ISSN 1063-780X, Plasma Physics Reports, 2007, Vol. 33, No. 10, pp. 859–870. © Pleiades Publishing, Ltd., 2007.
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NONLINEAR PHENOMENA

Nonlinear Theory of Ion-Acoustic Waves in an Ideal Plasma with Degenerate Electrons

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Received January 17, 2007; in final form, March 21, 2007

Abstract—A nonlinear theory is constructed that describes steady-state ion-acoustic waves in an ideal plasma in which the electron component is a degenerate Fermi gas and the ion component is a classical gas. The parameter ranges in which such a plasma can exist are determined, and dispersion relations for ion-acoustic waves are obtained that make it possible to find the linear ion-acoustic velocity. Analytic gas-dynamic models of ion sound are developed for a plasma with the ion component as a cold, an isothermal, or an adiabatic gas, and moreover, the solutions to the equations of all the models are brought to a quadrature form. Profiles of a subsonic periodic and a supersonic solitary wave are calculated, and the upper critical Mach numbers of a solitary wave are determined. For a plasma with cold ions, the critical Mach number is expressed by an explicit exact formula.

PACS numbers: 52.35.Fp

DOI: 10.1134/S1063780X07100078

1. INTRODUCTION

Being one of the main wave phenomena in a plasma, ion-acoustic waves have been studied for several decades. The nonlinear theory of these waves was first worked out in [1, 2], where their main properties were investigated by the Sagdeev pseudopotential method. It was found that steady-state ion-acoustic waves can exist as a periodic wave or as a solitary wave, whose speed is bounded below by the linear ion-acoustic velocity v_s and above by a value of about $1.58v_s$. An explicit exact expression for the upper limiting speed (the upper Mach number limit) was obtained later in [3]:

$$M = \sqrt{-1-2W_{-1}\left[-\frac{1}{2}\exp\left(-\frac{1}{2}\right)\right]} \approx 1.5852010065, \quad (1)$$

where $W_{-1}(x)$ is the negative branch of the Lambert W function [4]. In [1–3], it was assumed that the ion plasma component is cold and the electron component is isothermal and inertialess.

The nonlinear theory was further developed in more than several hundreds of papers by taking into account various physical factors, such as the influence of the ion temperature [5–7], the presence of two [8, 9] or more ion species [10] (in particular, negative ions [11, 13]) or two groups of electrons with different temperatures [14, 15], ion inertia [16–19], etc. The general conclusion was that accounting for, e.g., electron inertia does not change the range of possible Mach numbers of a solitary wave [19], but this range is very sensitive to the deviation of the electron distribution from a Boltzmann function [20].

In all of the above papers (as well as in most other papers on the subject), it was assumed the heated plasma components (electrons and ions) involved in the wave process obey an isothermal equation of state, i.e., that the electron and ion temperatures are constant. This is a simplifying assumption: it leaves open the question of an external source or sink of thermal energy because, in an isothermal process, the energy is consumed during compressions and is recovered during rarefactions.

Hence, more adequate and realistic models for describing nonlinear waves in a plasma may be those based on a gas-dynamic (adiabatic) approach in which the energy is assumed not to be exchanged with the surrounding environment. This approach makes it possible to account for temperature variations in different phases of the wave and the effect of these variations on the properties of the wave itself.

In recent years, the gas-dynamic approach has been applied to study ion-acoustic and other electromagnetic waves in a plasma [21–26]. In those papers, the nonlinear equations for the wave structure were analyzed in the adiabatic approximation in which the ion or dust plasma component is treated as a gas and is described by an adiabatic equation of state with an arbitrary adiabatic index γ_+ lying in the range $\gamma_+ \in [1; 3]$. An exact solution to the problem of the profile of an ion-acoustic wave in the gas-dynamic approximation was obtained in [27]. The analysis carried out in that paper made it

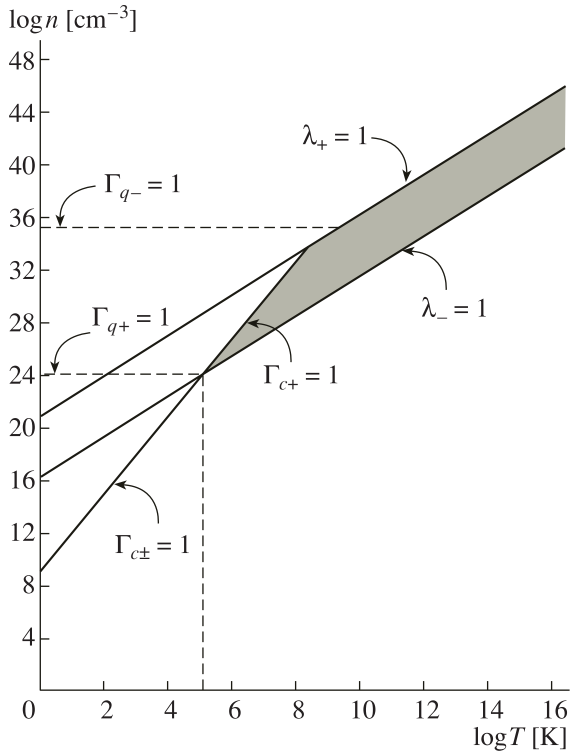

Fig. 1. Domains in the T–n plane where different types of plasmas exist. The domain of existence of the ideal plasma under consideration (that with a classical ion gas and a degenerate electron gas) is hatched.

possible to determine the parameter ranges in which a solitary wave can exist and the range of its possible velocities. In [21–27], it was shown that, for $\gamma_\pm > 1$, the maximum Mach number of such a wave can substantially exceed 1.58.

In the present paper, a new nonlinear problem is formulated and solved for the first time, namely, that of the structure of an ion-acoustic wave and a possible range of velocities of a solitary wave in a collisionless plasma in which the ion component is a classical gas and the electron component is a degenerate ideal Fermi gas.

The paper is organized as follows. Sections 2–4 are auxiliary ones—they are necessary for further presentation of the nonlinear theory. In Section 2, we determine the parameter ranges in which the plasma under consideration can exist. In Section 3, we show how the quantum nature of electron hydrodynamics can be described in the theory of electrostatic waves in a plasma in the Thomas–Fermi approximation. In Section 4, we derive dispersion relations in the linear theory of ion sound in a plasma with degenerate electrons and cold or hot ions. First of all, these relations are important for determining the linear ion-acoustic velocity, in terms of which the Mach number is expressed. In Section 5, we give a complete and exact solution to the problem of the structure of ion-acoustic waves and a possible range of velocities of a solitary wave in an ideal plasma with cold classical ions and degenerate electrons. In Section 6, a similar problem is solved for a plasma with hot isothermal ions, and, in Section 7, it is solved for a plasma with hot ions under conditions such that an ion-acoustic wave is a spatially developed adiabatic process. In Sections 6 and 7, it is assumed that, in a collisionless plasma, there is enough time for the wave to relax to a local thermodynamically equilibrium state due to uncorrelated Coulomb interactions between plasma particles. In the Conclusions, we briefly summarize the main results of this work.

2. MAIN PARAMETERS OF AN IDEAL PLASMA WITH DEGENERATE ELECTRONS

At first glance, it seems doubtful that the ideal plasma under consideration, which consists of a classical ion gas and a degenerate electron gas, can indeed exist. In fact, the ideal nature of the plasma requires that the pairwise interaction between the particles be weak, i.e., the electron and ion gases be rarefied, while the degenerate nature of the electrons requires that their density be sufficiently high, as well as the ion density (in view of the plasma quasineutrality).

The answers to these objections can be found in [28], where the existence diagram for a hydrogen plasma on the temperature–density (T–n) plane was calculated and it was found that, in one of the regions of the diagram, the electrons are degenerate and the plasma is ideal. Recall that the boundary criterion for the electrons to be degenerate can be given by the equality of their Fermi energy $\varepsilon_{F_\pm}$ to their temperature $kT_\pm$ in energy units, $\lambda_\pm = \varepsilon_{F_\pm}/kT_\pm = 1$, and that the boundary criterion for the plasma to be ideal can be expressed as the equality of the mean energy of pairwise interaction to the mean kinetic energy, $\Gamma_{c\pm} = e^2 n_\pm^{1/3}/kT_\pm = 1$ in the classical theory and $\Gamma_{q\pm} = e^2 n_\pm^{1/3}/\varepsilon_{F_\pm} = 1$ (for singly charged ions) in the quantum theory. Hereafter, the plus and minus subscripts refer to ions and electrons, respectively.

If we plot these criteria as lines in the T–n plane on a logarithmic scale, then we can see that they bound an inclined strip, which just corresponds to an ideal gas plasma with classical ions and degenerate electrons (see Fig. 1, referring to a hydrogen plasma).

In nature, such a plasma can exist in the inner layers of stars (e.g., white dwarfs [29]) and, under laboratory conditions, it can exist in laser fusion [30] and micropinch [31] experiments. A semiconductor plasma can also can be in a state with degenerate electrons and nondegenerate holes, provided that the effective mass of the electrons is much less than that of the holes.

3. DESCRIPTION OF A DEGENERATE ELECTRON GAS IN HYDRODYNAMIC THEORIES

In hydrodynamic theories of electrostatic waves in a plasma, it is necessary to know explicit expressions describing the dependence of each plasma component on the electrostatic potential φ in order to solve Poisson's equation. For a degenerate gas of free inertialess electrons, such expressions are known—they are obtained in the Thomas–Fermi approximation (see, e.g., [32, 33]):

$$n_- = n_0\left(1+\frac{e\varphi}{\varepsilon_F}\right)^{\frac{3}{2}} \approx n_0\left(1+\frac{3e\varphi}{2\varepsilon_F}\right), \qquad (2)$$

where $n_0 = (8\pi/3h^3)p_F^3$ is the unperturbed density in terms of the Fermi momentum and h is Planck's constant. In [34] (Section 2, problem 10), the same expressions are obtained in deriving the barometric formula for a degenerate ideal Fermi gas. Expressions (2) can also be obtained by considering, e.g., the equation of motion of a degenerate inertialess ($m_- \longrightarrow 0$) electron gas described by the Thomas–Fermi equation of state $P_- = (2/5)\varepsilon_F n_0(n_-/n_0)^{5/3}$ (without allowance for exchange interactions) from [32, 33].

In what follows, the first of expressions (2) will be used in the nonlinear theory and the second linearized expression will be utilized to derive the dispersion relation.

We describe a degenerate electron gas by assuming that the electrons are inertialess—an assumption that is valid for temperatures of up to 10^{12} K (about 100 MeV), when the relativistic mass of an electron is only ~10% of the rest mass of a proton. Note also that, according to the Lindhardt theory [33], the Thomas–Fermi approximation is valid when the Fermi wavelength is much less than the wavelength of the ion-acoustic waves; this yields the condition $\omega_{p+}/\omega_F \ll \sqrt{m_-/m_+}$ (where $\omega_F = \varepsilon_F/\hbar$), which is assumed to be satisfied and which enables us to ignore the spatial dispersion of the electron gas.

4. LINEAR THEORY OF ION-ACOUSTIC WAVES IN AN IDEAL PLASMA WITH DEGENERATE ELECTRONS

We write the standard equations of the hydrodynamic theory of ion-acoustic waves in a plasma with cold ions:

$$\frac{\partial v_+}{\partial t} + v_+\frac{\partial v_+}{\partial x} = -\frac{e}{m_+}\frac{\partial\varphi}{\partial x}; \qquad (3)$$

$$\frac{\partial n_+}{\partial t} + \frac{\partial}{\partial x}(n_+v_+) = 0; \qquad (4)$$

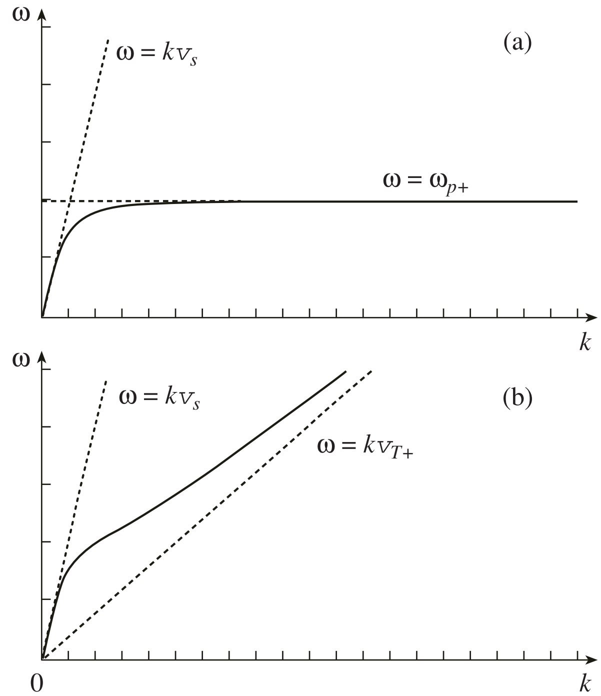

Fig. 2. Dispersion curves for ion-acoustic waves in a plasma with (a) cold ions and (b) hot isothermal ions.

$$\frac{\partial^2\varphi}{\partial x^2} = 4\pi e(n_- - n_+). \qquad (5)$$

In Eq. (3), we ignore the collisional term by virtue of the ideal nature of the plasma. The effects of collisions in a plasma are well known and will not be discussed here. After simple manipulations, we arrive at the following dispersion relation for small harmonic perturbations, with amplitudes varying as $\sim\exp[i(kx-\omega t)]$:

$$\frac{1}{\omega^2} = \frac{1}{\omega_{p+}^2} + \frac{3m_+}{2\varepsilon_F}\frac{1}{k^2}. \qquad (6)$$

This dispersion relation coincides with that obtained in [35]. Here, ω_{p+} is the ion plasma frequency in terms of the unperturbed density n_0. Equation (6) implies that, in a plasma with degenerate electrons, the linear ion-acoustic velocity is equal to

$$v_s = \sqrt{\frac{2\varepsilon_F}{3m_+}}. \qquad (7)$$

Dispersion curve (6) is shown in Fig. 2a.

If, instead of Eq. (3), the gas-dynamic equation for hot ions in the isothermal approximation is used,

$$\frac{\partial v_+}{\partial t} + v_+\frac{\partial v_+}{\partial x} = -\frac{e}{m_+}\frac{\partial\varphi}{\partial x} - \frac{\kappa T_+}{m_+}\frac{\partial}{\partial x}(\ln n_+) \qquad (8)$$

(where κ is Boltzmann's constant), then similar manipulations yield a dispersion relation with the same linear ion-acoustic velocity (7):

$$\frac{1}{\omega^2 - k^2 v_{T+}^2} = \frac{1}{\omega_{p+}^2} + \frac{1}{v_s^2 k^2}, \tag{9}$$

where $v_{T+} = \sqrt{\kappa_{T+}/m_+}$ is the ion thermal velocity. Dispersion curve (9) is shown in Fig. 2b. An analogous result is obtained when the gas-dynamic equation for hot ions in the adiabatic approximation in used instead of Eq. (8), the only difference being that the slope angle of the thermal asymptotic of the dispersion curve is $\sqrt{\gamma_+}$ times smaller.

Hence, the linear ion-acoustic velocity is described by formula (7) and, in speaking of the Mach number, we will mean a velocity normalized to the velocity v_s defined by this formula.

5. NONLINEAR THEORY OF ION-ACOUSTIC WAVES IN AN IDEAL PLASMA OF A CLASSICAL COLD ION GAS AND A DEGENERATE ELECTRON GAS

Proceeding from Eqs. (3)–(5), we can consider a steady-state ion-acoustic wave running in the x direction at a velocity V. To do this, we introduce a new self-similar variable,

$$\xi = x - Vt, \quad \frac{\partial}{\partial t} = -V\frac{d}{d\xi}, \quad \frac{\partial}{\partial x} = \frac{d}{d\xi}. \tag{10}$$

We thus pass over from the laboratory frame of reference to a comoving frame. In this frame, Eqs. (3)–(5) are reduced to the following set of ordinary differential equations:

$$-V\frac{dv_+}{d\xi} + v_+\frac{dv_+}{d\xi} = -\frac{e}{m_+}\frac{d\varphi}{d\xi}; \tag{11}$$

$$-V\frac{dn_+}{d\xi} + \frac{d(n_+v_+)}{d\xi} = 0; \tag{12}$$

$$\frac{d^2\varphi}{d\xi^2} = 4\pi e(n_- - n_+). \tag{13}$$

We integrate the continuity equation with allowance for the relationship $\lim_{v_+\to 0} n_+ = n_0$ and express v_+ in terms of n_+ to obtain

$$v_+ = V\left(1 - \frac{n_0}{n_+}\right). \tag{14}$$

We also integrate the equation of motion with allowance for the relationships $\lim_{v_+\to 0} n_+ = n_0$ and $\lim_{v_+\to 0} \varphi = 0$:

$$-Vv_+ + \frac{v_+^2}{2} = -\frac{e}{m_+}\varphi. \tag{15}$$

We insert relationship (14) into relationship (15) and express n_+ in terms of φ to get

$$n_+ = \frac{n_0}{\sqrt{1 - \frac{2e\varphi}{m_+V^2}}}. \tag{16}$$

Substituting explicit expressions (2) and (16) for the electron and ion densities into Poisson's equation (13), we obtain

$$\frac{d^2\varphi}{d\xi^2} = 4\pi e n_0\left[\left(1 + \frac{e\varphi}{\varepsilon_F}\right)^{\frac{3}{2}} - \left(1 - \frac{2e\varphi}{m_+V^2}\right)^{-\frac{1}{2}}\right]. \tag{17}$$

Equation (17) is an equation of motion of an oscillator in the one-dimensional pseudopotential $U(\varphi) = -\int_0^{\varphi} F(\varphi')d\varphi'$. In this equation, the electrostatic potential plays the role of a pseudo-coordinate and the self-similar coordinate ξ plays the role of a pseudo-time. In this context, the right-hand side of Eq. (17) can be regarded as a pseudo-force $F(\varphi)$.

Integrating Eq. (17) once under the condition $U(0) = 0$ gives the pseudo-energy conservation law

$$\frac{1}{2}\left(\frac{d\varphi}{d\xi}\right)^2 = 4\pi n_0\left[\frac{2\varepsilon_F}{5}\left(1 + \frac{e\varphi}{\varepsilon_F}\right)^{\frac{5}{2}} - \frac{2\varepsilon_F}{5} + m_+V^2\left(1 - \frac{2e\varphi}{m_+V^2}\right)^{\frac{1}{2}} - m_+V^2\right]. \tag{18}$$

It is now an easy matter to get a general solution to Eq. (17) with the constant of integration ξ_0:

$$\xi - \xi_0 = \int_0^{\varphi} \frac{d\varphi}{\sqrt{8\pi n_0\left[\frac{2\varepsilon_F}{5}\left(1 + \frac{e\varphi'}{\varepsilon_F}\right)^{\frac{5}{2}} - \frac{2\varepsilon_F}{5} + m_+V^2\left(1 - \frac{2e\varphi'}{m_+V^2}\right)^{\frac{1}{2}} - m_+V^2\right]}}. \tag{19}$$

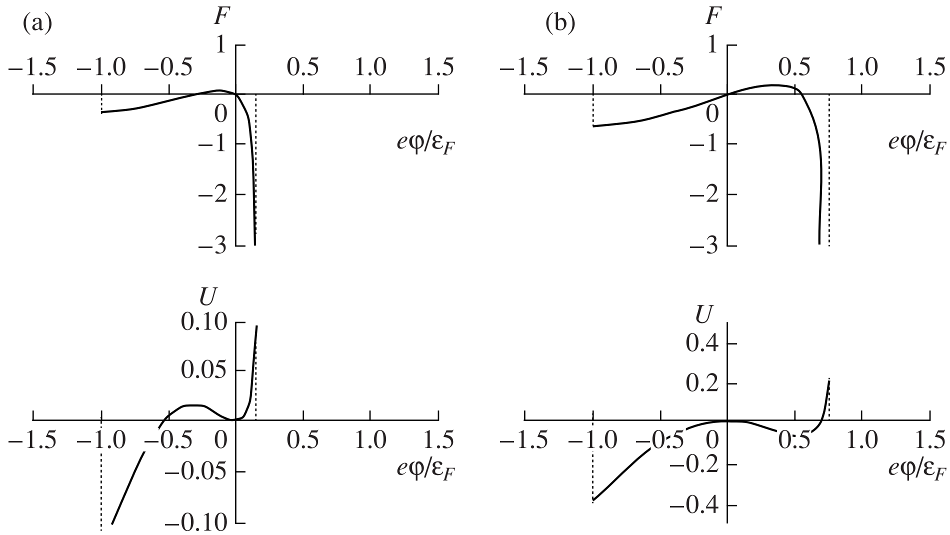

Fig. 3. Plots of the pseudo-force $F(\varphi)$ (on top) and pseudo-potential $U(\varphi)$ (on bottom) for m_+V^2/ε_F = (a) 0.3 ($M = 0.67 < 1$) and (b) 1.5 ($M = 1.5 > 1$). The vertical dashed lines show the largest possible peak-to-dip amplitude, (φ_{min}; φ_{max}), of the electrostatic potential in the wave.

However, solution (19) is unillustrative and inconvenient to analyze. Much more information on the properties of the solution can be provided by analyzing the pseudoforce $F(\varphi)$ (17) and pseudopotential $U(\varphi)$ (18).

Note first of all that the ranges of definition of the functions $F(\varphi)$ and $U(\varphi)$ are bounded from below and from above by the values $\varphi_{min} = -\varepsilon_F/e$ and $\varphi_{max} = m_+V^2/2e$, which give the largest possible peak-to-dip amplitude of the electrostatic potential in the wave.

The function $F(\varphi)$ can have either a zero and a negative root or a zero and a positive root, which correspond to the equilibrium points of the oscillator (see the upper plots in Fig. 3). Consequently, the function $U(\varphi)$ can also have either a minimum at zero and a maximum at $\varphi < 0$ or a maximum at zero and a minimum at $\varphi > 0$ (see the lower plots in Fig. 3). Periodic motion of the oscillator in the well of a pseudopotential corresponds to periodic ion-acoustic waves and the motion along the separatrix passing through the saddle point (or, for a zero velocity, through the maximum point of the function $U(\varphi)$) corresponds to a solitary ion-acoustic wave. It can easily be seen that the plots in Fig. 3a satisfy the condition $\int_0^{\Lambda} n_{\pm}(\xi)d\xi = n_0\Lambda$ for a periodic wave with the spatial period Λ but do not satisfy the boundary condition $\lim_{\xi\to\pm\infty}\varphi = 0$, which is a necessary condition for the existence of a solitary wave.

In order to determine the equilibrium points of the oscillator, it is necessary to solve the following equation for $\mu = e\varphi/\varepsilon_F$:

$$(1+\mu)^3(1-2\alpha\mu) = 1, \tag{20}$$

where $\alpha = \varepsilon_F/m_+V^2$. This equation has four roots,

$$\mu_1 = 0; \quad \mu_2 = \frac{1}{6}\left(\frac{\Theta^{1/3}}{\alpha} + \frac{6\alpha+1}{\alpha\Theta^{1/3}} - \frac{6\alpha-1}{\alpha}\right);$$

$$\mu_{3,4} = -\frac{1}{12}\left[\frac{\Theta^{1/3}}{\alpha} + \frac{6\alpha+1}{\alpha\Theta^{1/3}} + \frac{2(6\alpha-1)}{\alpha} \pm i\sqrt{3}\left(\frac{\Theta^{1/3}}{\alpha} - \frac{6\alpha+1}{\alpha\Theta^{1/3}}\right)\right], \tag{21}$$

where

$$\Theta = 54\alpha^2 + 9\alpha + 1 + 3\alpha\sqrt{3}\sqrt{108\alpha^2 + 28\alpha + 3}. \tag{22}$$

The cases of Figs. 3a and 3b are separated by the condition $\mu_2 = 0$, which holds for $\alpha = 3/2$ or accordingly, for the Mach number $M = 1$. Hence, as in the classical theory [1, 2], the periodic wave in Fig. 3a is always subsonic, while the solitary wave in Fig. 3b is always supersonic.

The highest possible Mach number of a solitary wave can be determined from the following geometric condition: the existence of a solitary wave requires that the rightmost point of the plot of the pseudopotential be above the abscissa. Otherwise, the wave breaks. The boundary state is determined from the equality $U(\varphi_{max}) = 0$, which leads to the following equation for $\lambda = \alpha^{-1}$:

$$\frac{4}{25}\left(1+\frac{\lambda}{2}\right)^5 = \left(\frac{2}{5}+\lambda\right)^2. \tag{23}$$

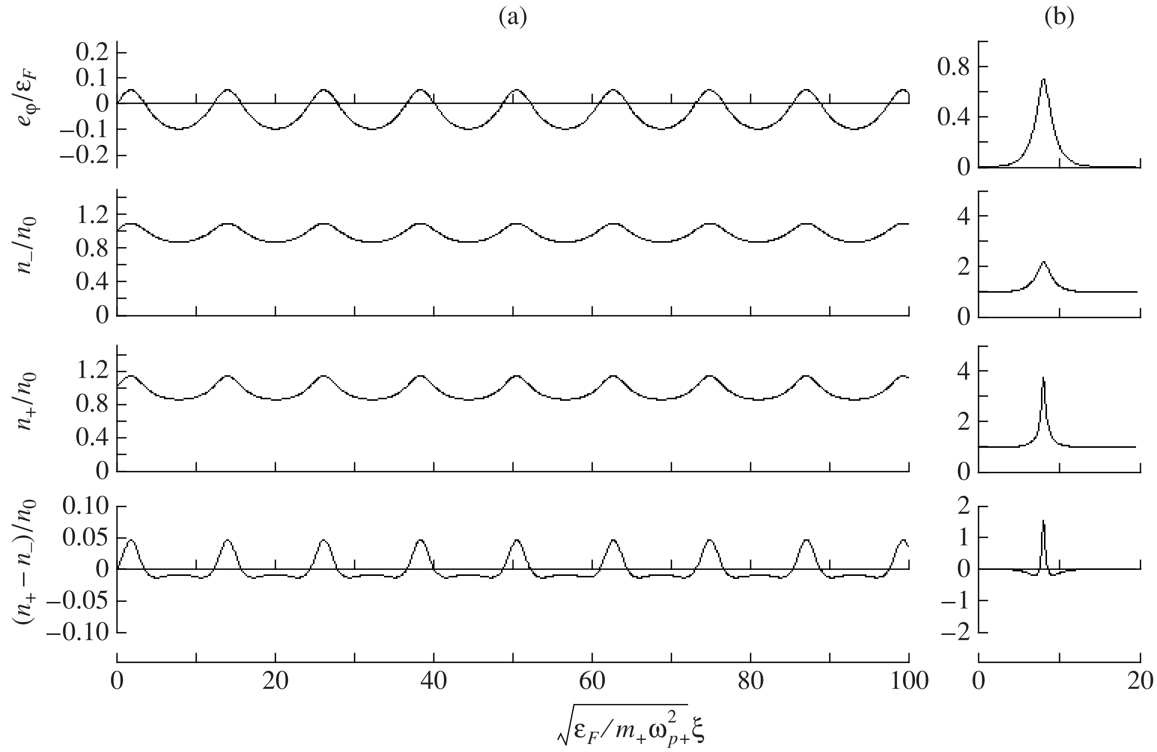

Fig. 4. Waveforms of the relative potential, relative electron density, relative ion density, and relative space charge (from top to bottom) in (a) a subsonic periodic ion-acoustic wave with $m_+V^2/\varepsilon_F = 0.4$ ($M \approx 0.77$) and (b) a supersonic solitary ion-acoustic wave with $m_+V^2/\varepsilon_F = 1.5$ ($M = 1.5$) in a plasma with cold ions and degenerate electrons.

This equation has five roots,

$$\lambda_1 = 0;$$

$$\lambda_2 = -\frac{5}{2} + \frac{3\sqrt{5}}{2} - \sqrt{\frac{1}{2}(-25 + 13\sqrt{5})} = -0.572237\ldots;$$

$$\lambda_3 = -\frac{5}{2} + \frac{3\sqrt{5}}{2} + \sqrt{\frac{1}{2}(-25 + 13\sqrt{5})} = 2.28044\ldots; \quad (24)$$

$$\lambda_{4,5} = \frac{1}{2}[-5 - 3\sqrt{5} \pm i\sqrt{2(25 + 13\sqrt{5})}]$$

$$= -0.5841\ldots \pm i5.19947\ldots.$$

The only physically meaningful root is λ_3, corresponding to the limiting Mach number $M = \sqrt{3\lambda_3/2} = 1.8495\ldots$. Hence, in an ideal plasma with classical ions and degenerate electrons, the range of possible Mach numbers of solitary ion-acoustic waves can be far wider (cf. expression (1)): $1 < M < 1.8495\ldots$.

This is the main result of the present section. To conclude it, we give in Fig. 4 an example of waveforms of the physical quantities for a subsonic periodic and a supersonic solitary wave. It should be noted that the space charge in the wave changes its sign at the points corresponding to the equilibrium points of the oscillator (see the lowermost plots in Fig. 4). We also note that, in the rarefaction phase of the wave, the electron density may become low enough to be described by the classical approach. In this case, the rarefaction region should be described by using the Boltzmann exponent in expressions (2), just as was done in [1, 2], and then by matching the classical solution with the quantum solution that is valid for the compression regions. This matching procedure was outlined in the above-mentioned problem 10 from Section 2 of [34].

6. NONLINEAR THEORY OF ION-ACOUSTIC WAVES IN AN IDEAL PLASMA OF A CLASSICAL ISOTHERMAL ION GAS AND A DEGENERATE ELECTRON GAS

We start with the set of Eqs. (4), (5), and (8). Recall that we have already substituted the equation of state of an ideal gas, $P_+ = n_+\kappa T_+$, with $T_+ = \text{const}$, into the last term on the right-hand side of Eq. (8).

As in the previous section, we introduce self-similar variable (10), in terms of which continuity and Poisson's equations (12) and (13) keep their forms and equation of motion (8) becomes

$$-V\frac{dv_+}{d\xi} + v_+\frac{dv_+}{d\xi} = -\frac{e}{m_+}\frac{d\varphi}{d\xi} - v_{T+}^2\frac{d(\ln n_+)}{d\xi}. \quad (25)$$

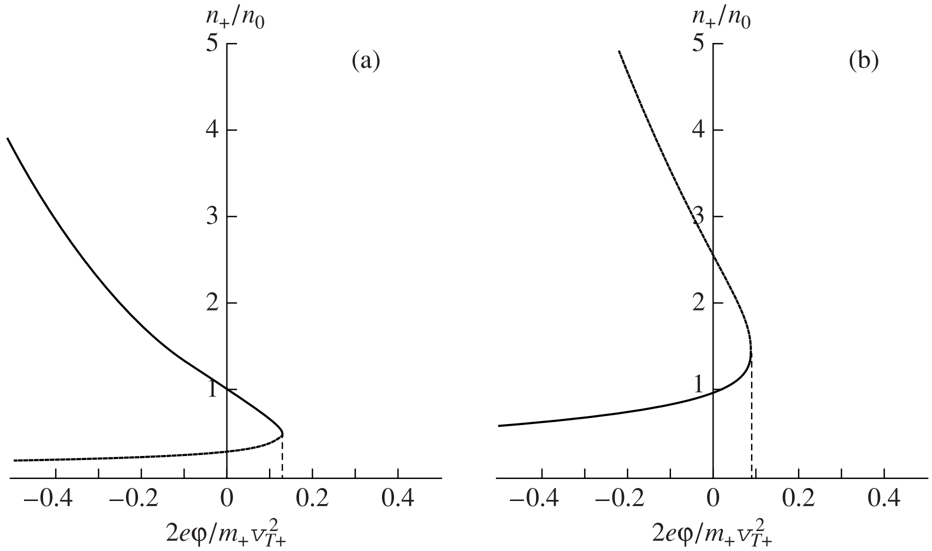

Fig. 5. Plots of the function $n_+(\varphi)$ for η = (a) 0.5 and (b) 1.5. The discarded branches are shown by dots. The vertical dashed lines correspond to φ_{max}.

The solution to the continuity equation of this new set of Eqs. (12), (13), and (25) is again given by formula (14), and the solution to Eq. (25) has the form

$$V^2\left[\left(\frac{n_0}{n_+}\right)^2 - 1\right] + 2v_{T+}^2\ln\frac{n_+}{n_0} = -\frac{2e}{m_+}\varphi. \tag{26}$$

Resolving solution (26) with respect to n_+ yields

$$n_+ = n_0\sqrt{\frac{-(V/v_{T+})^2}{W_{0,-1}\{-(V/v_{T+})^2\exp[-(V/v_{T+})^2 + (2e\varphi/m_+v_{T+}^2)]\}}}. \tag{27}$$

Since the argument of the Lambert W function is negative, we are formally forced to write two solutions to Eq. (26), which correspond to the two real branches of the W function. Accordingly, we must carry out a brief analysis in order to determine which of the branches should be discarded.

To do this, we consider the plot of the function $n_+(\varphi)$. The function is seen to consist of two branches having a conjugation point (Fig. 5); moreover, the upper branch always corresponds to the principal branch W_0 of the Lambert W function and the lower branch, to the negative branch W_{-1} of the W function. To the right from the conjugation point of the branches, the function $n_+(\varphi)$ is complex. The conjugation point determines the amplitude φ_{max} in the wave:

$$\varphi_{max} = \frac{m_+v_{T+}^2}{2e}\left(\frac{V^2}{v_{T+}^2} - 1 - 2\ln\frac{V}{v_{T+}}\right). \tag{28}$$

In this case, we again have $\varphi_{min} = -\varepsilon_F/e$.

One of the branches (27) can be chosen in a fairly simple way: of the two branches of the function $n_+(\varphi)/n_0$, we choose the branch that passes through unity at zero, because this is the only branch that, together with expressions (2), satisfies quasineutrality condition (27) for an unperturbed plasma. An analysis shows that, for $V/v_{T+} < 1$, the quasineutrality condition is satisfied by solution (27) with the principal branch W_0 of the Lambert W function (Fig. 5a) and, for $V/v_{T+} > 1$, it is satisfied by the solution with the negative branch W_{-1} of the W function (Fig. 5b).

We substitute expressions (2) and (27) into Poisson's equation (13) to obtain the differential equation

$$\frac{d^2\varphi}{d\xi^2} = 4\pi e n_0\left\{\left(1 + \frac{e\varphi}{\varepsilon_F}\right)^{\frac{3}{2}} - \sqrt{\frac{-V^2/v_{T+}^2}{W_{0,-1}[-V^2/v_{T+}^2\exp(-V^2/v_{T+}^2 + 2e\varphi/m_+v_{T+}^2)]}}\right\}, \tag{29}$$

which has the first integral (similar to conservation law (18) under the condition $U(0) = 0$)

$$\frac{1}{2}\left(\frac{d\varphi}{d\xi}\right)^2 = -U(\varphi) \equiv 4\pi n_0\left[\frac{2\varepsilon_F}{5}\left(1+\frac{e\varphi}{\varepsilon_F}\right)^{5/2} - \frac{2\varepsilon_F}{5}\right.$$

$$- m_+ v_{T+}^2 \frac{v_{T+}}{V}\sqrt{-W_{0,-1}[-V^2/v_{T+}^2 \exp(-V^2/v_{T+}^2 + 2e\varphi/m_+ v_{T+}^2)]} \tag{30}$$

$$\left. + m_+ v_{T+}^2 \frac{V}{v_{T+}}\sqrt{-\frac{1}{W_{0,-1}[-V^2/v_{T+}^2 \exp(-V^2/v_{T+}^2 + 2e\varphi/m_+ v_{T+}^2)]}}\right]$$

and the general solution in quadratures

$$\xi - \xi_0 = \int_0^{\varphi} \frac{d\varphi'}{\sqrt{-2U(\varphi')}}. \tag{31}$$

Solutions (30) and (31) are too involved and inconvenient for analytic consideration. We verified numerically that the profiles of the pseudopotential are similar to those in Fig. 3 and that the waveforms of the physical quantities in the wave are similar to those in Fig. 4. This is why we do not present here the plots of the solutions.

Equations (20) and (23), which determine the maximum and minimum possible Mach numbers of a solitary wave, are not solvable analytically. For this reason, the dependence of the highest possible Mach number of a solitary wave on v_{T+} was calculated numerically. The corresponding curve, which is displayed in Fig. 6, is similar to that for a classical plasma [27] (it has a minimum at $m_+ v_{T+}^2/2\varepsilon_F \approx 0.1$ and emerges from the point 1.8495…, corresponding to cold ions) but lies markedly higher.

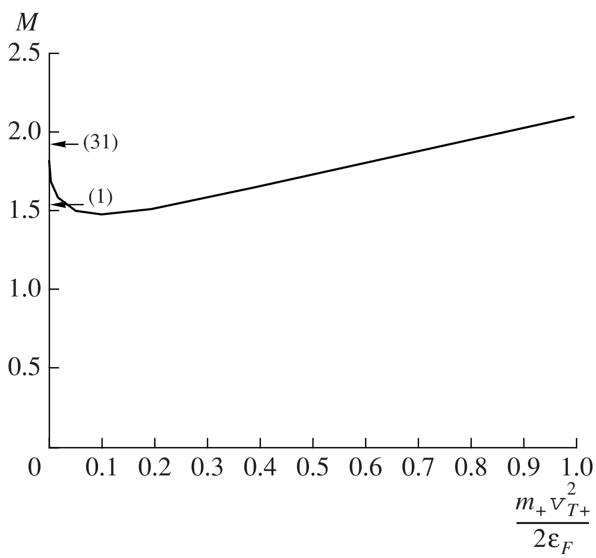

Fig. 6. Dependence of the highest possible Mach number of a solitary wave on v_{T+}. The arrows show maximum Mach numbers (1) and (31) for a cold classical plasma and a quantum plasma, respectively.

Hence, we have solved the problem of the structure of a steady-state ion-acoustic wave in an ideal plasma with degenerate electrons and classical isothermal ions. The main results of the present section are general solution (30) and (31) in quadratures and the plot in Fig. 6.

7. NONLINEAR THEORY OF ION-ACOUSTIC WAVES IN AN IDEAL PLASMA OF A CLASSICAL ADIABATIC ION GAS AND A DEGENERATE ELECTRON GAS

Here, we consider adiabatic ion-acoustic waves in an ideal plasma of a classical adiabatic ion gas and a degenerate electron gas. We make the same physical assumptions as in the previous sections but describe the ion gas not by the equation of state $P_+ = n_+\kappa T_+$ with $T_+ = \text{const}$ (see Section 6) but by the adiabatic equation

$$P_+ = n_0\kappa T_{0+}\left(\frac{n_+}{n_0}\right)^{\gamma_+}, \tag{32}$$

where T_{0+} is the temperature of the unperturbed ion gas. It is more justified to use this adiabatic equation of state because it eliminates the question of an external source or sink of energy in an ion-acoustic wave.

The basic set of equations consists of Eqs. (4) and (5) and the equation of motion

$$\frac{\partial v_+}{\partial t} + v_+\frac{\partial v_+}{\partial x} = -\frac{e}{m_+}\frac{\partial\varphi}{\partial x} - \frac{\gamma_+\kappa T_{0+}}{m_+}\left(\frac{n_+}{n_0}\right)^{\gamma_+-2}\frac{\partial}{\partial x}\left(\frac{n_+}{n_0}\right). \tag{33}$$

In the comoving frame of reference, this equation takes the form

$$-V\frac{dv_+}{d\xi} + v_+\frac{dv_+}{d\xi} = -\frac{e}{m_+}\frac{d\varphi}{d\xi} - \frac{\gamma_+}{\gamma_+-1}\frac{\kappa T_{0+}}{m_+}\frac{d}{d\xi}\left[\left(\frac{n_+}{n_0}\right)^{\gamma_+-1}\right]. \tag{34}$$

The solution to the continuity equation is again given by formula (14). Under the conditions $\lim_{v_+\to 0} n_+ = n_0$ and

$\lim_{v_+ \to 0} \varphi = 0$ and with formula (14), equation of motion (34) has the solution

$$\varphi = \frac{m_+ V^2}{2e}\left[1 - \left(\frac{n_0}{n_+}\right)^2\right] + \frac{\gamma_+}{\gamma_+ - 1}\frac{m_+ v_{T+}^2}{e}\left[1 - \left(\frac{n_+}{n_0}\right)^{\gamma_+ - 1}\right]. \tag{35}$$

For further analysis, we need the derivatives

$$\frac{d\varphi}{dn_+} = \frac{m_+}{en_0}\left[V^2\left(\frac{n_0}{n_+}\right)^3 - \gamma_+ v_{T+}^2\left(\frac{n_+}{n_0}\right)^{\gamma_+ - 2}\right]; \tag{36}$$

$$\frac{d^2\varphi}{dn_+^2} = \frac{m_+}{en_0^2}\frac{\gamma_+ v_{T+}^2 (n_+/n_0)^{\gamma_+ + 1} - 3V^2}{(n_+/n_0)^4}. \tag{37}$$

The plot of the function $\varphi(n_+)$ (35) is similar to that shown in Fig. 5. We again must discard the branch that does not satisfy the condition $\varphi(n_0) = 0$. The point φ_{max} can easily be determined by equating derivative (36) to zero and by solving the resulting equation for n_+:

$$n_{+max} = n_0\left(\frac{V^2}{\gamma_+ v_{T+}^2}\right)^{\frac{1}{\gamma_+ + 1}}; \tag{38}$$

$$\varphi_{max} = \frac{m_+}{e}\left\{\frac{V^2}{2}\left[1 - \left(\frac{V^2}{\gamma_+ v_{T+}^2}\right)^{-\frac{2}{\gamma_+ + 1}}\right] + \frac{\gamma_+}{\gamma_+ - 1} v_{T+}^2\left[1 - \left(\frac{V^2}{\gamma_+ v_{T+}^2}\right)^{\frac{\gamma_+ - 1}{\gamma_+ + 1}}\right]\right\}. \tag{39}$$

Recall that solution (35) cannot be explicitly resolved with respect to $n_+(\varphi)$. This is why, using the rule for differentiating a composite function,

$$\frac{d^2\varphi}{d\xi^2} = \frac{d\varphi}{dn_+}\frac{d^2 n_+}{d\xi^2} + \frac{d^2\varphi}{dn_+^2}\left(\frac{dn_+}{d\xi}\right)^2 \tag{40}$$

and substituting expressions (2) and (35)–(37), we rewrite Poisson's equation (13) as

$$\begin{aligned} &\frac{m_+}{en_0}[V^2(n_0/n_+)^3 - \gamma_+ v_{T+}^2 (n_+/n_0)^{\gamma_+ - 2}]\frac{d^2 n_+}{d\xi^2} \\ &+ \frac{m_+}{en_0^2}\frac{\gamma_+ v_{T+}^2 (n_+/n_0)^{\gamma_+ + 1} - 3V^2}{(n_+/n_0)^4}\left(\frac{dn_+}{d\xi}\right)^2 \\ &= 4\pi e n_0\left\{1 + \frac{m_+ V^2}{2\varepsilon_F}[1 - (n_0/n_+)^2] \right. \\ &\left. + \frac{\gamma_+}{\gamma_+ - 1}\frac{m_+ v_{T+}^2}{\varepsilon_F}[1 - (n_+/n_0)^{\gamma_+ - 1}]\right\}^{\frac{3}{2}} - 4\pi e n_+. \end{aligned} \tag{41}$$

The order of this equation can be lowered by making the replacement $p(n_+) = dn_+/d\xi$. As a result, we arrive at Bernoulli's differential equation

$$\frac{dp}{dn_+} = f_1(n_+)p + f_N(n_+)p^N, \tag{42}$$

in which the coefficients on the right-hand side are defined by

$$N = -1, \quad f_1(n_+) = -\frac{1}{n_+}\frac{\gamma_+ v_{T+}^2 (n_+/n_0)^{\gamma_+ + 1} - 3V^2}{(n_+/n_0)^4[V^2(n_0/n_+)^3 - \gamma_+ v_{T+}^2 (n_+/n_0)^{\gamma_+ - 2}]},$$

$$\text{and} \quad f_{-1}(n_+) = \frac{4\pi e n_0\left\{1 + \frac{m_+ V^2}{2\varepsilon_F}[1 - (n_0/n_+)^2] + \frac{\gamma_+}{\gamma_+ - 1}\frac{m_+ v_{T+}^2}{\varepsilon_F}[1 - (n_+/n_0)^{\gamma_+ - 1}]\right\}^{\frac{3}{2}} - 4\pi e n_+}{\frac{m_+}{en_0}[V^2(n_0/n_+)^3 - \gamma_+ v_{T+}^2 (n_+/n_0)^{\gamma_+ - 2}]}. \tag{43}$$

We use the fact that Bernoulli's equation always has a general solution in quadratures with a constant C_1 (see [36], Section 1.1.5, and also [37]):

$$p^2 = C_1 \exp\Phi(n_+) + 2\exp\Phi(n_+) \times \int \exp[-\Phi(n_+)] f_{-1}(n_+) dn_+, \tag{44}$$

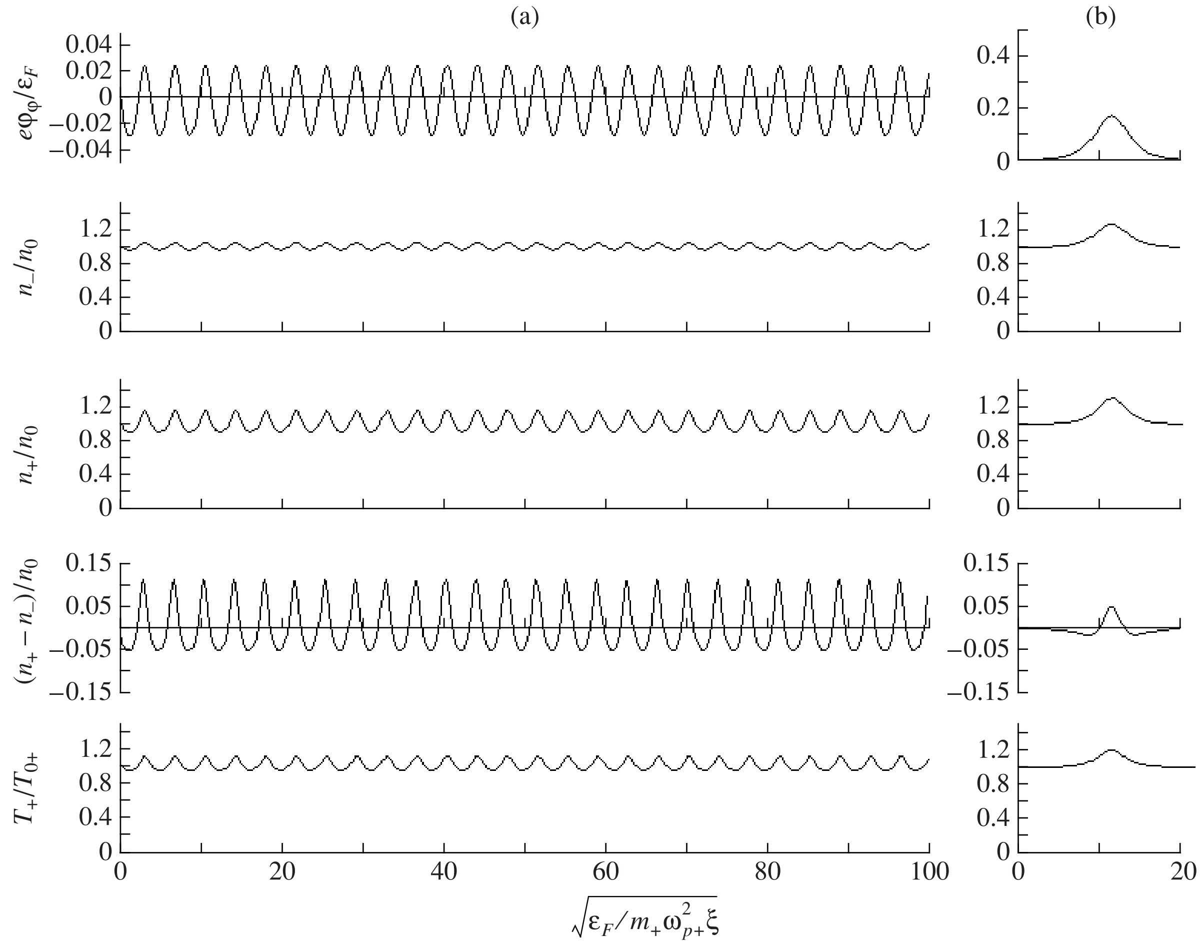

Fig. 7. Waveforms of the relative potential, relative electron density, relative ion density, relative space charge, and ion temperature (from top to bottom) in (a) a subsonic periodic wave with $M \approx 0.77$, $m_+ v_{T+}^2/2\varepsilon_F = 0.05$, and $\gamma_+ = 5/3$ and (b) a supersonic solitary wave with $M = 1.5$, $m_+ v_{T+}^2/2\varepsilon_F = 0.05$, and $\gamma_+ = 5/3$ in a plasma with adiabatic ions and degenerate electrons.

where

$$\Phi(n_+) = 2\int_{n_0}^{n_+} f_1(n_+)dn_+. \quad (45)$$

We return to the sought-for variable $n_+(\xi)$ to see that, under the conditions $n_+(0) = n_0$ and $\left.\frac{dn_+}{d\xi}\right|_{\xi = \xi_0} = 0$, the general solution to Poisson's equation (41) can be written as

$$\xi - \xi_0 = \int_{n_0}^{n_+} \frac{d\eta}{\sqrt{2\exp\Phi(\eta)\int_{n_0}^{\eta}\exp[-\Phi(\eta')]f_{-1}(\eta')d\eta'}}. \quad (46)$$

Since the solution given by formulas (45) and (46) is even more complicated than solution (19), as well as solutions (30) and (31), we illustrate it by particular examples. Figure 7 shows waveforms of the physical parameters of a periodic and a solitary ion-acoustic wave in a plasma with degenerate electrons and adiabatic ions. The temperature waveforms were calculated from the adiabatic equation as $T_+ = T_{0+}(n_+/n_0)^{\gamma_+ - 1}$. An analysis of the waveforms shows, in particular, that the amplitude and wavelength of a periodic wave decrease as the ion thermal velocity v_{T+} increases, all other conditions being the same.

The questions of the upper Mach number limit for a solitary wave and of how it depends on the ion temperature T_+ (or the ion thermal velocity v_{T+}) and adiabatic index γ_+ require a separate analysis. The results of relevant numerical calculations are illustrated in Figs. 8 and 9.

We can see that the dependences of the critical Mach number on the quantity $m_+ v_{T+}^2/\varepsilon_F$ are similar to that in

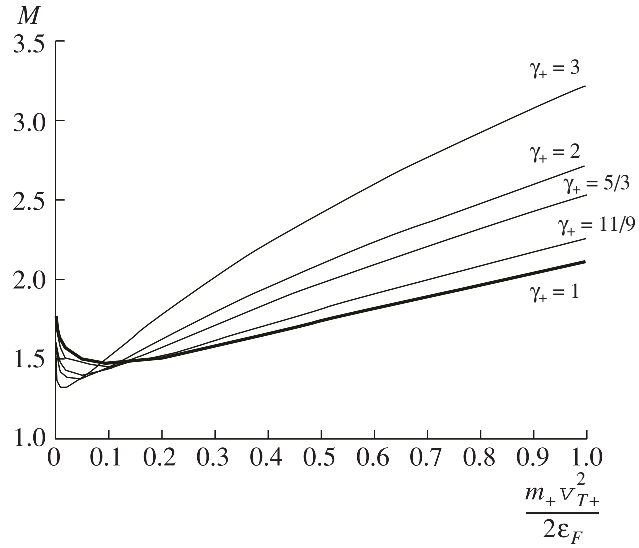

Fig. 8. Dependence of the highest possible Mach number of a solitary wave on v_{T+} for different values of γ_+. The heavy curve refers to a plasma with isothermal ions.

Fig. 6: they emerge from the point 1.8495… and have a minimum at $m_+ v_{T+}^2/\varepsilon_F \approx 0.2$. As γ_+ increases, the minimum decreases and is displaced toward smaller values of $m_+ v_{T+}^2/\varepsilon_F$. We can also see that, at high temperatures, the critical Mach number increases with γ_+. Figure 8 confirms that, in the limiting cases of an isothermal ($\gamma_+ \longrightarrow 1$) and a cold ($v_{T+} \longrightarrow 0$) plasma, the results obtained in this section coincide with the results obtained in Sections 6 and 5, respectively.

Since the adiabatic index for an ion gas can only take discrete values $\gamma_+ = (\varsigma + 2)/\varsigma$ (where $\varsigma = 1, 2, 3, \ldots$ is the number of degrees of freedom of an ion), the dependence of the critical Mach number on γ_+ for different values of $m_+ v_{T+}^2/\varepsilon_F$ can conveniently be represented as histograms (Fig. 9). Note by the way that, for pointlike atomic ions in one-dimensional problems, it is recommended to use $\gamma_+ = 3$. From the histograms we can see that, at low temperatures of the ion gas, the critical Mach number decreases with increasing γ_+, whereas at high ion temperatures, it increases with γ_+.

It is necessary to formulate the applicability limits of the adiabatic approach. As the ion-acoustic wave propagates, the temperature of the ion gas varies in such a way that it increases in the regions of adiabatic gas compression and decreases in the regions of adiabatic gas rarefaction. An ion-acoustic wave can be considered as an adiabatic process if the period of the wave is too short for heat to diffuse through a distance on the order of the wavelength λ. Numerical estimates of the thermal conductivity of a collisionless plasma in accordance with [38] and a comparison of the calculated wavelength with the effective spatial scale of the thermal conductivity show that, at least for the parameters of Fig. 7, heat diffusion can be ignored.

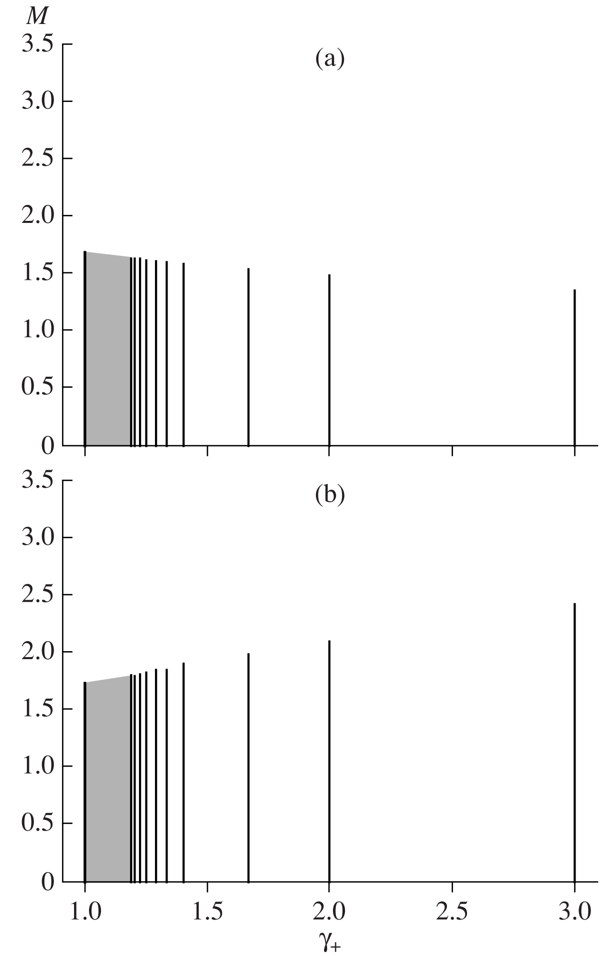

Fig. 9. Histograms illustrating the dependence of the highest possible Mach number of a solitary wave on γ_+ for $m_+ v_{T+}^2/\varepsilon_F$ = (a) 0.005 and (b) 0.5. The heavy bar refers to a plasma with isothermal ions. The region where the bars concentrate is hatched.

Hence, we have solved the problem of the structure of a steady-state ion-acoustic wave in an ideal plasma with degenerate electrons and classical adiabatic ions. The main results of the present section are general solution (45) and (46) in quadratures and the plots in Figs. 7–9.

8. CONCLUSIONS

We have constructed a nonlinear theory of steady-state ion-acoustic waves in an ideal plasma in which the electron component is a degenerate Fermi gas and the ion component is a classical gas.

We have determined the parameter ranges in which such a plasma can exist and have derived dispersion relations for ion-acoustic waves in such a plasma, which have enabled us to find the linear ion-acoustic velocity.

We have systematically developed analytic gas-dynamic models of ion sound in a plasma with an ion component as a cold, an isothermal, and an adiabatic gas. In all the models, the equations were solved by different mathematical methods and, moreover, all the solutions were brought to a quadrature form. We have calculated profiles of a subsonic periodic and a supersonic solitary wave and have determined the upper critical Mach numbers of a solitary wave. For a plasma with cold ions, the critical Mach number is expressed by an explicit exact formula. We have verified that, in the limiting cases $\gamma_+ \longrightarrow 1$ and $v_{T+} \longrightarrow 0$, the results obtained for a plasma with adiabatic ions coincide with those for a plasma with an isothermal and a cold ion component, respectively.

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Translated by O.E. Khadin