phi_I)-\frac{1}{k\eta}\sin(k\eta+\phi_I)\right]\quad\text{(inflationary era)}, \tag{5}$$

$$\mu_R(\eta)=C_R\sin(k\eta_R+\phi_r)\quad\text{(radiation era)}, \tag{6}$$

$$\mu_D(\eta)=C_D\left[\cos(k\eta_D+\phi_D)-\frac{1}{k\eta_D}\sin(k\eta_D+\phi_D)\right]\quad\text{(dust era)}, \tag{7}$$

where $C_{I,R,D}$, $\phi_{I,R,D}$ are constants of integration, $\eta_R=\eta-2\eta_1$, and $\eta_D=\eta+\eta_2-4\eta_1$.

It is possible to express C_R, ϕ_R and C_D, ϕ_D in terms of C_I, ϕ_I and C_R, ϕ_R, respectively, as $\mu(\eta)$ must be continuous at the transition times $\eta=\eta_1$ and $\eta=\eta_2$. Averaging the solution over the initial phase ϕ_I the amplification factor is found to be

$$R(k)=\frac{C_M}{C_I}\sim\begin{cases}1 & (k\gg-1/\eta_1)\\ k^{-2} & [-1/\eta_1\gg k\gg 1/(\eta_{D2})]\\ k^{-3} & [1/(\eta_{D2})\gg k].\end{cases}$$

It is usual to consider that the initial vacuum spectrum of RGWs $h_i(k)$ is proportional to k [8]. The rationale behind that is the following. One assimilates the quantum zero-point fluctuations of vacuum with classical waves of certain amplitudes and arbitrary phases; consequently it is permissible to equalize $\hbar\omega/2$ with the energy density of the gravitational waves, $c^4h^2/(G\lambda^2)$ times λ^3. Thereby the initial vacuum spectrum of RGWs is given by $h(k)\propto k$ [8,9].

Accordingly, the spectrum of RGWs in the dust era will be

$$h_f(k)=R(k)h_i(k)\sim\begin{cases}k & (k\gg-1/\eta_1)\\ k^{-1} & [-1/\eta_1\gg k\gg 1/(\eta_{D2})]\\ k^{-2} & [1/(\eta_{D2})\gg k].\end{cases} \tag{8}$$

C. Bogoliubov coefficients

The classical amplification mechanism may be seen as a spontaneous particle creation by the gravitational field acting

on the initial quantum vacuum [10,11,16,17]. This is possible as the GW equation is not conformally invariant when $a''/a \neq 0$ even in an isotropic background [15].

The GW equation may be interpreted as the massless Klein-Gordon equation [10]. Its solution can be written as

$$h_{ij}(\eta,\mathbf{x}) = \int [A_{(k)} h_{ij}^{(k)}(\mathbf{k},\mathbf{x}) + A_{(k)}^{\dagger} h_{ij}^{(k)}(\mathbf{k},\mathbf{x})] d^3k, \quad (9)$$

$$h_{ij}^{(k)}(\mathbf{k},\mathbf{x}) = \frac{1}{\sqrt{\pi}} e_{ij}(\mathbf{k}) \frac{\mu_{(k)}(\eta)}{a(\eta)} e^{i\mathbf{k}\cdot\mathbf{x}},$$

where $A_{(k)}$, $A_{(k)}^{\dagger}$ are the annihilation and creation operators, respectively, $e_{ij}(\mathbf{k})$ contains the two possible polarizations of the wave and $\mu_{(k)}(\eta)$ is a solution to Eq. (3), as in the classical approach, but with the additional condition

$$\mu_{(k)} \mu_{(k)}^{*\prime} - \mu_{(k)}^{*} \mu_{(k)}' = i, \quad (10)$$

which comes from the commutation relations of the operators $A_{(k)}$, $A_{(k)}^{\dagger}$ and the definition of the field $h_{ij}(\eta,\mathbf{x})$ [17,16].

Solution (9) can be expressed in terms of another family of orthogonal modes $\bar{\mu}_{(k)}(\eta)$ as, in contrast with the Mikowskian space-time, in a curved space-time there is no privileged family. The two families of modes are related by a Bogoliubov transformation

$$\mu_{(k)}(\eta) = \alpha \bar{\mu}_{(k)}(\eta) + \beta \bar{\mu}_{(k)}^{*}(\eta),$$

where α, β are the Bogoliubov coefficients. The number of particles of the family $\mu_{(k)}$, N_k, contained in the vacuum state of the modes $\bar{\mu}_{(k)}$ is given by

$$\langle \bar{0} | N_k | \bar{0} \rangle = \langle \bar{0} | A_{(k)}^{\dagger} A_{(k)} | \bar{0} \rangle = |\beta|^2.$$

It is necessary to ascertain which of the solutions to Eq. (3) correspond to real particles. An approach to that problem is known as the adiabatic vacuum approximation [16]. Basically, it assumes that the curved space-time is asymptotically Mikowskian in the limit $k \to \infty$. In that limit, the creation-destruction operators of each family exactly correspond to those associated with real particles and consequently $\alpha = 1$ and $\beta = 0$ as the two families of modes represent the same vacuum state in the Mikowskian space-time. This argument is identical to that made in Sec. II A, solutions to Eq. (3) with $k \gg a''/a$ are free waves and they do not experience amplification.

Following Allen [10] and Maia [11], we will now evaluate the number of RGWs created from the initial vacuum state in an expanding universe. In the three-stage cosmological model we are considering with the scale factor given by Eq. (4), the initial state is the vacuum associated with the modes of the inflationary stage $\mu_I(\eta)$, which are a solution to Eq. (3) compatible with condition (10). Taking into account the shape of the scale factor at this era the modes are

$$\mu_I = (\sqrt{\pi}/2) e^{i\psi_I} k^{-1/2} x^{1/2} H_{-3/2}^{(2)}(x), \quad (11)$$

where $x = k\eta$ and ψ_I is an arbitrary constant phase and $H_{-3/2}^{(2)}(x)$ is the Hankel function of order $-3/2$. The proper modes of the radiation era are

$$\mu_R = (\sqrt{\pi}/2) e^{i\psi_R} k^{-1/2} x_R^{1/2} H_{1/2}^{(2)}(x_R), \quad (12)$$

where $x_R = k(\eta - 2\eta_1)$ and ψ_R is again a constant phase.

The two families of modes are related by

$$\mu_I(\eta) = \alpha_1 \mu_R(\eta) + \beta_1 \mu_R^{*}(\eta). \quad (13)$$

From the continuity of $\mu(\eta)$ at the transition time η_1 we obtain

$$\alpha_1 = -1 + \frac{i}{k\eta_1} + \frac{1}{2(k\eta_1)^2}, \quad \beta_1 = \frac{1}{2(k\eta_1)^2}, \quad (14)$$

where we have neglected an irrelevant phase. The modes with frequency at the transition time $f = 2\pi k/a(\eta_1)$ larger than the characteristic time scale of transition are exponentially suppressed. The characteristic time scale is usually taken to be the inverse of the Hubble factor at the transition, H_1^{-1} in this case. The coefficients will be $\alpha_1 = 1$ and $\beta_1 = 0$ for RGWs with $k > 2\pi a_1 H_1$ and Eq. (14) when $k < 2\pi a_1 H_1$.

In the dust era ($\eta > \eta_2$) the solution for the modes is

$$\mu_D = (\sqrt{\pi}/2) e^{i\psi_D} k^{-1/2} x_D^{1/2} H_{3/2}^{(2)}(x_D), \quad (15)$$

where $x_D = k(\eta + \eta_2 - 4\eta_1)$ and is related to the radiation ones by

$$\mu_R(\eta) = \alpha_2 \mu_D(\eta) + \beta_2 \mu_D^{*}(\eta). \quad (16)$$

Similarly one obtains

$$\alpha_2 = -i\left(1 + \frac{i}{2k(\eta_2 - 2\eta_1)} - \frac{1}{8[k(\eta_2 - 2\eta_1)]^2}\right),$$

$$\beta_2 = \frac{i}{8[k(\eta_2 - 2\eta_1)]^2},$$

for $k < 2\pi a(\eta_2) H_2$ and $\alpha_2 = 1$, $\beta_2 = 0$ for $k > 2\pi a(\eta_2) H_2$ where H_2 is the Hubble factor evaluated at the transition η_2.

In order to relate the modes of the inflationary era to the modes of the dust era, we make use of the total Bogoliubov coefficients α_{Tr2} and β_{Tr2} [10,11]. For $k > 2\pi a_1 H_1$, we find that $\alpha_{Tr2} = 1$, $\beta_{Tr2} = 0$; in the range $2\pi a_1 H_1 > k > 2\pi a_2 H_2$, the coefficients are $\alpha_{Tr2} = \alpha_1$ and $\beta_{Tr2} = \beta_1$, and finally for $k < 2\pi a(\eta_2) H_2$ we obtain that

$$\beta_{Tr2} = -\frac{2(\eta_2 - 2\eta_1) + \eta_1}{8k^3 \eta_1^2 (\eta_2 - 2\eta_1)^2}.$$

Thus the number of RGWs at the present time η_0 created from the initial vacuum state is $\langle N_\omega \rangle = |\beta_{Tr2}|^2 \sim \omega^{-6}(\eta_0)$ for $\omega(\eta_0) < 2\pi(a_2/a_0) H_2$, $\omega^{-4}(\eta_0)$ for $2\pi(a_1/a_0) H_1$

$>\omega(\eta_0)>2\pi(a_2/a_0)H_2$, and 0 for $\omega(\eta_0)>2\pi(a_1/a_0)H_1$, where we have used the present value of the frequency, $\omega(\eta_0)=k/a_0$.

Assuming that each RGW has an energy $2\hbar\omega(\eta)$, it is possible to express the energy density of RGWs with frequencies in the range $[\omega(\eta),\omega(\eta)+d\omega(\eta)]$ as

$$d\rho_g(\eta)=2\hbar\omega(\eta)\left[\frac{\omega^2(\eta)}{2\pi^2c^3}d\omega(\eta)\right]\langle N_\omega\rangle=P(\omega(\eta))d\omega(\eta), \tag{17}$$

where $P(\omega(\eta))=[\omega^3(\eta)/\pi^2]\langle N_\omega\rangle$ denotes the power spectrum. As the energy density is a locally defined quantity, ρ_g loses its meaning for metric perturbations with wavelength $\lambda=2\pi/\omega(\eta)$ larger than the Hubble radius $H^{-1}(\eta)$. The present power spectrum of RGWs predicted by this model is

$$P(\omega)\sim\begin{cases}0 & [\omega(\eta_0)>2\pi(a_1/a_0)H_1]\\ \omega^{-1}(\eta_0) & [2\pi(a_2/a_0)H_2<\omega(\eta_0)\\ & \quad<2\pi(a_1/a_0)H_1]\\ \omega^{-3}(\eta_0) & [2\pi H_0<\omega(\eta_0)<2\pi(a_2/a_0)H_2].\end{cases} \tag{18}$$

We can compare these results with the classical ones of the previous subsection. The energy density of created RGWs in the classical approach is defined from the spectrum as [9]

$$\rho_g\sim\int_{2\pi a_0H_0}^{-1/\eta_1}h_f^2(k)kdk. \tag{19}$$

From Eqs. (17) and (19) it follows that

$$P(k)\sim kh_f^2(k).$$

Thus both descriptions agree with regard to the dependence on h_f and the ranges of the wave number of the spectrum of RGWs, as for a power law scale factor the Hubble factor is $H(\eta)=l(\eta a(\eta))^{-1}$.

III. RGWs IN A FRW UNIVERSE WITH AN ERA OF MINI BLACK HOLES AND RADIATION

In this section we evaluate the spectrum of RGWs using the method of the Bogoliubov coefficients in a more general model than the conventional three-stage model of the previous section.

As is well known, mini black holes can be created by quantum tunneling from the hot radiation [18]; some cosmological consequences of this effect have been studied [19,20]. We shall assume that these MBHs are created right after the inflationary period (once the reheating is accomplished) and coexist with the radiation until they evaporate. During that era the total energy density can be approximated by $\rho=\rho_{BH}+\rho_R$ and the total pressure is

$$p=p_{BH}+p_R=(\gamma-1)\rho, \tag{20}$$

where $1\leqslant\gamma<4/3$. MBHs follows the dust equation of state $p_{BH}=0$ as they can be considered nonrelativistic matter. If the density of MBHs is large enough to dominate the expansion of the Universe, then $\gamma\simeq1$. In the opposite case, the Universe expansion is dominated by the radiation, $\gamma\simeq4/3$, and the Universe undergoes the three stages of the previous section. From the Einstein equations and Eq. (20) one finds $a(\eta)\propto\eta^l$, where $l=2/(3\gamma-2)(\Rightarrow1<\ l\leqslant2)$ during the "MBHs+rad" era. The MBHs eventually evaporate in relativistic particles after some time span whose duration is model dependent.

A. Power spectrum

Assuming that the expansion of the Universe is *ab initio* dominated by the vacuum energy of some scalar field (the inflaton), then dominated by the mixture of MBHs and radiation, later radiation dominated (after the evaporation of the MBHs), and finally dust dominated up to the present time, the scale factor of each era is

$$a(\eta)=\begin{cases}-\dfrac{1}{H_1\eta} & (-\infty<\eta<\eta_1<0)\ \ \text{(inflation)}\\[2ex] \dfrac{[\eta_{BH}]^l}{l^lH_1(-\eta_1)^{l+1}} & (\eta_1<\eta<\eta_2)\ \ \text{("MBHs+rad" era)}\\[2ex] \dfrac{(\eta_{R2})^{l-1}}{H_1(-\eta_1)^{l+1}}\eta_R & (\eta_2<\eta<\eta_3)\ \ \text{(radiation era)}\\[2ex] \dfrac{(\eta_{R2})^{l-1}}{2H_1(-\eta_1)^{l+1}\eta_{D3}}[\eta_D]^2 & (\eta_3<\eta<\eta_0)\ \ \text{(dust era)},\end{cases}$$

where $\eta_{BH}=\eta-(l+1)\eta_1$, $\eta_R=\eta+[(1-l)/l]\eta_2-[(l+1)/l]\eta_1$, $\eta_D=\eta+\eta_3+2[(1-l)/l]\eta_2-2[(l+1)/l]\eta_1$, $\eta_{R2}=[\eta_2-(l+1)\eta_1]/l$, and $\eta_{D3}=2\{\eta_3+[(1-l)/l]\eta_2-[(l+1)/l]\eta_1\}$. As in the previous section, the transitions between stages are assumed to be instantaneous.

The shape of $\mu(\eta)$ can be found by solving Eq. (3) in each era. For the de Sitter era $\mu(\eta)$ is given by Eq. (11) as above. For the "MBHs+rad" era the solution of (3) is

$$\mu_{BH}=(\sqrt{\pi}/2)e^{i\psi_{BH}}k^{-1/2}x_{BH}^{1/2}H_{l-1/2}^{(2)}(x_{BH}),$$

where $x_{BH}=k\ \eta_{BH}$, and it is related to the modes of inflation by the Bogoliubov transformation (13) with μ_R replaced by μ_{BH}. By evaluating the Bogoliubov coefficients at the transition we obtain

$$\alpha_1^l,\ \beta_1^l\simeq\frac{l^22^l}{(-l\eta_1)^{l+1}}k^{-(l+1)},$$

when $k<2\pi a_1H_1$ and $\alpha_1^l=1$, $\beta_1^l=0$ when $k>2\pi a_1H_1$.

The solution for the radiation era is again (12) with $x_R=k\ \eta_R$. The coefficients that relate Eq. (12) with μ_{BH} are

$$\alpha_2^l = -\frac{1}{2}\sqrt{\frac{\pi l x_{R2}}{2}}\left[\left(\frac{1}{x_{R2}} - i\right)H^{(2)}_{l-1/2}(l x_{R2}) - H^{(2)}_{l+1/2}(l x_{R2})\right]e^{i x_{R2}},$$

$$\beta_2^l = \frac{1}{2}\sqrt{\frac{\pi l x_{R2}}{2}}\left[\left(\frac{1}{x_{R2}} + i\right)H^{(2)}_{l-1/2}(l x_{R2}) - H^{(2)}_{l+1/2}(l x_{R2})\right]e^{-i x_{R2}},$$

when $k<2\pi a_2 H_2$ and $\alpha_2^l=1$, $\beta_2^l=0$ when $k>2\pi a_2 H_2$.

The modes of the dust era are given by Eq. (15) with $x_D = k\,\eta_D$ and are related to the modes of the radiation era by the coefficients

$$\alpha_3 = -i\left(1 + \frac{i}{x_{D3}} - \frac{1}{2x_{D3}^2}\right), \quad \beta_3 = i\frac{1}{2x_{D3}^2},$$

when $k<2\pi a_3 H_3$ and $\alpha_3=1$, $\beta_3=0$ when $k>2\pi a_3 H_3$.

In order to evaluate the spectrum of RGWs the total Bogoliubov coefficients are needed. The coefficients relating the initial vacuum state with the modes of the radiation state can be evaluated with the help of the relationships

$$\alpha_{Tr2}^l = \alpha_2^l \alpha_1^l + \beta_2^{l*}\beta_1^l, \quad \beta_{Tr2}^l = \beta_2^l \alpha_1^l + \alpha_2^{l*}\beta_1^l,$$

and one obtains

$$\alpha_{Tr2}^l,\ \beta_{Tr2}^l \simeq \begin{cases} 1,\ 0 \quad (k>2\pi a_1 H_1) \\ \alpha_1^l,\ \beta_1^l \quad (2\pi a_1 H_1 > k > 2\pi a_2 H_2) \\ \dfrac{l^{-l+2}(2l^2-3l+1)}{8(-l\eta_1)^{l+1}(\eta_{R2})^{l-1}}k^{-2l} \quad (k<2\pi a_2 H_2). \end{cases} \tag{21}$$

Finally, the total coefficients that relate the inflationary modes with the modes of the dust era evaluated from

$$\alpha_{Tr3}^l = \alpha_3 \alpha_{Tr2}^l + \beta_3^* \beta_{Tr2}^l, \quad \beta_{Tr3}^l = \beta_3 \alpha_{Tr2}^l + \alpha_3^* \beta_{Tr2}^l$$

are found to be

$$\alpha_{Tr3}^l,\ \beta_{Tr3}^l \simeq \begin{cases} \alpha_{Tr2}^l,\ \beta_{Tr2}^l \quad (k>2\pi a_3 H_3) \\ \dfrac{l^{-l+2}(2l^2-3l+1)}{8(-l\eta_1)^{l+1}(\eta_{R2})^{l-1}\eta_{D3}}k^{-(2l+1)} \\ \quad (k<2\pi a_3 H_3). \end{cases} \tag{22}$$

We are now in a position to calculate the current spectrum of RGWs. Taking into account that

$$-\eta_1 = (a_1 H_1)^{-1}, \quad \eta_{R2} = (a_2/a_1)^{1/l}(a_1 H_1)^{-1},$$

$$\eta_{D3} = 2(a_3/a_2)(a_2/a_1)^{1/l}(a_1 H_1)^{-1},$$

and $\omega = k/a_0$, the power spectrum $P(\omega)$ can be written as

$$P \simeq 0 \quad \left(\omega > 2\pi\frac{a_1}{a_0}H_1\right),$$

$$P \simeq \frac{l^{2-2l}2^{2l}}{\pi^2}\left(\frac{a_1}{a_0}\right)^{2l+2} H_1^{2l+2}\omega^{-(2l-1)} \quad \left(2\pi\frac{a_1}{a_0}H_1 > \omega > 2\pi\frac{a_2}{a_0}H_2\right),$$

$$P \simeq \frac{l^{2-4l}(2l^2-3l+1)^2}{64\pi^2}\left(\frac{a_1}{a_0}\right)^{4l}\left(\frac{a_1}{a_2}\right)^{2-2/l} H_1^{4l}\omega^{-(4l-3)} \quad \left(2\pi\frac{a_2}{a_0}H_2 > \omega > 2\pi\frac{a_3}{a_0}H_3\right),$$

$$P \simeq \frac{l^{2-4l}(2l^2-3l+1)^2}{16\pi^2}\left(\frac{a_1}{a_0}\right)^{4l+4}\left(\frac{a_0}{a_3}\right)^{2} H_1^{4l+2}\omega^{-(4l-1)} \quad \left(2\pi\frac{a_3}{a_0}H_3 > \omega > 2\pi H_0\right). \tag{23}$$

Comparing the power of ω in Eq. (18) and (23) for $\omega < 2\pi(a_1/a_0)H_1$, we conclude that the four-stage scenario leads to a higher number of RGWs created at low frequencies than the three-stage scenario. This fact can be explained intuitively with the classical amplification approach. In the three-stage scenario the RGWs are parametrically amplified as long as $k^2 < a''(\eta)/a(\eta)$. For $\eta = \eta_1$, $a''(\eta)/a(\eta)$ vanishes and there is no more amplification. On the other hand, in the four-stage scenario the RGWs with $\omega < 2\pi(a_1/a_0)H_1$ are amplified until η_1 by the same term $a''(\eta)/a(\eta) = 2/\eta^2$ than in the three-stage model and from η_1 to η_2 by the term $l(l-1)/\eta_{BH}^2$. Consequently they have a larger amplitude in the radiation era.

Figure 1 shows spectrum (23) for $l=2$ and $l=1.1$. As is apparent, the four-stage scenario gives rise to a much lower power spectrum than the three-stage scenario assuming that in each case the spectrum has the maximum value allowed by the CMBR bound. The higher the MBHs contribute to the energy density, the lower the final power spectrum.

In this four-stage cosmological scenario, the Hubble factor $H(\eta)$ decreases monotonically, while the energy density of the RGWs for $\eta > \eta_3$ can be approximated by $\rho_g(\eta) \sim H^{-4l+2}(\eta)$, thereby it increases with expansion [11]. Obviously this scenario will break down before $\rho_g(\eta)$ becomes comparable to the energy density of matter and/or radiation since from that moment on the linear approximation on which our approach is based ceases to be valid. The parameters of our model will be constrained such that this cannot happen before the current era.

B. Evaluation of the parameters

At this stage it is expedient to evaluate the parameters occurring in Eq. (23). The redshift a_0/a_3, relating the present value of the scale factor with the scale factor at the

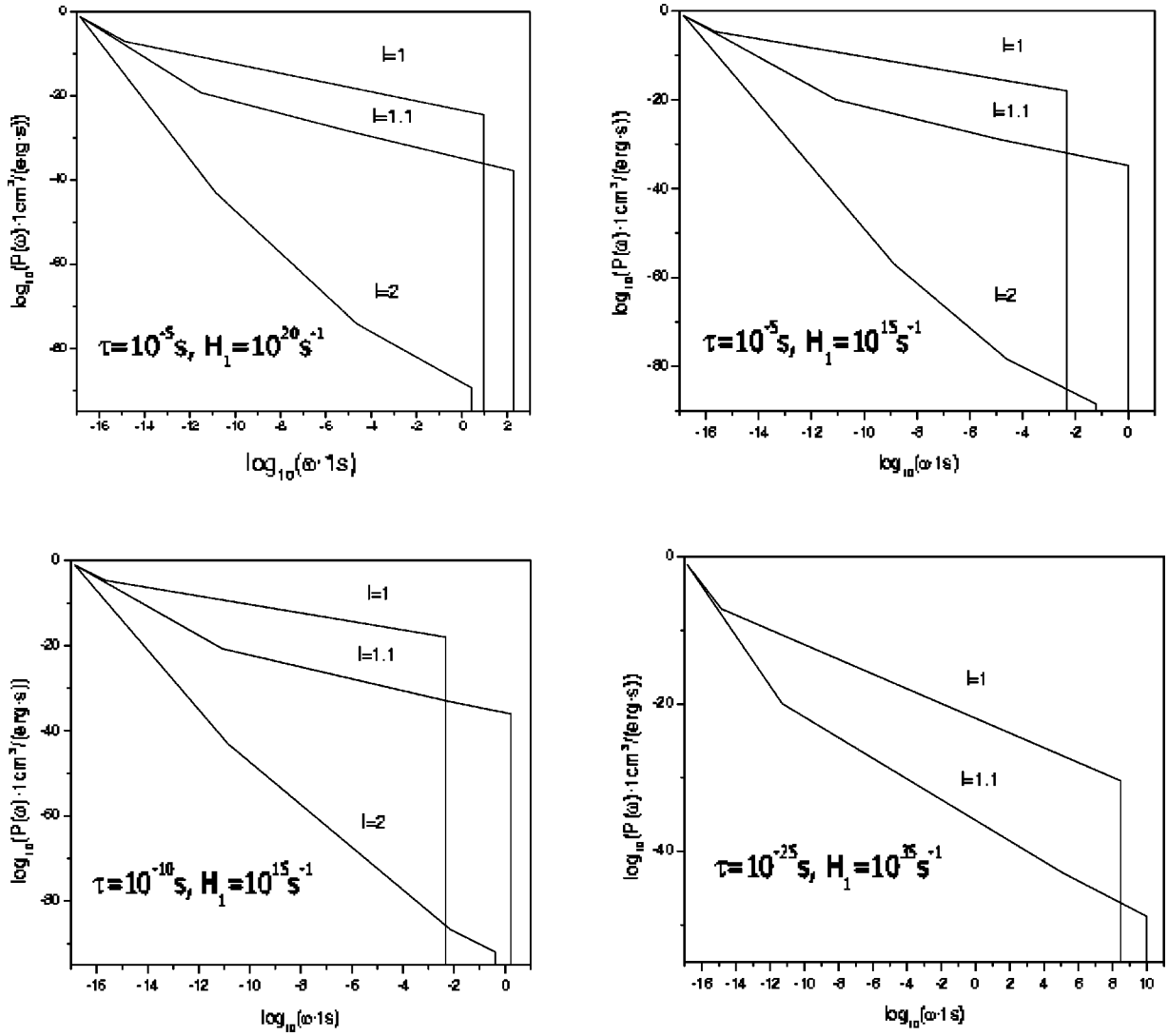

FIG. 1. Spectrum of RGWs in an expanding universe with a "MBHs+rad" era for certain values of l, τ, and H_1. The spectrum predicted for the three-stage model of the previous section is plotted for comparison, $l=1$. It is assumed that each spectrum has the maximum value allowed by the CMBR anisotropy data at the frequency $\omega=2\pi H_0=2.24\times10^{-18}\ \mathrm{s}^{-1}$. In the bottom-right panel the power spectrum with $l=2$ is excluded as it yields a CMBR anisotropy larger than that observed.

transition radiation-dust may be taken as 10^4 [21]. The Hubble factor H_1 is connected to the density at the inflationary era by

$$\rho_1=\frac{c}{\hbar}\frac{3m_{Pl}^2}{8\pi}H_1^2,$$

where we have restored momentarily the fundamental constants. In any reasonable model the energy density at that time must be larger than the nuclear density ($\sim10^{35}$ erg/cm^3) and lower than the Planck density ($\sim10^{115}$ erg/cm^3) [22], therefore

$$10^3\ \mathrm{s}^{-1}<H_1<10^{43}\ \mathrm{s}^{-1}. \tag{24}$$

Using the expression of the scale factor at the "MBHs+rad" era in terms of the proper time, one obtains

$$\frac{a_2}{a_1}=\left(1+H_1\frac{l+1}{l}\tau\right)^{l/(l+1)}, \tag{25}$$

where τ is the time span of the "MBHs+rad" era which depends on the evaporation history of the MBHs. The simplest model assumes that the MBHs are all created instantaneously at some given time with the same mass and terminate their evaporation simultaneously. If the MBHs evaporated freely, their mass would evolve according to $(dM/dt)_{free}=-g_*m_{Pl}^4/(3M^2)$ [23], where g_* is the number of particles for the black hole to evaporate into. However, it is natural to assume that the MBHs are surrounded by an atmosphere of particles in quasi thermal equilibrium with them. Therefore we have $|(dM/dt)_{atm}|\ll|(dM/dt)_{free}|$ for a comparatively long period whence the span of the "MBHs+rad" era is larger than if the MBHs evaporated freely [24]. Other possible model consists in considering that MBHs are

created according to some nucleation rate during the era, instead of being all nucleated instantaneously. These possibilities give us some freedom on evaluating τ.

However, the "MBHs+rad" era span should be longer than the duration of the transition at η_1 (as the transition is assumed instantaneous) in calculating the spectrum of RGWs. To evaluate the adiabatic vacuum cutoff for the frequency we have considered that the transition between whatever two successive stages has a duration of the same order as the inverse of the Hubble factor. This places the additional constraint

$$\tau > H_1^{-1}. \tag{26}$$

Finally, a_1/a_0 can be evaluated from the evolution of the Hubble factor until the present time

$$H_0 = \left(\frac{a_0}{a_3}\right)^{1/2}\left(\frac{a_2}{a_1}\right)^{(l-1)/l}\left(\frac{a_1}{a_0}\right)^2 H_1.$$

The current value of the Hubble factor is estimated to be $2.24\times 10^{-18}\ \mathrm{s}^{-1}$ [25] and

$$\left(\frac{a_1}{a_0}\right)^2 = \left(\frac{a_3}{a_0}\right)^{1/2}\left(1+H_1\frac{l+1}{l}\tau\right)^{(1-l)/(l+1)}\frac{H_0}{H_1}. \tag{27}$$

The only free parameters considered here are l, τ, and H_1, with the restrictions $1<l\leqslant 2$, Eq. (26), and Eq. (24). The two first free parameters depend on the assumption made on the MBHs, although it is possible to obtain rigorous constraints on H_1 and τ from the density of the RGWs.

C. Restrictions on the "MBHs+rad" era

It is obvious that ρ_g cannot be arbitrarily large, in fact the RGWs are seen as linear perturbations of the metric. The linear approximation holds only for $\rho_g(\eta)\ll\rho(\eta)$, $\rho(\eta)$ being the total energy density of the Universe. Several observational data place constraints on ρ_g. The regularity of the pulses of stable millisecond pulsars sets a constraint at frequencies of order 10^{-8} Hz [26]. Likewise, there is a certain maximum value for ρ_g compatible with the primordial nucleosynthesis scenario. But the most severe constraints come from the high isotropy degree of the CMBR. We will focus on the latter constraint. Metric perturbations with frequencies between 10^{-16} and 10^{-18} Hz at the last scattering surface can produce thermal fluctuations in the CMBR due to the Sachs-Wolfe effect [27]. These thermal fluctuations cannot exceed the observed value of $\delta T/T\sim 5\times 10^{-6}$.

A detailed analysis of the CMBR bound yields [5,28]

$$\Omega_g h_{100}^2 < 7\times 10^{-11}\left(\frac{H_0}{f}\right)^2 \quad (H_0<f<30\times H_0), \tag{28}$$

where $\Omega_g = fP(f)/\rho_0$, $\rho_0 = 3\ \mathrm{cm}_{Pl}^2 H_0^2/(8\pi\hbar)$ and $H_0 = h_{100}\times 100\ \mathrm{km/(s\times Mpc)}$, $h_{100}=0.7$. The CMBR bound for spectrum (23) evaluated at the frequency $\omega = 2\pi H_0$ reads [30]

$$1 > (2\pi)^{-4l} l^{2-4l}(2l^2-3l+1)^2\left(\frac{a_1}{a_0}\right)^{4l+4}\left(\frac{a_0}{a_3}\right)^2 \times\left(\frac{H_1}{3.72\times 10^{19}\ \mathrm{s}^{-1}}\right)^3\left(\frac{H_1}{H_0}\right)^{4l-1},$$

and consequently

$$f(l,H_1,\tau) = -107.69 + l\left[28.10 + 2\log_{10}\left(\frac{H_1}{1s^{-1}}\right)\right] - (2l-2)\log_{10}\left(1+\frac{l+1}{l}H_1\tau\right) + (-4l+2) \times\log_{10}(l) + 2\log_{10}(2l^2-3l+1) < 0. \tag{29}$$

We next consider different values for l and τ.

(i) When $l=1.1$, relation (29) reads

$$f(1.1,H_1,\tau) = -76.71 + 2.20\log_{10}\left(\frac{H_1}{1s^{-1}}\right) - 0.2\log_{10}\left(1+1.91H_1\tau\right) < 0, \tag{30}$$

see Fig. 2. From it we observe the following.

(1) For $\tau<\tau_c^{l=1.1} = 1.22\times 10^{-35}$ s, condition (30) is satisfied if $H_1 < 8.13\times 10^{34}\ \mathrm{s}^{-1}$ and conflicts with Eq. (26), which in the most favorable case is $H_1 = 8.13\times 10^{34}\ \mathrm{s}^{-1}$ for $\tau=\tau_c^{l=1.1}$. For $l=1.1$, there is no compatibility with the observed CMBR anisotropy when $\tau<\tau_c^{l=1.1}$. Thus, this range of τ is ruled out.

(2) For $\tau>\tau_c^{l=1.1}$, one obtains $H_1 < H_c^{l=1.1}(\tau)$ from condition (30). $H_c^{l=1.1}(\tau)$ is always larger than τ^{-1} in the range considered, e.g., $H_c^{l=1.1}(\tau=10^{-30}\ \mathrm{s}) = 2.40\times 10^{35}\ \mathrm{s}^{-1}$. Taking into account Eq. (26) one obtains that condition (30) is satisfied for $\tau^{-1} < H_1 < H_c^{l=1.1}(\tau)$.

(ii) When $l=2$ (expansion dominated by the MBHs) we have

$$f(2,H_1,\tau) = -51.89 + 4\log_{10}\left(\frac{H_1}{1s^{-1}}\right) - 2\log_{10}\left(1+1.5H_1\tau\right) < 0, \tag{31}$$

see Fig. 2. Inspection of Eq. (31) and Fig. 2 reveals the following.

(1) For $\tau<\tau_c^{l=2} = 6.75\times 10^{-14}$ s, one obtains $H_1 < 10^{13}\ \mathrm{s}^{-1}$ which is totally incompatible with condition (26), $H_1 > 1.48\times 10^{13}\ \mathrm{s}^{-1}$ for $\tau=\tau_c^{l=2}$ in the most favorable case. Thus, the region $\tau<\tau_c^{l=2}$ is ruled out as predicts an excess of anisotropy in the CMBR.

(2) For $\tau>\tau_c^{l=2}$, one obtains $H_1 < H_c^{l=2}(\tau)$ from condition (31). $H_c^{l=2}(\tau)$ is always larger than $1.48\times 10^{13}\ \mathrm{s}^{-1}$ for τ in the range considered, e.g., $H_c^{l=2}(\tau=10^{-10}\ \mathrm{s}) = 1.35\times 10^{16}\ \mathrm{s}^{-1}$. Conditions (26) and (31) are both satisfied in this range for $\tau^{-1} < H_1 < H_c^{l=2}(\tau)$.

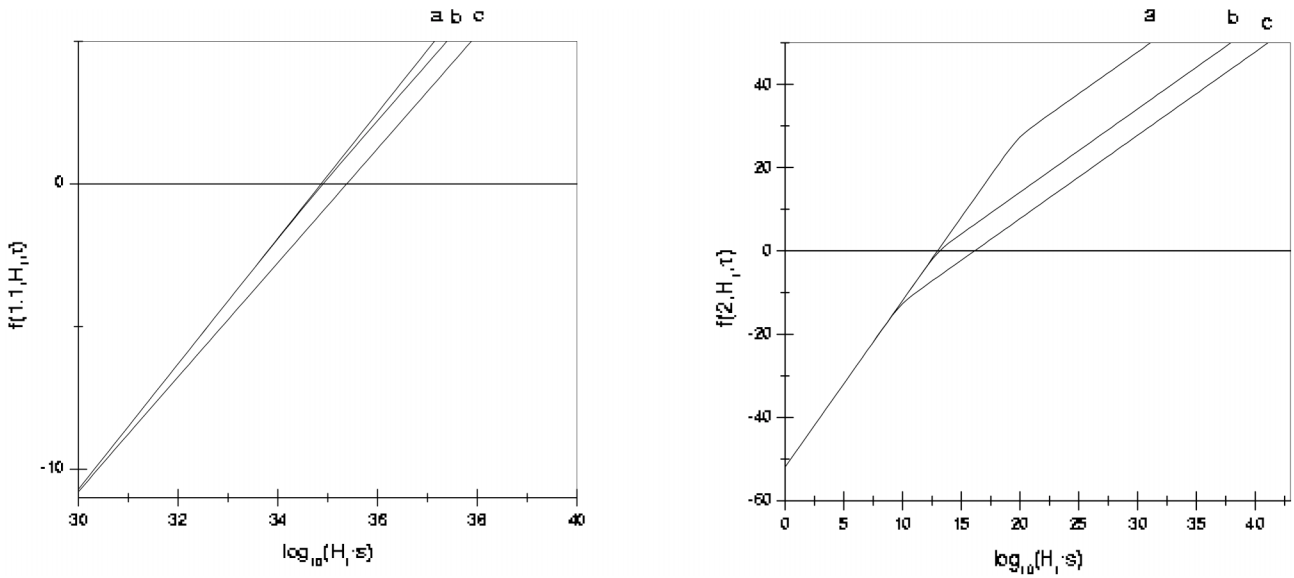

FIG. 2. The left panel depicts $f(1.1,H_1,\tau)$ vs $\log_{10}H_1$ for (a) $\tau=10^{-40}$ s, (b) $\tau=\tau_c^{l=1.1}=1.23\times10^{-35}$ s, and (c) $\tau=10^{-30}$ s. The right panel depicts $f(2,H_1,\tau)$ for (a) $\tau=10^{-20}$ s, (b) $\tau=\tau_{c1}^{l=2}=6.75\times10^{-14}$ s, and (c) $\tau=10^{-10}$ s. Conditions (29) and (26) are satisfied for certain ranges of τ and H_1 in each case.

We may conclude that condition (29) leads to different allowed ranges for H_1 and τ for each l considered, although their interpretation is rather similar. For $\tau<\tau_c^l$ the condition of minimum duration of the "MBHs+rad" era (26) and the upper bound given by the CMBR anisotropy are incompatible. However, for $\tau>\tau_c^l$ these two conditions are compatible for $\tau^{-1}<H_1<H_c^l(\tau)$.

IV. CONCLUDING REMARKS

We have calculated the power spectrum of RGWs in a universe that begins with an inflationary phase, followed by a phase dominated by a mixture of MBHs and radiation, then a radiation dominated phase (after the MBHs evaporated), and finally a dust dominated phase. The spectrum depends just on three free parameters, namely, H_1 the Hubble factor at the transition inflation-"MBHs+rad" era, τ the cosmological time span of the "MBHs+rad" era, and the power l, $a(\eta)\propto\eta^l$ being the scale factor of the "MBHs+rad" era with $1<l\leqslant2$.

The upper bound on the spectrum of RGWs obtained from the CMBR anisotropy places severe constraints on H_1 and τ. For each value of l considered, there is a minimum value of τ, τ_c^l, compatible with the CMBR anisotropy. There is a range of τ, $\tau>\tau_c^l$, for which $\tau^{-1}<H_1<H_c^l(\tau)$ satisfies the CMBR upper bound.

The four-stage scenario predicts a much lower power spectrum of RGWs than the conventional three-stage scenario. We may therefore conclude that if LISA fails to detect a spectrum at the level expected by the three-stage model, rather than signaling the recycle model of Khoury *et al.* [7] should supersede the standard big-bang inflationary model, it may indicate that a MBHs+radiation era between inflation and radiation dominance truly took place. Likewise, once the spectrum is successfully measured we will be able to learn from it the proportion of MBHs and radiation in the mixture phase.

In reality the current epoch is not dust dominated as the recent supernovae type Ia data seem to suggest that the Universe is dominated by a dark energy component and cold dark matter implying an epoch of accelerated expansion [29]. This implies a new transition in the scale factor shape at some time η_4 with $\eta_3<\eta_4<\eta_0$. The spectrum of RGWs with $k_0<k<k_4$ is changed. Thus, the constraints on H_1 and τ obtained from the CMBR bound stay unchanged as the RGWs created at the transition dust-accelerated era were not present at the last scattering.

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