Physica B 405 (2010) 2915–2919

Contents lists available at ScienceDirect

Physica B

journal homepage: www.elsevier.com/locate/physb

Relative local density of states for homogeneous lossy materials

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ARTICLE INFO

Keywords:
LDOS
Lossy materials
Homogeneous media

ABSTRACT

We derive the appropriate form of the relative local density of states (RLDOS) for the two-dimensional Helmholtz equation in a homogeneous lossy medium, and give the corresponding result for the three-dimensional equation. The RLDOS enables the calculation of the enhancement or suppression of the energy radiated in the form of transverse electromagnetic waves by a localized source in the medium, and so would be useful in studies of either electromagnetic communication from inside a cloaking system or detection of the cloaking system by radiation from thermally excited atoms within it.

1. Introduction

Much current research is devoted to the study of structured systems such as photonic crystals (PC), plasmonic materials and metamaterials. In the case of PCs, one of the first two papers launching the field [1] argued that structured dielectric systems could exhibit exciting new physics by creating band gaps for photons in which spontaneous emission from atoms within the system could be suppressed. Thus, there has been ongoing interest in this field in the key quantity determining the dynamics of radiative sources in photonic crystals: the spatially resolved, or local, density of states (LDOS) denoted by $\rho(\mathbf{r})$ [2]. In three-dimensional problems, it quantifies the coupling of an atom, with transition frequency ω at position $\mathbf{r}$, to the modes of the photonic crystal and thus reveals how a photonic crystal affects the emission rate of an atom. For infinite structures, the LDOS vanishes inside a complete band gap, and thus an excited two-level atom with a corresponding transition frequency cannot decay, with a bound photon-atom state being formed instead [3].

In the case of lossy materials, there has been little or no attention to date paid to the radiating properties of sources within them. Accordingly, this paper is concerned with the development of methods by which to calculate the radiation dynamics of sources in lossy materials and, in particular, the characterization of the enhancement or suppression of the local density of states.

It is well known that for lossless materials, the LDOS is proportional to the imaginary part of the trace of Green's function, with the constant of proportionality determined by the frequency and the phase velocity of radiation in the medium. The extension of this Green's function representation for the probability density of electromagnetic states to dissipative materials is quite problematic due to the complex permittivity.

Nevertheless, it is possible to define a quantity which has a number of the required characteristics of the local density of states, and which provides a mechanism by which to measure the relative enhancement or suppression of the radiation from embedded sources.

We thus introduce a quantity $\tilde{\rho}$ which we term the *relative local density of states* (RLDOS)—a quantity which is proportional to the power emitted by an embedded source [4]—and which is defined by

$$\tilde{\rho}(\mathbf{r}) = \lim_{\mathbf{r}\to\mathbf{r}_0} \mathrm{Im}(\mathrm{Tr}\,\mathbf{G}(\mathbf{r},\mathbf{r}_0)), \tag{1}$$

where Tr denotes the trace operation and $\mathbf{G}(\mathbf{r},\mathbf{r}_0)$ denotes the electric Green's tensor (or Green's dyadic function) at a field point $\mathbf{r}$ corresponding to a current source located at $\mathbf{r}_0$. In the case of non-magnetic lossless materials, the RLDOS $\tilde{\rho}$ and the LDOS ρ are related by [5]

$$\rho(\mathbf{r}) = \frac{2\omega\nu^2}{\pi c^2}\tilde{\rho}(\mathbf{r}), \tag{2}$$

where ν denotes the refractive index of the medium in which the source is embedded.

The RLDOS, while not providing an absolute value for the local density of states, is nevertheless able to characterize the suppression or enhancement of emission from sources and thus is a useful quantity for this purpose. For example, if one was to try to communicate from within a cloaking structure to the external medium, the RLDOS allows us to estimate the power requirements of the source. Correspondingly, the detection of an electromagnetic cloak might require the detection of altered radiation properties of atoms within the cloak, or adjacent to it, with this method being particularly effective for resonant cloaking [6]—since in this case the resonant interaction would

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doi:10.1016/j.physb.2010.01.003

generate local fields of very high intensity which would, in the presence of ohmic dissipation, generate strong thermal emission, detectable in the infrared.

In what follows, we consider the calculation of the RLDOS in two-dimensional and three-dimensional uniform, isotropic media. While this might be thought to be an easy task, in fact it poses considerable technical difficulties. These problems are resolved using Fourier techniques and splitting Green's functions into its longitudinal and transverse parts (see Section 2), with only the latter coupling to radiated transverse electromagnetic waves. The result for two-dimensional case, derived in Section 3, is appealingly simple, and is shown to be polarization independent, even though the constructions of Green's functions on which the derivations rest are dependent on polarization.

2. Longitudinal and transverse parts of dyadic Green's function for two-dimensional problems

For an infinite, isotropic, and homogeneous medium with relative permittivity $\varepsilon_r(\omega)$ and permeability $\mu_r(\omega)$, we have to use the Maxwell equations in the form

$$\nabla\times\mathbf{H}(\mathbf{r},t)=\mathbf{j}^{\rm ext}(\mathbf{r},t)+\frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t), \tag{3}$$

$$\nabla\times\mathbf{E}(\mathbf{r},t)=-\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t), \tag{4}$$

$$\nabla\cdot\mathbf{B}(\mathbf{r},t)=0, \tag{5}$$

$$\nabla\cdot\mathbf{D}(\mathbf{r},t)=\rho^{\rm ext}(\mathbf{r},t), \tag{6}$$

where, for isotropic media,

$$\mathbf{D}=\varepsilon_0\varepsilon_r\mathbf{E},\quad \mathbf{B}=\mu_0\mu_r\mathbf{H}. \tag{7}$$

Note that here, and in what follows, we will omit the argument ω for the functions $\varepsilon_r(\omega)$, $\mu_r(\omega)$, and $\nu(\omega)$, to simplify the notations. However, our main interest in this paper is to study non-magnetic materials so that $\mu_r(\omega)=1$.

By means of the spatial Fourier transform [7]

$$\tilde{\mathbf{V}}(\mathbf{k})=\int d\mathbf{r}\,e^{-i\mathbf{k}\cdot\mathbf{r}}\mathbf{V}(\mathbf{r}),\quad \mathbf{V}(\mathbf{r})=\frac{1}{(2\pi)^3}\int d\mathbf{k}\,e^{i\mathbf{k}\cdot\mathbf{r}}\tilde{\mathbf{V}}(\mathbf{k}), \tag{8}$$

for three-dimensional vectors, we can write the Maxwell equations (3)–(6) in the form

$$i\mathbf{k}\times\tilde{\mathbf{H}}(\omega,\mathbf{k})=\tilde{\mathbf{j}}^{\rm ext}(\omega,\mathbf{k})-i\omega\varepsilon_0\varepsilon_r\tilde{\mathbf{E}}, \tag{9}$$

$$\mathbf{k}\times\tilde{\mathbf{E}}(\omega,\mathbf{k})=\omega\tilde{\mathbf{B}}(\omega,\mathbf{k})=\omega\mu_0\mu_r\tilde{\mathbf{H}}(\omega,\mathbf{k}), \tag{10}$$

$$\mathbf{k}\cdot\tilde{\mathbf{B}}(\omega,\mathbf{k})=0, \tag{11}$$

$$\mathbf{k}\cdot\tilde{\mathbf{D}}(\omega,\mathbf{k})=\tilde{\rho}^{\rm ext}(\omega,\mathbf{k}). \tag{12}$$

Note that, when the relative permittivity (ε_r) and permeability (μ_r) are piecewise constant, they are not affected by the spatial Fourier transforms. Thus, the electric field will satisfy the equation

$$\frac{\omega^2}{c^2}\nu^2\tilde{\mathbf{E}}(\omega,\mathbf{k})+\mathbf{k}\times[\mathbf{k}\times\tilde{\mathbf{E}}(\omega,\mathbf{k})]=-i\omega\mu_0\mu_r\tilde{\mathbf{j}}^{\rm ext}(\omega,\mathbf{k}), \tag{13}$$

where $c^2=1/(\varepsilon_0\mu_0)$ and $\nu=\sqrt{\varepsilon_r\mu_r}$ is the refractive index of the medium. From Eq. (5) we also have $\mathbf{B}(\mathbf{r},t)=\nabla\times\mathbf{A}(\mathbf{r},t)$ and consequently $\tilde{\mathbf{E}}(\omega,\mathbf{k})=i\omega\tilde{\mathbf{A}}(\omega,\mathbf{k})$. Hence, the vector potential is the solution of the equation

$$\frac{\omega^2}{c^2}\nu^2\tilde{\mathbf{A}}(\omega,\mathbf{k})+\mathbf{k}\times[\mathbf{k}\times\tilde{\mathbf{A}}(\omega,\mathbf{k})]=-\mu_0\mu_r\tilde{\mathbf{j}}^{\rm ext}(\omega,\mathbf{k}) \tag{14}$$

or, in matrix form

$$\left[\left(\frac{\omega^2}{c^2}\nu^2-|\mathbf{k}|^2\right)\delta_{ij}+k_ik_j\right]\tilde{A}_j(\omega,\mathbf{k})=-\mu_0\mu_r\tilde{j}_i^{\rm ext}(\omega,\mathbf{k}). \tag{15}$$

Note that here, and in what follows, we use Einstein summation convention that repeated indices are implicitly summed over. By definition, Green's function for Eq. (15) is the solution of the equation

$$\left[\left(\frac{\omega^2}{c^2}\nu^2-|\mathbf{k}|^2\right)\delta_{ij}+k_ik_j\right]\tilde{G}_{jl}(\omega,\mathbf{k})=-\delta_{il} \tag{16}$$

or, in matrix form,

$$\mathbf{M}\tilde{\mathbf{G}}(\omega,\mathbf{k})=-\mathbf{I}, \tag{17}$$

where $\tilde{\mathbf{G}}(\omega,\mathbf{k})=[\tilde{G}_{jl}(\omega,\mathbf{k})]$ and $\mathbf{I}$ is the unit dyad. The matrix $\mathbf{M}$ in Eq. (17) can easily be inverted and we may write Green's function for an infinite, isotropic, and homogeneous medium in the form

$$\tilde{G}_{jl}(\omega,\mathbf{k})=-\frac{k_jk_l-\omega^2\varepsilon\mu\delta_{jl}}{\varepsilon\mu\omega^2[|\mathbf{k}|^2-\omega^2\varepsilon\mu]}. \tag{18}$$

In the case of two-dimensional problems all the physical quantities are independent of x_3, and we have $\mathbf{k}=(k_1,k_2,0)$ and $\mathbf{r}=(x_1,x_2,0)$. For this kind of problems the vector potential is the solution of a particular form of Eq. (14) obtained by setting $k_3=0$. Hence, the matrix of Eq. (16) takes the block diagonal form

$$\mathbf{M}=\begin{bmatrix}\left(\frac{\omega^2}{c^2}\nu^2-k^2\right)+k_1^2 & k_1k_2 & 0\\ k_2k_1 & \left(\frac{\omega^2}{c^2}\nu^2-k^2\right)+k_2^2 & 0\\ 0 & 0 & \frac{\omega^2}{c^2}\nu^2-k^2\end{bmatrix}, \tag{19}$$

where $k=(k_1^2+k_2^2)^{1/2}$. The separation of matrix $\mathbf{M}$ into two blocks corresponds to the two polarizations: $H_\parallel$ (associated with the two-by-two block) and $E_\parallel$ (associated with the one-by-one block). Also, the vector potential for the $H_\parallel$ polarization is $\mathbf{A}^{\rm H}=(A_1,A_2,0)$ while the vector potential for the $E_\parallel$ polarization is $\mathbf{A}^{\rm E}=(0,0,A_3)$.

Now, Green's function is obtained by inverting the matrix $\mathbf{M}$. The longitudinal part of Green's function becomes (see Eq. (A.7))

$$\tilde{\mathbf{G}}^{\rm L}(\omega,\mathbf{k})=\tilde{\mathbf{G}}^{\rm HL}(\omega,\mathbf{k})\oplus\tilde{\mathbf{G}}^{\rm EL}(\omega,\mathbf{k}), \tag{20}$$

where the symbol $\oplus$ denotes the direct sum of matrices, and

$$\tilde{\mathbf{G}}^{\rm HL}(\omega,\mathbf{k})=\left[-\frac{k_ik_j}{k^2\varepsilon\mu\omega^2}\right]_{i,j=1,2},\quad \tilde{\mathbf{G}}^{\rm EL}(\omega,\mathbf{k})=[0].$$

Note that the longitudinal part of the spectral form of Green's function, for $E_\parallel$ polarization is equal to zero. Therefore, the longitudinal part of the spatial form of Green's function, for $E_\parallel$ polarization, is also equal to zero. Also, the transverse part of Green's function is (see Eq. (A.8))

$$\tilde{\mathbf{G}}^{\rm T}(\omega,\mathbf{k})=\tilde{\mathbf{G}}^{\rm HT}(\omega,\mathbf{k})\oplus\tilde{\mathbf{G}}^{\rm ET}(\omega,\mathbf{k}), \tag{21}$$

where

$$\tilde{\mathbf{G}}^{\rm HT}(\omega,\mathbf{k})=\left[\frac{\delta_{ij}-k_ik_j/k^2}{\varepsilon\mu[k^2/(\varepsilon\mu)-\omega^2]}\right]_{i,j=1,2}$$

and

$$\tilde{\mathbf{G}}^{\rm ET}(\omega,\mathbf{k})=\left[\frac{1}{\varepsilon\mu[k^2/(\varepsilon\mu)-\omega^2]}\right].$$

To find the spatial form of the transverse and longitudinal parts of Green's tensor, we will use here a method based on the raising and lowering operators for the Bessel functions [8] which, expressed in terms of Cartesian and polar coordinates, have the forms

$$\mathrm{R}^{\pm} = \mp\frac{1}{q}\left(\frac{\partial}{\partial x_1}\pm \mathrm{i}\frac{\partial}{\partial x_2}\right) = \mp\frac{\mathrm{e}^{\mathrm{i}\theta}}{q}\left(\frac{\partial}{\partial r}\pm \mathrm{i}\frac{1}{r}\frac{\partial}{\partial\theta}\right), \tag{22}$$

where $q=\omega v/c=(\omega/c)\sqrt{\varepsilon_r\mu_r}=\omega\sqrt{\varepsilon\mu}$. These operators satisfy the relations

$$(\mathrm{R}^{\pm})^n H_0(qr) = H_{\pm n}(qr)\,\mathrm{e}^{\pm\mathrm{i}n\theta}. \tag{23}$$

Starting with the integral representation of the Hankel function of order zero [9]

$$H_0(qr) = -\frac{\mathrm{i}}{\pi^2}\int_{\mathbb{R}^2}\mathrm{d}\mathbf{k}\,\frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}}{k^2-q^2}, \tag{24}$$

and the relations

$$(\mathrm{R}^{\pm})^2 H_0(qr) = H_2(qr)\,\mathrm{e}^{\pm 2\mathrm{i}\theta} = -\frac{\mathrm{i}}{\pi^2 q^2}\int_{\mathbb{R}^2}\mathrm{d}\mathbf{k}(k_1^2-k_2^2\pm 2\mathrm{i}k_1k_2)\frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}}{q^2-k^2}, \tag{25}$$

and taking into account that

$$\nabla^2 H_0(qr) = -q^2H_0(qr)+4\mathrm{i}\delta(\mathbf{r}) = \frac{\mathrm{i}}{\pi^2}\int_{\mathbb{R}^2}\mathrm{d}\mathbf{k}(k_1^2+k_2^2)\frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}}{k^2-q^2}, \tag{26}$$

we deduce that

$$\frac{1}{(2\pi)^2q^2}\int_{\mathbb{R}^2}\mathrm{d}\mathbf{k}k_1^2\frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}}{q^2-k^2} = \frac{\mathrm{i}}{8}[-H_0(qr)+H_2(qr)\cos(2\theta)]-\frac{1}{2q^2}\delta(\mathbf{r}) \tag{27}$$

and

$$\frac{1}{(2\pi)^2q^2}\int_{\mathbb{R}^2}\mathrm{d}\mathbf{k}k_2^2\frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}}{q^2-k^2} = -\frac{\mathrm{i}}{8}[H_0(qr)+H_2(qr)\cos(2\theta)]-\frac{1}{2q^2}\delta(\mathbf{r}). \tag{28}$$

Here, we have also made use of the longitudinal delta function [10]

$$\delta_{ij}^{\mathrm{L}}(\mathbf{r}) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\mathrm{d}\mathbf{k}\,\frac{k_ik_j}{k^2}\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}. \tag{29}$$

Thus, the spatial form of the transverse part of Green's function for the $H_\parallel$ polarization is

$$\mathbf{G}^{\mathrm{HT}}(\omega,\mathbf{r}) = \frac{1}{(2\pi)^2}\int\mathrm{d}\mathbf{k}\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\tilde{\mathbf{G}}^{\mathrm{HT}}(\omega,\mathbf{k}) = \frac{1}{q^2}[\delta_{ij}^{\mathrm{L}}(\mathbf{r})]-\frac{1}{2q^2}\delta(\mathbf{r})\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{\mathrm{i}}{8}\begin{bmatrix}H_0(qr)+H_2(qr)\cos(2\theta) & H_2(qr)\sin(2\theta)\\ H_2(qr)\sin(2\theta) & H_0(qr)-H_2(qr)\cos(2\theta)\end{bmatrix}, \tag{30}$$

It is easy to recognize in Eq. (30) expressions similar to the components of the dyadic Green's function $\mathbf{G}^{\mathrm{TE}}$ for the free space [11]. Note that the superscript "TE" stands for "transverse electric" which is the same as "H parallel" polarization.

The longitudinal delta function in two dimensions has the form

$$\delta_{ij}^{\mathrm{L}}(\mathbf{r}) = \frac{1}{2}\delta_{ij}\delta(\mathbf{r})+\frac{r^2\delta_{ij}-2x_ix_j}{2\pi r^4}. \tag{31}$$

Consequently, the spatial form of the transverse part of Green's function for the $H_\parallel$ polarization is

$$\mathbf{G}^{\mathrm{HT}}(\omega,\mathbf{r}) = \frac{1}{q^2}\left[\frac{r^2\delta_{ij}-2x_ix_j}{2\pi r^4}\right] + \frac{\mathrm{i}}{8}\begin{bmatrix}H_0(qr)+H_2(qr)\cos(2\theta) & H_2(qr)\sin(2\theta)\\ H_2(qr)\sin(2\theta) & H_0(qr)-H_2(qr)\cos(2\theta)\end{bmatrix}, \tag{32}$$

where the first term is a two-by-two matrix.

3. RLDOS for a homogeneous lossy medium

The transverse part of Green's function is singular for $|\mathbf{k}|^2-\omega^2\varepsilon\mu=0$, while the longitudinal part is non-singular. This is interpreted in terms of the emission of electromagnetic radiation [7]. Electromagnetic waves satisfy a dispersion relation $\omega^2=|\mathbf{k}|^2/(\varepsilon\mu)$, so that they correspond to the singular (transverse) part. The longitudinal part of the Fourier transformed field is not involved in the emission of radiation and has no singularity when the transverse dispersion relation is satisfied. It is associated with the inductive (non-radiative) part of the field (a static field is purely longitudinal in Fourier space). The radiation field is associated with the singular terms in the transverse part of the field. Consequently, to calculate the RLDOS we need the inverse Fourier transform of $\tilde{\mathbf{G}}^{\mathrm{T}}(\omega,\mathbf{k})$ spectral form, into coordinate space, via Eq. (8).

3.1. $H_\parallel$ polarization

The trace of Eq. (32) has the form

$$\mathrm{Tr}\,\mathbf{G}^{\mathrm{HT}}(\omega,\mathbf{r}) = \frac{\mathrm{i}}{4}H_0(q\,r), \tag{33}$$

so that the RLDOS for $H_\parallel$ polarization is[1]

$$\tilde{\rho}_0 = \left|\mathrm{Im}\left(\lim_{r\to 0}\frac{\mathrm{i}}{4}H_0(qr)\right)\right| \approx \left|\mathrm{Im}\left(\frac{\mathrm{i}}{4}-\frac{1}{2\pi}\log(qr)\right)\right| = \left|\frac{1}{4}-\frac{1}{2\pi}\arg v\right|. \tag{34}$$

3.2. $E_\parallel$ polarization

For the $E_\parallel$ polarization the transverse part of the spectral form of Green's function is

$$\tilde{G}^{\mathrm{ET}}(\omega,\mathbf{k}) = \frac{1}{\varepsilon\mu[k^2/(\varepsilon\mu)-\omega^2]} = \frac{1}{k^2-q^2} \tag{35}$$

so that, the transverse part of the spatial form of Green's function is

$$G^{\mathrm{ET}}(\omega,\mathbf{r}) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\mathrm{d}\mathbf{k}\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\tilde{G}^{\mathrm{ET}}(\omega,\mathbf{k}) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\mathrm{d}\mathbf{k}\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\,\frac{1}{k^2-q^2} = \frac{\mathrm{i}}{4}H_0(qr). \tag{36}$$

Hence, the RLDOS for $E_\parallel$ polarization is the same as for that of $H_\parallel$ polarization

$$\tilde{\rho}_0 = \left|\frac{1}{4}-\frac{1}{2\pi}\arg v\right|. \tag{37}$$

This result has an appealing connection with a figure of merit sometimes used for metamaterial systems which have been characterized by an effective refractive index. The figure of merit is simply the real part of the effective index divided by the imaginary part, so that from Eq. (37), as the figure of merit increases the RLDOS tends to its free space value. This means that this figure of merit acts in an appropriate way in characterizing the degree of "tacitude" of a cloaking system, because the closer the RLDOS is to that of free space, the lower the chances of detection of the cloaking system through examination of radiation from atoms in its environment.

[1] We use the absolute values to ensure that RLDOS is always non-negative.

4. The three-dimensional case

For an infinite, isotropic, and homogeneous medium with relative permittivity ε_r and permeability μ_r, the spectral domain electric Green's function satisfies Eq. (16) and has the form (18). Now, the longitudinal part of Green's function is (see Eq. (A.7))

$$\tilde{G}^{L}_{ij}(\omega,\mathbf{k}) = -\frac{k_i k_j k_l k_m}{|\mathbf{k}|^4}\tilde{G}_{lm}(\omega,\mathbf{k}) = -\frac{k_i k_j}{|\mathbf{k}|^2\varepsilon\mu\omega^2}. \tag{38}$$

The transverse part can be easily obtained from

$$\tilde{G}^{T}_{ij}(\omega,\mathbf{k}) = \tilde{G}_{ij}(\omega,\mathbf{k}) - \tilde{G}^{L}_{ij}(\omega,\mathbf{k}) = \frac{\delta_{ij}-k_i k_j/|\mathbf{k}|^2}{\varepsilon\mu[|\mathbf{k}|^2/(\varepsilon\mu)-\omega^2]}. \tag{39}$$

Then, we have the spatial form of the transverse Green's function [12]

$$G^{T}_{ij}(\omega,\mathbf{r}) = \frac{1}{(2\pi)^3}\int d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{r}}\tilde{G}^{T}_{ij}(\omega,\mathbf{k}) = \frac{1}{4\pi\omega^2\varepsilon\mu}\left\{\left(\delta_{ij}-\frac{3x_i x_j}{r^2}\right)\frac{1}{r^3}\right.$$
$$\left. + q^3\left[\left(\frac{1}{qr}+\frac{i}{(qr)^2}-\frac{1}{(qr)^3}\right)\delta_{ij} - \left(\frac{1}{qr}+\frac{3i}{(qr)^2}-\frac{3}{(qr)^3}\right)\frac{x_i x_j}{r^2}\right]e^{iqr}\right\}, \tag{40}$$

where $q=\omega v/c=(\omega/c)[\eta(\omega)+i\kappa(\omega)]$ and $v=\sqrt{\varepsilon_r\mu_r}=c\sqrt{\varepsilon\mu}$ is the refractive index of the medium. Also, $q=\omega v/c=(\omega/c)\sqrt{\varepsilon_r\mu_r}=\omega\sqrt{\varepsilon\mu}$.

The spatial form of the longitudinal Green's function can be obtained in terms of the three-dimensional, longitudinal delta function [10], that is [10,12]

$$G^{L}_{ij}(\omega,\mathbf{r}) = -\frac{1}{4\pi\omega^2\varepsilon\mu}\left[\frac{4\pi}{3}\delta_{ij}\delta(\mathbf{r}) + \left(\delta_{ij}-\frac{3x_i x_j}{r^2}\right)\frac{1}{r^3}\right] \tag{41}$$

so that, the spatial form of Green's function is

$$G_{ij}(\omega,\mathbf{r}) = \frac{1}{4\pi\omega^2\varepsilon\mu}\left\{-\frac{4\pi}{3}\delta_{ij}\delta(\mathbf{r}) + q^3\left[\left(\frac{1}{qr}+\frac{i}{(qr)^2}-\frac{1}{(qr)^3}\right)\delta_{ij}\right.\right.$$
$$\left.\left. -\left(\frac{1}{qr}+\frac{3i}{(qr)^2}-\frac{3}{(qr)^3}\right)\frac{x_i x_j}{r^2}\right]e^{iqr}\right\}. \tag{42}$$

By substituting $v=\eta(\omega)+i\kappa(\omega)$ in Eq. (40), we obtain the trace of the transverse Green's function in the form

$$\mathrm{Tr}[G^{T}_{ij}(\omega,\mathbf{r})] = \frac{1}{2\pi}\frac{\cos[\omega\eta(\omega)r/c]+i\sin[\omega\eta(\omega)r/c]}{r}e^{-\omega\kappa(\omega)r/c}. \tag{43}$$

The imaginary part of the trace of transverse Green's function is

$$\mathrm{Im}(\mathrm{Tr}[G^{T}_{ij}(\omega,\mathbf{r})]) = \frac{1}{2\pi}\frac{\sin[\omega\eta(\omega)r/c]}{r}e^{-\omega\kappa(\omega)r/c}, \tag{44}$$

and we have the formula

$$\tilde{\rho}(\mathbf{r}) = \left|\lim_{r\to 0}\mathrm{Im}(\mathrm{Tr}[G^{T}_{ij}(\omega,\mathbf{r})])\right| = \left|\frac{\omega\eta(\omega)}{2\pi c}\right|, \tag{45}$$

for the RLDOS. In vacuum Eq. (45) becomes

$$\tilde{\rho}_{\text{vacuum}}(\mathbf{r}) = \frac{\omega}{2\pi c}. \tag{46}$$

A curious feature of the result (45) is that it does not depend on the imaginary part of v, $\kappa(\omega)$. There is thus a clear difference between the two-dimensional case and the three-dimensional case. A corresponding result has been given by Guérin et al. [13], while Narayanaswamy and Chen [14] have presented their formula in an implicit expression.

5. Conclusions

One feature of the results derived here may be puzzling to readers. We have treated an infinite, homogeneous and lossy material, and discussed the coupling of sources to electromagnetic waves. However, those waves would all be absorbed in such a medium before reaching infinity. Nevertheless, in any treatment involving a finite lossy material, or a finite material incorporating any heterogeneities, the results obtained here are useful. Any corrections to Green's functions involving boundaries away from the source point will entail fields which are finite there, and are thus readily taken into account. The only exception to this statement would be encountered with a source point situated exactly on a boundary between different media, but such boundary points should be treated through the limit of ordinary points lying to one side or the other of the boundary.

Green's functions we have evaluated can be used as the starting point for the treatment of more general situations, e.g., the study of the radiation of sources embedded in structured media, with the difficulties encountered in handling field divergences around a source point having been dealt in this article.

For non-magnetic materials the RLDOS is always positive, and so possesses a key characteristic of the genuine local density of states. However, for structures such as metamaterials, in which the permeability can have a negative imaginary part, the imaginary part of Green's function can also become negative and so is neither a reasonable measure of source emissivity nor a relative density of states. The extension of this work to encompass metamaterials is thus a delicate matter, requiring careful conceptualization from first principles.

Acknowledgement

This work was produced with the assistance of the Australian Research Council through its Discovery Grants Scheme.

Appendix A. Projection operators

To solve Eq. (15) it is convenient to separate the vector $\tilde{\mathbf{A}}$ into longitudinal (parallel to $\mathbf{k}$) and transverse (perpendicular to $\mathbf{k}$) parts. The longitudinal part is

$$\tilde{\mathbf{A}}^{L} = (\mathbf{k}\cdot\tilde{\mathbf{A}})\frac{\mathbf{k}}{|\mathbf{k}|^2} = -\frac{\mu_0 c^2}{\omega^2}(\mathbf{k}\cdot\tilde{\mathbf{J}})\frac{\mathbf{k}}{|\mathbf{k}|^2}, \tag{A.1}$$

having the components

$$\tilde{A}^{L}_i = \frac{k_i k_j}{|\mathbf{k}|^2}\tilde{A}_j. \tag{A.2}$$

The transverse part can be obtained from

$$\tilde{\mathbf{A}}^{T} = -\frac{\mathbf{k}\times(\mathbf{k}\times\tilde{\mathbf{A}})}{|\mathbf{k}|^2} = \tilde{\mathbf{A}} - (\mathbf{k}\cdot\tilde{\mathbf{A}})\frac{\mathbf{k}}{|\mathbf{k}|^2} = \tilde{\mathbf{A}} - \tilde{\mathbf{A}}^{L}, \tag{A.3}$$

and has the components

$$\tilde{A}^{T}_i = \tilde{A}_i - \frac{k_i k_j}{|\mathbf{k}|^2}\tilde{A}_j = \left(\delta_{ij}-\frac{k_i k_j}{|\mathbf{k}|^2}\right)\tilde{A}_j. \tag{A.4}$$

Accordingly, using the projection operators[2]

$$\Pi^{L} = [\Pi^{L}_{ij}] = \left[\frac{k_i k_j}{|\mathbf{k}|^2}\right] \quad \text{and} \quad \Pi^{T} = [\Pi^{T}_{ij}] = \left[\delta_{ij}-\frac{k_i k_j}{|\mathbf{k}|^2}\right], \tag{A.5}$$

[2] Note that the matrices of the operators Π^L and Π^T are symmetric, and satisfy the relations: $\Pi^L\Pi^L=\Pi^L$, $\Pi^T\Pi^T=\Pi^T$, $\Pi^L\Pi^T=\Pi^T\Pi^L=\mathbf{0}$, and $\Pi^L+\Pi^T=\mathbf{I}$, where $\mathbf{I}$ is the identity operator.

we may also separate Green's function into longitudinal and transverse parts

$$\tilde{G}_{ij}(\omega,\mathbf{k}) = \tilde{G}^{\mathrm{L}}_{ij}(\omega,\mathbf{k}) + \tilde{G}^{\mathrm{T}}_{ij}(\omega,\mathbf{k}), \tag{A.6}$$

where

$$\tilde{G}^{\mathrm{L}}_{ij}(\omega,\mathbf{k}) = (\Pi^{\mathrm{L}}\tilde{\mathbf{G}}\Pi^{\mathrm{L}})_{ij} = \frac{k_i k_j k_l k_m}{|\mathbf{k}|^4}\tilde{G}_{lm}(\omega,\mathbf{k}), \tag{A.7}$$

$$\tilde{G}^{\mathrm{T}}_{ij}(\omega,\mathbf{k}) = (\Pi^{\mathrm{T}}\tilde{\mathbf{G}}\Pi^{\mathrm{T}})_{ij} = \left(\delta_{il} - \frac{k_i k_l}{|\mathbf{k}|^2}\right)\left(\delta_{jm} - \frac{k_j k_m}{|\mathbf{k}|^2}\right)\tilde{G}_{lm}(\omega,\mathbf{k}), \tag{A.8}$$

Note that Eq. (A.6) is satisfied by Eqs. (A.8) and (A.7) only by dyads of the form $(\delta_{ij} - \alpha k_i k_j)$, where α is an arbitrary constant.

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