A zero-one law for a class of random walks and a converse to Gauss' mean value theorem

By WILLIAM A. VEECH*

1. Introduction

Let there be given a positive function, $\delta(\cdot)$, on a bounded region $\Omega \subseteqq \mathbf{R}^n$ such that for each $x \in \Omega$ $\delta(x) \leqq d(x, \partial\Omega)$, where $d(x, \partial\Omega)$ denotes the distance to the boundary of Ω. We associate to δ a collection of balls $B(x)$, $x \in \Omega$, requiring for each x that $B(x)$ have center x and radius $\delta(x)$. By our restriction on δ, $B(x) \subseteqq \Omega$ for every x. A Lebesgue measurable function f on Ω shall be called *δ-harmonic* if for each $x \in \Omega$ f is integrable on $B(x)$, and

$$m(B(x))f(x) = \int_{B(x)} f(y)m(dy)$$

where $m(\cdot)$ is Lebesgue measure.

If f is a positive harmonic function on Ω, then by the mean value theorem for harmonic functions, f is δ-harmonic for arbitrary δ. (Without the positivity assumption on f it is necessary to assume $\delta(x) < d(x, d\Omega)$, $x \in \Omega$.) In the present paper we shall give a collection of hypotheses on f, δ, and Ω, in the presence of which it is possible to prove that if f is δ-harmonic, then f is harmonic. It is in this sense that we obtain a "converse" to Gauss' mean value theorem.

A familiar result from classical potential theory asserts that if Ω is a region which is regular for the Dirichlet problem and if f is a function which is continuous on $\Omega \cup \partial\Omega$, then if f is δ-harmonic on Ω for *some* δ, f is harmonic. (cf. [7], and compare with [22].) The first nontrivial converse to the mean value theorem was given by Feller [11]: If Ω is the unit disc in the plane, if $\delta(x) = d(x, \partial\Omega)$, and if f is a bounded δ-harmonic function on Ω, then f is harmonic. See [1] for a proof of Feller's theorem.

There can be no converse to the mean value theorem which makes no assumption on f or δ. For a well-known example, let Ω be any nonempty region in $\mathbf{R}^n$, and let L be a hyperplane passing through the interior of Ω. Define f to be 1 on one side of $L \cap \Omega$, -1 on the other side, and 0 on $L \cap \Omega$. If δ is any function such that $\delta(x) \leqq \min\big(d(x, L), d(x, \partial\Omega)\big)$, $x \notin L$, and $\delta(x) \leqq d(x, \partial\Omega)$, $x \in L$, then f is δ-harmonic.

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There are at least two plausible explanations for the counterexample just given. The first is that the f we have defined is not continuous. Indeed, any f which is δ-harmonic for every δ above will be continuous on $\Omega \cap L^c$, and hence by Koebe's theorem [22] it will be harmonic on $\Omega \cap L^c$. If in addition f is continuous at every point of L, then f will be harmonic on Ω. The second explanation, which from our point of view is the more natural one, is that any $\delta(\cdot)$ satisfying the inequalities of the preceding paragraph must tend to 0 as $x \to L$, $x \notin L$. Thus δ cannot be bounded away from 0 on compact subsets of Ω. One consequence of Theorem 1 below is that if Ω is a bounded "Lipschitz domain", and if δ is bounded away from 0 on compact subsets of Ω, then every bounded, δ-harmonic function on Ω is harmonic. (It is possible that this result remains true if $L_0 = \{y \mid \lim_{x\to y} \inf \delta(x) = 0\}$ has capacity 0, but we have thus far been unable to decide this. In the examples above, $L_0 \supseteqq L \cap \Omega$ has positive capacity.) What follows is our main result:

THEOREM 1. *Let Ω be a bounded Lipschitz domain $\mathbf{R}^n$, $n \geqq 1$, and let f be δ-harmonic on Ω. A sufficient condition for f to be harmonic is that f and δ obey the growth restrictions* (i) *and* (ii):

(i) *There exists an harmonic function g on Ω such that $|f| \leqq g$.*

(ii) *δ is bounded away from 0 on every compact subset of Ω.*

A converse to the mean value theorem which lies between Feller's theorem and Theorem 1 has been given in [3]. There it is proved for Ω with differentiable boundary, f bounded, δ measurable, and for some constant $\alpha > 0$, $\delta(x) \geqq \alpha d(x, \partial\Omega)$, that if f is δ-harmonic, f is harmonic. The same theorem appears later in [13] with no restriction on $\partial\Omega$ but a stronger assumption on δ; namely, $\delta(x) \leqq \beta d(x, \partial\Omega)$ for some constant $\beta < 1$. In a private communication S. Orey has informed me that in joint work with Heath this latter assumption has been removed.

One of the reasons we are able to deal with δ-harmonicity under the rather weak hypothesis (ii) above is the following "minimal theorem" or "density theorem" which may be of independent interest. In its statement B_n is the unit ball in $\mathbf{R}^n$.

THEOREM 2. *For each n there exists a function $\varphi_n(\beta) > 0$, $0 < \beta \leqq 1$, with the following property: Given an arbitrary Lebesgue measurable set $E \subseteqq B_n$, $m(E) > 0$, there exists a point $x \in E$ such that $m(E \cap Q) \geqq \varphi_n(\beta) m(Q)$, $\beta = m(E)m(B_n)^{-1}$, for every ball Q such that $x \in Q \subseteqq B_n$.*

In an earlier (2 dimensional) version of this paper we have given a density theorem in the plane for squares which directly implies the density theorem for discs ([21], Theorem 2). If Lemma 9.1 of the present paper is true with "balls" replaced by "cubes", then our argument in Section 9 yields the corresponding theorem for cubes. Richard Hunt has shown us an extension and improvement of Theorem 2 of [21] which does contain the density theorem for cubes, and which gives the best possible estimate $\varphi_n(\beta) = C_n\beta^n$. (Private communication.) Hunt's result will appear elsewhere.

It may be possible to replace assumption (i) of Theorem 1 by the assumption $f \geqq 0$. There is some evidence of this in the probabilistic Martin boundary theory ([9], [15]) which can be taken over to the g-processes of Section 2. It is possible to show, using Theorem 1, that the g-harmonic functions [6] comprise a set of "harmonic measure" 1 in the boundary of the g-process. The case $f \geqq 0$ is in fact equivalent to the assertion that the exceptional set is empty. Since our results in this direction are incomplete, they will not be included here, but we do hope to have more to say on the matter at a future time.

We begin our proof of Theorem 1 in Section 2 with the simple observation that it is, in a natural way, equivalent to a zero-one law for certain random walks on Ω. This sort of observation has been made before; see for example [8, p. 442]. Sections 3–10 are devoted to proving the zero-one law.

2. The g processes

In the present section Ω, unless otherwise specified, is an arbitrary bounded region in $\mathbf{R}^n$. We begin by reducing Theorem 1 to the case in which both f and δ are Borel functions.

PROPOSITION 2.1. *Let f be a nonnegative δ-harmonic function (δ arbitrary). There exist Borel functions f_0 and δ_0 such that $f_0 \geqq 0$, $f_0 = f$ a.e., $\delta \leqq \delta_0$, and f_0 is δ_0-harmonic.*

Proof. Define $\Omega^* \subseteqq \mathbf{R}^{n+1}$ to be the set of pairs (x, δ), $x \in \Omega$, $0 < \delta \leqq d(x, \partial\Omega)$. Ω^* is a Borel set, and we define $F((x, \delta))$ on Ω^* as the average of f over the ball of radius δ centered at x (possibly $F = \infty$). By Fatou's lemma $\liminf_{(x',\delta')\to(x,\delta)} F((x', \delta')) \geqq F((x, \delta))$, and therefore F is Borel. There exists a Borel set $E \subseteqq \Omega$, $m(E) = m(\Omega)$ such that $f|_E$ is Borel. Define $\delta(x)$ on E by $\varepsilon(x) = \sup\{\delta \mid F((x, \delta)) = f(x)\}$. Then $F((x, \varepsilon(x))) = f(x)$, $x \in E$, by the monotone convergence theorem, and by definition $\varepsilon(x) \geqq \delta(x)$, $x \in E$. Now the set $\{x \in E \mid \varepsilon(x) > a\}$ is the projection onto the x coordinate of the Borel set $\{(x, \delta) \mid \delta > a\} \cap \{(x, \delta) \mid x \in E$ and $F((x, \delta)) = f(x)\}$. Thus, $\varepsilon(\cdot)$ is Lebesgue measurable (because analytic sets are Lebesgue measurable [5]), and there is a Borel set $E_1 \subseteqq E$ such that $m(E_1) = m(E) = m(\Omega)$, and $\varepsilon|_{E_1}$ is Borel.

Now define δ_0 on Ω by

$$\delta_0(x) = \begin{cases} \varepsilon(x) & x \in E_1 \\ d(x, \partial\Omega) & x \notin E_1 . \end{cases}$$

We have $\delta_0 \geqq \delta$ everywhere, δ_0 is Borel, and if $f_0(x) = F\big((x, \delta_0(x))\big)$, f_0 is Borel. By construction $f_0 = f$ a.e. and f_0 is δ_0-harmonic. (If F_0 is to f_0 as F is to f, then $F = F_0$ everywhere.)

Remark 2.2. In the construction above if $f \leqq g$ for some positive harmonic g, then $f_0(x) = F\big((x, \delta_0(x))\big) \leqq g(x)$ for all x, and the same estimate applies to f_0. To apply Proposition 2.1 assume f, δ, and g are as in the statement of Theorem 1. Let $h = f + g$. Then $0 \leqq h \leqq 2g$, and h is also δ-harmonic. If h_0 and δ_0 are as in Proposition 2.1, then $h_0 \leqq 2g$, also. Suppose we are able to prove h_0 is harmonic. Then for each $x \in \Omega$, $f(x) + g(x) = h(x)$, which is the average of $f + g$ over $B(x)$, is, since $f + g = h_0$ a.e. also the average of h_0 over $B(x)$. But the latter is just $h_0(x)$ by the mean value theorem, and therefore $f(x) + g(x) = h_0(x)$ for all x. Thus, $f = h_0 - g$ is harmonic.

In all that follows f and δ are Borel functions which satisfy the hypotheses (i) and (ii) of Theorem 1. Also, g is a positive harmonic function such that $|f| \leqq g$ on Ω. Using $\delta(\cdot)$ and $g(\cdot)$ we set up a kernel, P_g, on $\Omega \times \Omega$, defining

$$P_g(x, y) = \begin{cases} \dfrac{g(y)}{g(x)m\big(B(x)\big)} & y \in B(x) \\ 0 & y \notin B(x) . \end{cases}$$

P_g is a Borel function, and for each $x \in \Omega$, $P_g(x, y)dy$ $\big(dy = m(dy) =$ Lebesgue measure$\big)$ is a probability measure on Ω. Kernels and related objects ("g-harmonic functions") of this sort have been studied by various authors; see for example [6], [8, 9], [15], and [11]. Define $X = \Omega^N$, $N = \{1, 2, \cdots\}$, and let $x_1, x_2, \cdots$ be the coordinate functions on X. If ν is a Borel probability measure on Ω, then we define for every n and $A_1, \cdots, A_n \subseteqq \Omega$ Borel the measure $\mu_\nu^g(A)$ of the cylinder set $A = \{\omega \in X \mid x_j(\omega) \in A_j \; 1 \leqq j \leqq n\}$ to be

$$\mu_\nu^g(A) = \int_{A_1}\int_{A_2} \cdots \int_{A_n} \prod_{j=1}^{n-1} P_g(x_j, x_{j+1})\nu(dx_1)dx_2 \cdots dx_n .$$

By the Kolmogorov extension theorem μ_ν^g extends to a probability measure on the σ-algebra $\mathcal{B} = \mathcal{B}(x_1, x_2, \cdots)$ generated by the coordinate functions. In all that follows ν will be a point mass at some point x, and so we write μ_x^g for μ_ν^g.

Let $\mathcal{H} = \mathcal{H}(\Omega)$ be the cone of positive harmonic functions on Ω, and let $x_0 \in \Omega$ be any fixed point. We define $\mathfrak{M} = \mathfrak{M}_{x_0}$ to be the extreme points of the convex set $\{h \in \mathcal{H} \mid h(x_0) = 1\}$. If $\mathcal{H}$ is given the topology of local uniform convergence, $\mathfrak{M}$ is a Borel set, and the representation theorem of R. S. Martin [19] asserts that there exists a positive (unique) Borel measure $\lambda = \lambda(g)$ on $\mathfrak{M}$ such that for every $x \in \Omega$

$$g(x) = \int_{\mathfrak{M}} h(x)\lambda(dh) .$$

Using a direct computation on cylinder sets together with a monotone class argument [12], it is easy to prove that for each $E \in \mathcal{B}$, $x \in \Omega$, $h \to \mu_x^h(E)$ is a Borel function on $\mathfrak{M}$, and

$$g(x)\mu_x^g(E) = \int_{\mathfrak{M}} h(x)\mu_x^h(E)\lambda(dh) . \tag{2.1}$$

Let T be the left shift on X, $T\omega = \big(x_2(\omega), x_3(\omega), \cdots\big)$. Still another computation on cylinder sets together with a monotone class argument shows for each $A \in \mathcal{B}$ that

$$\mu_x^g(T^{-1}A) = \int_\Omega P_g(x, y)\mu_y^g(A)dy . \tag{2.2}$$

Define $P_g^{(1)} = P_g$, $P_g^{(n+1)}(x, y) = \int_\Omega P_g^{(n)}(x, z)P_g(z, y)dz$, $n = 1, 2, \cdots$. Then (2.2) generalizes to

$$\mu_x^g(T^{-n}A) = \int_\Omega P_g^{(n)}(x, y)\mu_y^g(A)dy . \tag{2.3}$$

Define $\mathcal{B}_I = \{A \in \mathcal{B} \mid A = T^{-1}A\}$. $\mathcal{B}_I$ is a σ-field called the *invariant σ-field* [4], and (2.3) implies for $A \in \mathcal{B}_I$

$$\mu_x^g(A) = \int_\Omega P_g^{(n)}(x, y)\mu_y^g(A)dy \qquad (n \geqq 1) . \tag{2.4}$$

PROPOSITION 2.3. *If x, $z \in \Omega$, then μ_x^g and μ_z^g are mutually absolutely continuous on $\mathcal{B}_I$.*

Proof. Assume $\mu_x^g(A) = 0$. Our assumption (ii) on $\delta(\cdot)$ implies $\{y \mid P_g^{(n)}(x, y) > 0\} \uparrow \Omega$, and therefore by (2.4) $\mu_y^g(A) = 0$ a.e. on Ω. Then setting $x = z$, $n = 1$ in (2.4) we see $\mu_z^g(A) = 0$. Thus μ_z^g is absolutely continuous on $\mathcal{B}_I$ with respect to μ_x^g.

Define $A \in \mathcal{B}_I$ to be the set

$$A = \left\{\omega \mid \lim_{n\to\infty} \frac{f\big(x_n(\omega)\big)}{g\big(x_n(\omega)\big)} = F(\omega) \text{ exists}\right\} .$$

Note that F is $\mathcal{B}_I$ measurable on A. Since f/g is stationary for the kernel P_g, and uniformly bounded on Ω, the process $f\big(x_n(\omega)\big)/g\big(x_n(\omega)\big)$ is a bounded martingale. Therefore, by the martingale theorem [10] $\mu_x^g(A) = 1$, $x \in \Omega$. Moreover, we have for all x the representation

$$f(x) = g(x)\int_X F(\omega)\mu_x^g(d\omega) .$$

Applying (2.1) we see

$$\begin{aligned} f(x) &= g(x)\int_X F(\omega)\mu_x^g(d\omega) \\ &= g(x)\int_{\mathfrak{M}} \frac{h(x)}{g(x)}\lambda(dh)\Big\{\int_X F(\omega)\mu_x^h(d\omega)\Big\} \qquad (2.5) \\ &= \int_{\mathfrak{M}} h(x)\Big\{\int_X F(\omega)\mu_x^h(d\omega)\Big\}\lambda(dh) . \end{aligned}$$

Now suppose it is known for every $h \in \mathfrak{M}$ that $\mathcal{B}_I$ is μ_x^h trivial. Then by Proposition 2.3 there is for each h such that $\mu_x^h(A) = 1$, $x \in \Omega$, a constant, $F_0(h)$, such that $F = F_0(h)$ a.e. μ_x^h, all x. Thus, (2.5) becomes

$$f(x) = \int_{\mathfrak{M}} h(x)F_0(h)\lambda(dh) \qquad (2.6)$$

and f is harmonic. Therefore, Theorem 1 will be a consequence of

THEOREM 2.4. *If Ω is a bounded Lipschitz domain in $\mathbf{R}^n$, $n \geqq 1$, then $\mathcal{B}_I$ is μ_x^h trivial for every $h \in \mathfrak{M}(\Omega)$.*

Again we assume Ω is an arbitrary bounded region in $\mathbf{R}^n$. If $h \in \mathfrak{M}(\Omega)$, Brelot has shown h has a "pole" $P \in \partial\Omega$ with the following property [6, Theorem 21 and §§ 10, 11]: For every $x \in \Omega$ and $\varepsilon > 0$ there exists an open set $\mathcal{O} \supseteqq \partial\Omega - \{P\}$ and a superharmonic function h_ε on Ω such that $h_\varepsilon = h$ on $\mathcal{O} \cap \Omega$, $h_\varepsilon \leqq h$ on Ω, and $h_\varepsilon(x) \leqq \varepsilon h(x)$. If we form the process $h_\varepsilon(\xi_n)/h(\xi_n)$ for μ_x^h, $\xi_n = x_n(\omega)$, it is a bounded supermartingale which converges a.e. to a function $H_\varepsilon(\omega)$ satisfying

$$\begin{aligned} \varepsilon \geqq \frac{h_\varepsilon(x)}{h(x)} &\geqq \int_X H_\varepsilon(\omega)\mu_x(d\omega) \\ &\geqq \mu_x\{\omega \mid \xi_n \in \mathcal{O} \text{ infinitely often}\} . \end{aligned}$$

Since ε is arbitrary we see that for almost all sample paths for μ_x^h there is at most one cluster point on $\partial\Omega$, namely P. Compare the following proposition with [8], Theorem 5.1.

PROPOSITION 2.5. *If $h \in \mathfrak{M}(\Omega)$, almost every μ_x^h path converges to the pole of h on $\partial\Omega$.*

Proof. Let $M = \sup \|x\|$, $x \in \Omega$, and let $x^{(k)}$ be the k^{th} coordinate of $x \in \mathbf{R}^n$. Each of the processes $h(x_j)^{-1}$, $(M + x_j^{(k)})h(x_j)^{-1}$, $k = 1, \cdots, n$, is a nonnegative martingale for μ_x^h, hence convergent a.e. to a finite limit. If $\lim_{j\to\infty} h(x_j)^{-1} > 0$, then for each k $\lim_{j\to\infty} x_j^{(k)} = \lim_{j\to\infty} (x_j^{(k)} h(x_j)^{-1}) h(x_j)$ exists, hence $\lim_{j\to\infty} x_j$ exists. If $\lim_{j\to\infty} h(x_j)^{-1} = 0$, then $\lim_{j\to\infty} d(x_j, \partial\Omega) = 0$, and therefore by the discussion preceding the proposition $\lim_{j\to\infty} x_j = P$ exists. Thus, we have that almost every μ_x^h path is *convergent*. Next, notice that by Harnack's theorem there exists a dimensional constant L such that

$$E(\| x_{j+1} - x_j \| \mid x_1, \cdots, x_j) \geqq L\delta(x_j) .$$

Since $\lim x_j$ exists in $L^1(\mu_x^h)$, it must be that $\delta(x_j)$ converges to 0 in measure. By our condition (ii) on δ it must be that $\lim x_j \in \partial\Omega$ a.e.. Using once more the observation made preceding the proposition, we have $\lim x_n = P$, a.e., as claimed.

We remark that if g is an arbitrary positive harmonic function on Ω, then (2.1) and Proposition 2.5 imply convergence for μ_x^g paths. The distribution of $\lim x_n$ on $\partial\Omega$ is just the probability measure whose generalized Poisson integral is g if, say, Ω is a Lipschitz domain.

Definition 2.6. Let Ω be a bounded region, and suppose $P \in \partial\Omega$. A positive harmonic function h on Ω is said to be a *kernel function* at P if $\lim_{x\to y} h(x) = 0$ for all $y \in \partial\Omega$, $y \neq P$.

A simple modification of the argument in Proposition 2.5 shows that if h is a kernel function at P, then μ_x^h paths converge to P with probability 1.

It is proved in [16] that for a Lipschitz domain every $h \in \mathcal{M}$ is a kernel function (at its pole). (As mentioned in [16] this is a consequence of the uniform estimate (2.5) of [16] and the general theory of R. S. Martin.) Theorem 2.4 will follow from

THEOREM 2.7. *If Ω is a bounded Lipschitz domain, and if h is a kernel function at some $P \in \partial\Omega$, then $\mathcal{B}_I$ is μ_x^h trivial.*

Remark 2.8. Let h be as in Theorem 2.7, and suppose a δ-harmonic function, h_0, satisfies $0 \leqq h_0 \leqq h$. Then by the usual martingale argument h_0/h can be represented as the integral of a $\mathcal{B}_I$ measurable function with respect to μ_x^h. By Theorem 2.7 and the mutual absolute continuity on $\mathcal{B}_I$, there is constant c such that $h_0 = ch$. In other words, h is an *extremal* in the cone of positive δ-harmonic functions. A special case of this fact is the Hunt-Wheeden uniqueness theorem:

THEOREM 2.9 (*R. Hunt and Wheeden* [16]). *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^n$, and let h_1 and h_2 be kernel functions at a point $P \in \partial\Omega$. Then $h_1 = ch_2$ for some constant c.*

Proof. Apply Remark 2.8 for some δ, say $\delta(x) = (1/2)d(x, \partial\Omega)$, and $h = h_1 + h_2$.

3. An auxiliary function

Let $G(x, y)$ be the Greens function for the open unit ball in $\mathbf{R}^n$, $n \geqq 2$ fixed. If B_ρ is the ball of radius ρ and G_ρ the corresponding Greens function, we have the relation $G_\rho(x, y) = \rho^{-(n-2)}G(x/\rho, y/\rho)$ [14].

There exists a number $r_0 > 0$, fixed in what follows such that $a \geqq b + 1$, where

$$a = a(r_0) = \min_{\|x\|, \|y\| \leqq r_0} G(x, y)$$
$$b = b(1/3) = \max_{\substack{\|x\| \leqq r_0 \\ \|y\| \geqq 1/3}} G(x, y)$$

a and b are also fixed.

Let $A \subseteqq B_\rho$ be a measurable set. We define $\varphi_{A;\rho}(\cdot)$ on B_ρ to be the Greens potential

$$\varphi_{A;\rho}(x) = \frac{1}{\rho^2}\int_A G_\rho(x, y)m(dy) .$$

By a change of variables we have

$$\begin{aligned} \varphi_{A;\rho}(x) &= \frac{1}{\rho^2}\int_A G_\rho(x, y)m(dy) \\ &= \frac{1}{\rho^2}\int_A \frac{1}{\rho^{(n-2)}}G\Big(\frac{x}{\rho}, \frac{y}{\rho}\Big)m(dy) \qquad (3.1) \\ &= \int_{A_0} G\Big(\frac{x}{\rho}, z\Big)m(dz) \end{aligned}$$

where $A_0 = \{y/\rho \mid y \in A\}$. If $n = 2$, $G(x, \cdot)$ is locally L^p for $p < \infty$, while if $n > 2$, $G(x, \cdot)$ is locally L^p for $p < n/(n - 2)$. Let $p = 2$ if $n = 2$ and $p = (n - 1)/(n - 2)$ if $n > 2$. The dual (Holder) exponents are $q = 2$ if $n = 2$ and $q = n - 1$ if $n > 2$. Define

$$\Lambda_n = \sup_{\|x\| \leqq 1} \| G(x, \cdot) \|_p$$

where p is as above. $\Lambda_n < \infty$ because when $n=2$, $G(x, y) \leqq |\log \|x - y\|| + \log 2$, and when $n \geqq 3$, $G(x, y) \leqq \|x - y\|^{2-n}$. Using (3.1), we find

$$\varphi_{A;\rho}(x) \leqq \Lambda_n m(A_0)^{1/q} \qquad (3.2)$$

where q is the dual exponent above.

Let σ_n be the volume of the unit ball in $\mathbf{R}^n$. Then $m(A_0) = \rho^{-n}m(A) = \sigma_n m(A)/m(B_\rho)$, and (3.2) can be rewritten as

$$\varphi_{A;\rho}(x) \leqq \Lambda_n \sigma_n^{1/q}\left(\frac{m(A)}{m(B_\rho)}\right)^{1/q} . \tag{3.3}$$

Let $\gamma_n = \Lambda_n \sigma_n^{1/q}$, a constant which depends only upon n.

If $A \subseteqq B_{r_0\rho}$, then by (3.2) we have for $||x|| \geqq \rho/3$

$$\varphi_{A;\rho}(x) \leqq bm(A_0) = b\sigma_n \frac{m(A)}{m(B_\rho)} . \tag{3.4}$$

Let $\delta_n = b\sigma_n$. δ_n depends only upon n. If $x \in B_{r_0\rho}$, then by the choice of r_0

$$\begin{aligned} \varphi_{A;\rho}(x) &\geqq a\sigma_n \frac{m(A)}{m(B_\rho)} \\ &> (b+1)\sigma_n \frac{m(A)}{m(B_\rho)} \\ &= \delta_n \frac{m(A)}{m(B_\rho)} + \sigma_n \frac{m(A)}{m(B_\rho)} . \end{aligned} \tag{3.5}$$

Fix any $x \in B_{r_0\rho}$, and suppose $\lambda(\cdot)$ is a probability measure on B_ρ with the property that

$$\varphi_{A;\rho}(x) = \int_{B\rho} \varphi_{A;\rho}(y)\lambda(dy) \tag{3.6}$$

(x fixed). We will obtain a simple estimate for $u = \lambda(B_{\rho/3})$. Using (3.2–3.6) we find

$$\begin{aligned} & u\gamma_n \left(\frac{m(A)}{m(B_\rho)}\right)^{1/q} + (1-u)\delta_n \frac{m(A)}{m(B_\rho)} \\ &\geqq \varphi_{A;\rho}(x) \\ &\geqq \delta_n \frac{m(A)}{m(B_\rho)} + \sigma_n \frac{m(A)}{m(B_\rho)} . \end{aligned} \tag{3.7}$$

Let $\tau = \tau(A) = m(A)/m(B_\rho)$, and solve (3.7) for u to find

$$u \geqq \frac{\sigma_n \tau}{\lambda_n \tau^{1/q} - \delta_n \tau} \tag{3.8}$$

the right side being a positive function of τ, $0 \leqq \tau \leqq r_0^n$, which we denote by $u = u_n(\tau)$.

4. Local behavior of the P_g process

It will be convenient to restate Theorem 2 in a more directly applicable form. (See § 9 for the proof of Theorem 2.) If E is a measurable subset of B_n we define the "minimal function" of E by

$$\alpha(x, E) = \inf_{Q \in b(x)} \frac{m(E \cap Q)}{m(Q)}$$

where for each x $b(x)$ denotes all balls in B_n which contain x. Theorem 2 asserts that $\|\alpha\|_\infty \geqq \varphi_n(\beta)$, $\beta = m(E)m(B_n)^{-1}$.

THEOREM 4.1. *If E is a Lebesgue measurable subset of B_n, define for $0 < \lambda \leqq 1$*

$$E_\lambda = \{x \in E \mid \alpha(x, E) \geqq \varphi_n(\lambda\beta)\}$$

where $\beta = m(E)m(B_n)^{-1}$. Then $m(E_\lambda) \geqq (1 - \lambda)\beta$.

Proof. Clear.

Remark 4.2. We think of a set such as E_λ as being the "core" of E.

Remark 4.3. If B is any ball in $\mathbf{R}^n$, and if $E \subseteqq B$ is measurable, then it makes sense to define $\alpha(x, E)$, $x \in B$. (Let $b(x)$ be the balls in B which contain x.) If $E_\lambda = \{\alpha \geqq \varphi_n(\lambda\beta)\}$, then $m(E_\lambda) \geqq (1 - \lambda)m(E)$.

Remark 4.4. We fix $g \equiv 1$ and $P(x, y) = P_g(x, y)$. If $\mathcal{O}_1$, $\mathcal{O}_2 \subseteqq \Omega$ are Borel sets, and if $E \subseteqq \mathcal{O}_1$ is the set $E = \{x \in \mathcal{O}_1 \mid P(x, \mathcal{O}_2) > 0\}$, then E is a Borel set. Here $P(x, \mathcal{O}_2) = \int_{\mathcal{O}_2} P(x, y)dy$. Indeed, the function $x \rightarrow P(x, \mathcal{O}_2)$ is Borel, and E is just the intersection of two Borel sets. Curiously, if we replace E by $E_0 = \{x \in \mathcal{O}_1 \mid B(x) \cap \mathcal{O}_2 \neq \emptyset\}$, $B(x) =$ ball of radius $\delta(x)$ centered at x, then it appears the best one can say is that E_0 is Lebesgue measurable (an analytic set).

In preparation of Lemma 4.5, define $B_\rho(z)$ to be the ball of radius ρ centered at z for some $\rho < d(x, \partial\Omega)$, and suppose $A \subseteqq B_{r_0\rho}(z)$, r_0 as in Section 3, is a Borel set. We define $\tau = m(A)/m(B_\rho(z))$ and $\sigma = \sigma(\rho, z) = \sup \delta(y)$, $y \in B_{\rho/3}(z)$.

LEMMA 4.5. *Let the notations and assumptions be as above. For every $x \in B_{r_0\rho}(z)$ we have*

$$\mu_x\{\omega \mid x_m(\omega) \in A, \text{ some } m \geqq 1\} \geqq u_n(\tau)\varphi_n(\beta/2) \tag{4.1}$$

where $\beta = m(A)/(m(B_n)(\rho + \sigma)^n)$, u_n is as in Section 3, and φ_n is as in Theorem 2.

Proof. We regard $B_\rho(z)$ as a subset of the ball $B = B_{\rho+\sigma}(z)$. Let $A_0 \subseteqq A$ be a Borel set such that if $x \in A_0$, then $\alpha(x, A) \geqq \varphi_n(\beta/2)$ and $m(A_0) \geqq (1/2)m(A)$. A_0 exists by Theorem 4.1. By definition, if D is a ball in B which intersects A_0, then

$$m(D \cap A) \geqq \varphi_n(\beta/2)m(D) . \tag{4.2}$$

In particular, if $x \in B_{r_0\rho}(z)$ is such that $B(x) \cap A_0 \neq \emptyset$, then the one step probability of travelling to A is at least $\varphi_n(\beta/2) \geqq \varphi_n(\beta/2)u_n(\tau)$. If x does not have this property, define

$$G = \{y \in B_\rho(z) \mid P(y, F) > 0,\ F = A_0 \cup B_\rho(z)^c\}\ .$$

Because $\lim_{m\to\infty} x_m(\omega) \in \partial\Omega$ a.e. μ_x and $\rho < d(x, \partial\Omega)$ by assumption, it must be that $x_m(\omega) \notin B_\rho(z)$ for some $m > 1$ with probability 1. It follows that the stopping time

$$T(\omega) = \inf\{m \geqq 1 \mid x_m(\omega) \in G\}$$

is finite a.e. μ_x. Define stopping times T_m by $T_m(\omega) = \min(m, T(\omega))$, and let $y_m(\omega)$ be the process $y_m(\omega) = x_{T_m(\omega)}(\omega)$. $\{y_m\}$ is a martingale. In fact, the transition probabilities for $\{y_m\}$ are the same as those for $\{x_m\}$, except on G where the process stops. With this in mind we see that the process

$$g_m(\omega) = \varphi_{A;\rho}(y_m(\omega))$$

is a bounded martingale where $\varphi_{A;\rho}$ is the function defined in Section 3, with the definition translated to $B_\rho(z)$. Let $y_\infty(\omega) = \lim_{m\to\infty} y_m(\omega)$ ($\in G$ with probability 1), and let $\lambda_x(\cdot)$ be the distribution of y_∞ on G. We have

$$\varphi_{A;\rho}(x) = \int_G \varphi_{A;\rho}(y)\lambda_x(dy)$$

and therefore by the previous section, $\lambda(B_{\rho/3}(z)) > u_n(\tau)$. Now if $y \in B_{\rho/3}(z) \cap G$, then $P(y, A_0) > 0$, and therefore, $P(y, A) \geqq \varphi_n(\beta/2)$. We conclude from the strong Markov property [18] that

$$\mu_x\{\omega \mid x_{T+1}(\omega) \in A\} \geqq \varphi_n(\beta/2)u_n(\tau)$$

and the lemma is proved.

We continue to suppose A is a Borel set in $B_{r_0\rho}(z)$, $\rho < d(z, \partial\Omega)$. Let $A' = A^c \cap \Omega$, and define for $x \in B_{r_0\rho}(z)$, $y \in \Omega$, $n = 1, 2, \cdots$

$$Q_1(x, y) = P(x, y)$$

$$Q_n(x, y) = \int_{A'}\int_{A'} \cdots \int_{A'} \prod_{j=1}^n P(x_j, x_{j+1})dx_2 \cdots dx_n$$

where in the second formula $x_1 = x$, $x_{n+1} = y$. If $g > 0$ is harmonic, there are similarly defined quantities $Q_n^g(x, y)$. The integral

$$\int_A \sum_{n=1}^\infty Q_n^g(x, y)m(dy) = R_g(x, A)$$

gives the probability $\mu_x^g\{\omega \mid x_n(\omega) \in A$ for some $n \geqq 2\}$. By Harnack's theorem and the fact

$$\prod_{j=1}^n P_g(x_j, x_{j+1}) = \frac{g(x_{n+1})}{g(x_1)} \prod_{j=1}^n P(x_j, x_{j+1})$$

there exists a constant $M = M_n(r_0)$, depending only upon the dimension (n) and r_0 such that

$$\frac{1}{M}R_g(x, A) \leqq R_1(x, A) \leqq MR_g(x, A) \tag{4.3}$$

for $x \in B_{r_0\rho}(z)$ and $A \subseteqq B_{r_0\rho}(z)$. We have proved:

LEMMA 4.6. *With notations as in Lemma* 4.5 *and* (4.3) *we have for* $x \in B_{r_0\rho}(z)$ *and* $A \subseteqq B_{r_0\rho}(z)$ *Borel*

$$\mu_x^g\{\omega \mid x_m(\omega) \in A \ \text{for some } m \geqq 2\} \geqq \frac{1}{M}\varphi_n(\beta/2)u_n(\tau) . \tag{4.4}$$

Recall that $\tau = m(A)/m(B_\rho(z))$, $\sigma = \sup \delta(y)$, $y \in B_{\rho/3}(z)$, and $\beta = m(A)m(B_n)^{-1}(\rho + \sigma)^{-n}$.

5. Regions of type (R, k, Σ, t)

Let Ω, $\delta(\cdot)$, and r_0 be as before. A region $\mathcal{O} \subseteqq \Omega$ has *type* (R, k, Σ, t) if

(i) $d(z, \partial\Omega) > R$ for all $z \in \mathcal{O}$

(ii) $\delta(y) \leqq \Sigma$, $y \in B_{R/3}(z)$, $z \in \mathcal{O}$

(iii) There exists a point $w \in \mathcal{O}$ and for every $z \in \mathcal{O}$ a sequence $D_1, \cdots, D_k$ of balls, $D_j = B_R(z_j)$, some $z_j \in \mathcal{O}$, such that

(a) $z \in B_{r_0R}(z_1)$, $w = z_k$

(b) $m(B_{r_0R}(z_j) \cap B_{r_0R}(z_{j+1})) \geqq tm(B_R(0))$ $1 \leqq j \leqq k - 1$.

If $\mathcal{O}$ has type (R, k, Σ, t), we associate to it the number $\beta = tm(B_R)m(B_n)^{-1}(R + \Sigma)^{-n}$. Notice that if $A_j = B_{r_0R}(z_j) \cap B_{r_0R}(z_{j+1}) \subseteqq B_{r_0R}(z_j)$, the number β_j associated to A_j in Lemma 4.5 is at least as large as β. Note also that β is actually a function of t and R/Σ.

LEMMA 5.1. *Let $\mathcal{O}$ have type (R, k, Σ, t), and let $\omega \in \mathcal{O}$ be as in the definition. If g is a positive harmonic function on Ω, then for every $x \in \mathcal{O}$*

$$\mu_x^g\{\omega \mid x_m(\omega) \in B_{r_0R}(w) \ \text{for some } m \geqq 1\} \geqq \left(\frac{1}{M}\varphi_n(\beta/2)u_n(t)\right)^{k-1} \tag{5.1}$$

where M is as in (4.4).

Proof. The proof is a simple induction based on the strong Markov property and (4.4). If $k = 1$, there is nothing to prove. Let us assume (5.1) is known for pairs x, ω which can be connected by balls $D_1, \cdots, D_k$ as in the definition. Given balls $D_1, \cdots, D_{k+1}$, $D_k = B_R(z_k)$, connecting x to w we have by the induction hypothesis that

$$\mu_x^g\{\omega \mid x_m(\omega) \in B_{r_0R}(z_k), \ \text{some } m \geqq 1\} \geqq \left(\frac{1}{M}\varphi_n(\beta/2)u_n(t)\right)^{k-2} .$$

Let $T(\omega) = \min\{m \mid x_m(\omega) \in B_{r_0R}(z_k)\}$, and define $T_l(\omega) = \min(l, T(\omega))$. By the strong Markov property and (4.4) we have for each l

$$\begin{aligned}&\mu^g_x\{\omega \mid x_m(\omega) \in B_{r_0R}(w), \ \text{some } m \geqq 1\}\\ &\geqq \mu^g_x\{\omega \mid T_l(\omega) = T(\omega) \ \text{and} \ x_{m+T}(\omega) \in B_{r_0R}(w) \ \text{some } m \geqq 1\}\\ &\geqq \mu^g_x\{\omega \mid T_l(\omega) = T(\omega)\}\Big(\frac{1}{M}\varphi_n(\beta/2)u_n(t)\Big) .\end{aligned}$$

Letting $l \to \infty$, the Lemma obtains.

Remark 5.2. It has been convenient but hardly necessary to use the density theorem for Lemma 5.1. The reason the density theorem is unnecessary is that it is obviously possible to calculate directly the core of a region which is the intersection of two balls. Lemma 5.3 below contains the important application of (4.4).

LEMMA 5.3. *Let the notations and assumptions be as in Lemma* 5.1. *If* $A \subseteqq B_{r_0R}(w)$ *is a Borel set such that* $m(A) \geqq sm(B_R)$, *and if* $\beta' = sm(B_R)m(B_n)^{-1}(R + \Sigma)^{-n}$, *then for all* $x \in \mathcal{O}$

$$\begin{aligned}&\mu^g_x\{\omega \mid x_m(\omega) \in A, \ \textit{some } m \geqq 1\}\\ &\geqq M^{-k}\big(u_n(t)\varphi_n(\beta/2)\big)^{k-1}u_n(s)\varphi_n(\beta'/2) .\end{aligned} \tag{5.2}$$

Proof. Apply Lemma 5.1 and (4.4) together with the strong Markov property as in the proof of Lemma 5.1.

Suppose F is a Borel function on Ω, $0 \leqq F \leqq 1$, such that $P_gF = F$. That is, for all $x \in \Omega$

$$F(x) = \int_\Omega P_g(x, y)F(y)dy .$$

We wish to obtain a uniform bound (either an upper bound or a lower bound) for F on any region $\mathcal{O} \subseteqq \Omega$ of type (R, k, Σ, t).

Let w be the distinguished point in $\mathcal{O}$, and define complementary sets $A_i \subseteqq B_{r_0R}(w)$, $i = 1, 2$, by

$$\begin{aligned}A_1 &= \{y \in B_{r_0R}(w) \mid F(y) \leqq 1/2\}\\ A_2 &= \{y \in B_{r_0R}(w) \mid F(y) > 1/2\} .\end{aligned}$$

If $i = 1$ or 2 is such that $m(A_i) \geqq (1/2)\, m(B_{r_0R})$, the number $s = s_i$ associated to A_i in Lemma 5.3 is at least $(1/2)r_0^n$. For the same set A_i it is clear that $s \geqq t/2$, and therefore $\beta' \geqq 2^{-n}\beta$. It follows from (5.2), the fact β is a function of t and R/Σ, and the preceding discussion that for all $x \in \mathcal{O}$

$$\mu^g_x\{\omega \mid x_m(\omega) \in A_i \ \text{for some } m\} \geqq \eta(R/\Sigma, k, t) \tag{5.3}$$

where η is some positive function on $(R/\Sigma, k, t)$ space.

Let us suppose it was $i = 1$ in the above. Define $T(\omega)$ on X to be the least value of m, if any, such that $x_m(\omega) \in A_1$ and as usual define $T_l(\omega) = \min(l, T(\omega))$. The process $F_m(\omega) = F(x_{T_m})$ is a bounded martingale, [10],

p. 300, and therefore for each m

$$\begin{aligned} F(x) &= \int_X F_m(\omega)\mu_x^g(d\omega) \\ &\leqq \frac{1}{2}\lambda_m + 1 - \lambda_m \\ &= 1 - \frac{1}{2}\lambda_m \end{aligned}$$

where $\lambda_m = \mu_x^g\{\omega \mid T_m(\omega) = T(\omega)\}$. Letting $m \to \infty$ we see

(5.4) $$F(x) \leqq 1 - \frac{1}{2}\eta(R/\Sigma, k, t) \quad (x \in \mathcal{O}) .$$

Similarly, if $m(A_2) \geqq (1/2)m(B_{r_0R})$, we find

(5.5) $$F(x) \geqq \frac{1}{2}\eta(R/\Sigma, k, t) \quad (x \in \mathcal{O}) .$$

Remark 5.4. The function $\eta(\cdot)$ is uniformly bounded away from 0 on any set of the form

$$\{(R/\Sigma, k, t) \mid R/\Sigma \geqq c,\ k \leqq b,\ t \geqq a\}$$

where a, $c > 0$ and $b < \infty$. This remark will be useful in Section 6, where in certain arguments a number of sets $\mathcal{O}$ occur having such simultaneous bounds on the associated $(R/\Sigma, k, t)$.

6. Lipschitz domains

Let $\Omega \subseteqq \mathbf{R}^m$, $m \geqq 2$, be a bounded Lipschitz domain. This means by definition that for each $P \in \partial\Omega$ there is a local coordinate (z, t), $z \in \mathbf{R}^{m-1}$, $t \in \mathbf{R}$, a Lipschitz function $b(z)$, $z \in \mathbf{R}^{m-1}$, and a neighborhood U of $P = (z_0, t_0)$ such that

$$\Omega \cap U = \{(z, t) \mid b(z) < t\} \cap U .$$

As noted in Section 2, the results of [16] imply that if Ω is Lipschitz, then every $g \in \mathfrak{M}(\Omega)$ is a kernel function at some $P \in \partial\Omega$. We fix P and a kernel function g at P. In all that follows μ_x refers to μ_x^g, when no confusion can arise, and we recall that μ_x almost every sample path converges to P.

The following lemma is proved in Section 8.

LEMMA 6.1. *There exists a function* $\gamma(\alpha) = \gamma_m(\alpha) < 0$, $0 < \alpha < 1$, $m =$ *dimension, with the following property: Given balls* B, D_1, D_2 *such that*

(a) $D_1 \subseteqq D_2$ *concentrically;*

(b) $r_1 \leqq \alpha r_2$, $r_j =$ *radius* D_j, $j = 1, 2$;

(c) *center* $B \notin D_2$

and a postive harmonic function g *on* B, *then*

$$\int_{B \cap \Delta} g(z)dz \geqq \gamma(\alpha) \int_{B \cap D_2} g(z)dz \tag{6.1}$$

where $\Delta = D_1^c \cap D_2$. *Moreover,* $\lim_{\alpha \to 0} \gamma(\alpha) = 1$.

Definition 6.2. If $P \in \partial\Omega$ is as above, we define for $d > 0$ an open set $S(d, P)$

$$S(d, P) = \{x \in \Omega \mid d/2 < \|x - P\| < d\} .$$

In what follows d will be assumed small enough to insure that $S(d, P) \subseteqq U$, where U is the neighborhood of P at the beginning of this section.

LEMMA 6.3. *There exists a number* $\gamma > 0$ *such that if* $x \in \Omega$ *and* $\|x - P\| > d$, *then the* $\mu_x(=\mu_x^g)$ *probability that the first visit to* $B_d(P)$ *is made in* $S(d, P)$ *is at least* γ.

Proof. Let $x' \in \Omega$ be such that $x' \notin B_d(P)$, but $B(x') \cap B_d(P) \neq \emptyset$. Taking $\gamma = \gamma(1/2)$ in Lemma 6.1 we see that the conditional ($\mu_{x'}$) probability of moving in one step to $S(d, P)$, given that one moves in one step to $B_d(P)$, is at least $\gamma(1/2)$. Since the process visits $B_d(P)$ with μ_x probability 1, the distribution of the first visit to $B_d(P)$ must assign mass at least γ to $S(d, P)$. The lemma is proved.

LEMMA 6.4. *Let* d_n, $n \geqq 1$, *be a sequence of positive numbers decreasing to* 0, *and let* $S = \bigcup_{n=1}^{\infty} S(d_n, P)$. *For any* $x \in \Omega$ *the* μ_x *process visits* S *infinitely often with probability* 1.

Proof. Let $\gamma(\cdot)$ be as in Lemma 6.1, and let α_n be a sequence decreasing to 0 so rapidly that $\gamma(\alpha_n) > 1 - (1/2)^n$. By choosing a subsequence of $\{d_n\}$ and then renumbering, if necessary, we can assume $d_n/d_{n+1} < \alpha_n$ for all n. Define $\tau_n(\omega) = \inf\{l \mid x_l(\omega) \in B_{d_n}(P)\}$. The requirement $d_n/d_{n+1} < \alpha_n$ insures that $\tau_n = \tau_{n+1}$ on a set whose measure is at most $(1/2)^n$. (Argue as in Lemma 6.3 using $\gamma(\alpha_n)$ instead of $\gamma(1/2)$.) Now fix n, $l > 0$, and let $E(n, l)$ be the set

$$E(n, l) = \{\omega \mid \tau_n(\omega) < \tau_{n+1}(\omega) < \cdots < \tau_{n+l}(\omega) \text{ and } x_{\tau_{n+j}}(\omega) \notin S(d_{n+j}, P),\ 0 \leqq j \leqq l\} .$$

Obviously, $\mu_x(E(n, l)) \leqq (1 - \gamma)^{l+1}$, where γ is as in Lemma 6.3. If we choose n_0 and l so large that $2^{-(n_0-1)} < \varepsilon$ and $(1 - \gamma)^{l+1} < \varepsilon$, then for all $n \geqq n_0$ the set $E(n, l) \cup \{\omega \mid \tau_{n+j}(\omega) = \tau_{n+j+1}(\omega)$, some $0 \leqq j \leqq l - 1\}$ has measure at most 2ε. Thus, the set of ω such that $x_n(\omega) \in S$ infinitely often has measure at least $1 - 2\varepsilon$, and since ε is arbitrary, the lemma follows.

Definition 6.5. [17]. A set $A \subseteqq \Omega$ is said to be *thin* at P if for every $x \in \Omega$ and $\varepsilon > 0$ there is a neighborhood V of P and a superharmonic function $g_1 \leqq g$ on Ω such that $g_1 = g$ on $V \cap A$ and $g_1(x) < \varepsilon g(x)$.

LEMMA 6.6. *If A is a Borel set which is thin at P, then μ_x almost every sample path visits A at most a finite number of times.*

Proof. Use the argument following the statement of Theorem 2.4 with $\mathcal{O}$ replaced by $A \cap V$.

Definition 6.7. Given $\varepsilon, d > 0$, $\varepsilon < 1/2$, define a set $T(\varepsilon, d) \subseteqq \Omega$ by $T(\varepsilon, d) = \{x = (z, t) \in \Omega \cap U \mid x \in S(d, P)$ and $b(z) < t < b(z) + \varepsilon d\}$. If M is the Lipschitz constant for b at P, and if $P = (z_0, t_0)$, then for all $(z, t) \in T(\varepsilon, d)$ we have

$$|t - b(z)| \leqq \frac{2\varepsilon}{1 - 2\varepsilon}(M + 1)\,||z - z_0|| \,. \tag{6.2}$$

This is because $(z, t) \in T(\varepsilon, d)$ implies $||z - z_0|| \geqq (1/2 - \varepsilon)(M + 1)^{-1}d$.

Remark 6.8. Suppose T is a subset of Ω which is contained in a set S of the form

$$S = \{(z, t) \in \Omega \cap U \mid b(z) < t < b(z) + c(z)\} \tag{6.3}$$

and suppose

$$\lim_{z \to z_0} \frac{c(z)}{||z - z_0||} = 0 \,.$$

Then by Lemma 5.4 of [16] there is a sequence $q_1, q_2, \cdots$ of nonnegative superharmonic functions on Ω such that $q_n \leqq g$ on Ω, $q_n = h$ on $S \cap S(1/2^n, P)$, q_n is harmonic away from $S \cap S(1/2^n, P)$, and $\lim_{n\to\infty} q_n(x) = 0$ for all $x \in P$. We will make use of this fact in the Lemma to follow.

LEMMA 6.9. *Let $\{\varepsilon_n\}$ and $\{d_n\}$ be sequences of positive numbers tending to 0. There exists a sequence $n_1 < n_2 < \cdots$ such that $T' = \bigcup_{k=1}^{\infty} T(\varepsilon_{n_k}, d_{n_k})$ is thin at P.*

Proof. We may suppose $T(\varepsilon_n, d_n)$ is defined for all n and also that $S(d_n, P) \cap S(d_{n'}, P) = \emptyset$, $n \neq n'$. Let $T = \bigcup_{n=1}^{\infty} T(\varepsilon_n, d_n)$. Then by (6.2) and the fact $\varepsilon_n \to 0$ we see that T is contained in a set S as in (6.3). Let $q_1, q_2, \cdots$ be as in the discussion preceding this lemma. Fix $x \in \Omega$. There exist sequences m_k and n_k such that for each k

$$\left(\frac{1}{2}\right)^{m_k} \geqq d_{n_k} > \frac{1}{2} d_{n_k} > \left(\frac{1}{2}\right)^{m_k + 2}$$

and $q_{m_k}(x), q_{m_k+1}(x) < (1/2)^k$. Define $Q_l(y) = \sum_{k=l}^{\infty} \left(q_{m_k}(y) + q_{m_k+1}(y)\right)$. If $y \in \Omega$, there is at most one term in the series defining Q_l which is not harmonic at y. Since $Q_l(x) < \infty$, it must be by Harnack's theorem that $Q_l < \infty$ on Ω, and Q_l is superharmonic on Ω. Setting $T' = \bigcup_{k=1}^{\infty} T(\varepsilon_{n_k}, d_{n_k})$, we claim T'

is thin at P. For on each set $T' \cap B_{(1/2)^{m_l}}(P) = T'_l$ we have $Q_l \geqq g$, and since $Q_l(y) \to 0$ for all y, T' is thin at P.

Combining Lemmas 6.6 and 6.9 we have

LEMMA 6.10. *With notations as in Lemma 6.9, μ_x almost every sample path visits T' at most a finite number of times.*

LEMMA 6.11. *Let the notations and assumptions be as in Lemmas 6.9–6.10. Define $W = \bigcup_n \big(S(d_n, P) \cap T(\varepsilon_n, d_n)^c\big)$. If $x \in \Omega$, then μ_x almost every sample path visits W infinitely often.*

Proof. Apply Lemmas 6.4 and 6.10.

Our first application of Lemma 6.11 is to prove the μ_x sample sequences do not converge *tangentially* to P. It is not true however that the convergence is nontangential. Writing $x_n(\omega) = \big(z_n(\omega), t_n(\omega)\big)$ when $x_n \in U$, we let $A \in \mathcal{B}$ be the set

$$A = \left\{\omega \mid \lim_{n\to\infty} \frac{t_n - b(z_n)}{|| z_n - z_0 ||} = 0\right\}$$

where as before $P = (z_0, t_0)$. We will prove $\mu_x(A) = 0$.

Given n and $\varepsilon > 0$, define a set $A(n, \varepsilon) \in \mathcal{B}$ by

$$A(n, \varepsilon) = \left\{\omega \in A \;\middle|\; \frac{t_k - b(z_k)}{|| z_k - z_0 ||} < \varepsilon,\ k \geqq n\right\}.$$

Fixing x, there exists an n such that

$$\mu_x\big(A(n, \varepsilon)\big) \geqq (1 - \varepsilon)\mu_x(A)\,. \tag{6.4}$$

Now choose $d = d(n, \varepsilon)$ so small that if

$$A(d) = \{\omega \in A \mid || x_k - P || < d,\ \text{some } k < n\}$$

then

$$\mu_x\big(A(d)\big) < \varepsilon\,. \tag{6.5}$$

Next, let ε_k be a sequence of positive numbers such that $\sum_k \varepsilon_k < \infty$. Associate to each ε_k an n_k and $d = d_k$ as in (6.4) and (6.5). Define $F \subseteqq \Omega$ as

$$F = \bigcup_k S(d_k, P) \cap T(d_k, \varepsilon_k)^c\,.$$

If $\omega \in A$ and if n is such that $x_n(\omega) \in F$, then for some k one of the following two statements must be true:

(i) $n < n_k$ and $|| x_n(\omega) - P || < d_k$.

(ii) $n \geqq n_k$ and $t_n - b(z_n) \geqq \varepsilon_k || z_n - z_0 ||$.

In other words for this ω there exists a k such that $\omega \in A(d_k)$ or $\omega \notin A(n_k, \varepsilon_k)$.

Now by (6.4), (6.5), and our choice of ε_k the set of $\omega \in A$ which belong to $A(d_k) \cup A(n_k, \varepsilon_k)^c$ for an infinite number of k has measure 0. On the other hand, μ_x almost every sample path visits F infinitely often by Lemma 6.11. We conclude that $\mu_x(A) = 0$. We have proved

LEMMA 6.12. *For each $x \in \Omega$ μ_x almost every sample path satisfies*

$$\limsup_{n\to\infty} \frac{t_n - b(z_n)}{\| z_n - z_0 \|} > 0 .$$

7. Proof of Theorem 2.7

Let $\Omega \subseteqq \mathbf{R}^n$ be a bounded Lipschitz domain, and let g be a kernel function at $P \in \partial\Omega$. We are to prove that if $E \in \mathcal{B}_I$ then $\mu_x(E) = 0$ or 1, where $\mu_x = \mu_x^g$.

We base our proof on the following familiar fact (see, e.g., [4]): For all n $E(\mathcal{X}_E \mid x_1, \cdots, x_n) = \mu_{x_n}(E)$, $x_n = x_n(\omega)$ a.e., where $\mathcal{X}_E$ is the characteristic function of E, and therefore by the martingale theorem

$$\lim_{n\to\infty} \mu_{x_n}(E) = \mathcal{X}_E(\omega) \tag{7.1}$$

a.e. Of course the right side of (7.1) is 0 or 1 a.e. μ_x.

Notations will be as in Section 6. Recall that M is the Lipschitz constant for b in a neighborhood of P. If ε, $d > 0$ are sufficiently small, then the open set $\mathcal{O}(\varepsilon, d)$ will be a subset of $\Omega \cap U$, where $\mathcal{O}(\varepsilon, d)$ is defined by

$$\mathcal{O}(\varepsilon, d) = \{w = (z, t) \mid \| z - z_0 \| < d \text{ and } b(z) + \varepsilon d < t < t_0 + (M + 1)d\} .$$

Notice that $\mathcal{O}(\varepsilon, d)$ contains $S(d, P) \cap T(\varepsilon, d)^c$.

If ε and d are small, and if $w \in \mathcal{O}(\varepsilon, d)$, say $w = (z, t)$, then any $w' \in \partial\Omega$ such that $\| w' - w \| = d(w, \partial\Omega)$ will be of the form $w' = (z', b(z'))$. We claim $d(w, \partial\Omega) \geqq \varepsilon d/(M + 1)$. For if $\| w - w' \| < \varepsilon d/(M + 1)$, then in particular $\| z - z' \| < \varepsilon d/(M + 1)$ and $| b(z) - t | \leqq | t - b(z') | + | b(z') - b(z) | < \varepsilon d/(M + 1) + M \| z - z' \| \leqq \varepsilon d$. This contradicts the definition of $\mathcal{O}(\varepsilon, d)$. Define $R = R(\varepsilon, d) = 2^{-1}\varepsilon d (M+1)^{-1}$. By what has just been proved, $d(w, \partial\Omega) > R$ for all $w \in \mathcal{O}(\varepsilon, d)$.

Next, define $\Sigma = \Sigma(\varepsilon, d) = (2M + 1)d$. If $x \in \mathcal{O}(\varepsilon, d)$, $x = (z, t)$, then $d(x, \partial\Omega) \leqq | t - b(z) | < (2M + 1)d$. Notice that R/Σ is a function of ε and M. Next, we will choose values of k and t, depending only upon ε and M, such that $\mathcal{O}(\varepsilon, d)$ has type (R, k, Σ, t).

The distinguished point $w_0 \in \mathcal{O}(\varepsilon, d)$ will be taken to be $w_0 = (z_0, t_0 + (M + 3/4)d)$. Given $w \in \mathcal{O}(\varepsilon, d)$, $w = (z, t)$, join w to w_0 by the following rectilinear path, $L = L(w, w_0)$. The first portion of L is the vertical segment connecting w to $(z, t_0 + (M + 3/4)d) = w'$. This segment lies in $\mathcal{O}(\varepsilon, d)$ if

$\varepsilon < 3/4$, as does the horizontal segment connecting w' to w_0. The length of L is less than $(2M+1)d + d = 2(M+1)d$. Next, choose points $x_1, x_2, \cdots, x_k$ successively on L with $x_1 = w$ and $\|x_{i+1} - x_i\| = r_0 R$ (r_0 as in previous sections), $1 \leqq i \leqq k-1$, and $x_k = w_0$, $\|x_{k-1} - x_k\| \leqq r_0 R$. There is a dimensional constant $t > 0$ such that $m(B_{r_0 R}(x_i) \cap B_{r_0 R}(x_{i+1})) \geqq tm(B_R)$ (of course t also depends on r_0, whose definition depends only upon the dimension). In this construction $k \leqq 2(M+1)d(r_0 R)^{-1} = 4(M+1)^2(\varepsilon r_0)^{-1}$. Set $k = [4(M+1)^2(\varepsilon r_0)^{-1}] + 1$, $[\cdot]$ = greatest integer function. Then k depends only upon ε and M. Therefore, $\mathcal{O}(\varepsilon, d)$ has type (R, k, Σ, t) depending only upon ε, M and the dimension. Because of this we shall denote the function η of equations (5.4) and (5.5) by $\eta = \eta(\varepsilon, M)$. We have that if F is Borel on Ω, and if $P_g F = F$, $0 \leqq F \leqq 1$, then either

$$F(x) \geqq \frac{1}{2}\eta(\varepsilon, M) \tag{7.2}$$

for all $x \in \mathcal{O}(\varepsilon, d)$, or else

$$F(x) \leqq 1 - \frac{1}{2}\eta(\varepsilon, M) \tag{7.3}$$

all $x \in \mathcal{O}(\varepsilon, d)$. We shall apply (7.2) and (7.3) to the function $F(x) = \mu_x(E)$ (see 2.4).

For fixed ε it is clear the set of d for which (7.2) holds on $\mathcal{O}(\varepsilon, d)$ is closed, and the same is true for (7.3). Therefore, unless there is for a given ε a sequence $d_n \to 0$ such that *both* (7.2) and (7.3) hold on $\mathcal{O}(\varepsilon, d_n)$, it will be the case for all sufficiently small d, say $d < d(\varepsilon)$, that the same one of (7.2) or (7.3) holds on $\mathcal{O}(\varepsilon, d)$. There are two cases to be considered.

(a) For every small ε there exists $d = d(\varepsilon)$ as above. In this case choose a sequence $\varepsilon_k \to 0$ such that, say, (7.2) holds on $\mathcal{O}(\varepsilon_k, d)$, $d < d(\varepsilon_k)$. Define

$$V_k = \bigcup\nolimits_{d < d(\varepsilon_k)} \mathcal{O}(\varepsilon_k, d) .$$

V_k contains all $(z, t) \in \Omega$ in some neighborhood of P such that $|t - b(z)| > \varepsilon_k \|z - z_0\|$. Therefore, by the discussion at the end of Section 6, for μ_x almost all sample paths, $x_n(\omega)$, there exists a k such that $x_n \in V_k$ infinitely often. For such a path we have by (7.1) and (7.2) $\mathcal{X}_E(\omega) \geqq (1/2)\eta(\varepsilon_k, M) > 0$, so $\omega \in E$. Thus $\mu_x(E) = 1$. Similarly, $\mu_x(E) = 0$ in the case of (7.3).

(b) There exists a sequence $\varepsilon_k \to 0$ and for every k a sequence $d_{kl} \to 0$ such that for every l both (7.2) and (7.3) hold on $\mathcal{O}(\varepsilon_k, d_{kl})$. Define

$$\mathcal{O}(\varepsilon_k) = \bigcup\nolimits_l \mathcal{O}(\varepsilon_k, d_{kl}) .$$

By (7.1) and the remark following (7.1) we see that μ_x almost every sample path visits $\mathcal{O}(\varepsilon_k)$ at most a finite number of times. This being so, it is possible to choose a sequence m_k such that

$$\mu_x\{\omega \mid x_n(\omega) \in \bigcup_{l \geqq m_k} \mathcal{O}(\varepsilon_k, d_{kl}) \text{ for some } n\} \leqq \left(\frac{1}{2}\right)^k .$$

Then if $\mathcal{O} = \bigcup_k \mathcal{O}(\varepsilon_k, d_{km_k})$, μ_x almost every sample path visits $\mathcal{O}$ at most a finite number of times. But by an earlier remark $\mathcal{O}$ contains

$$\bigcup_k \left(S(d_{km_k}, P) \cap T(\varepsilon_k, d_{km_k})^c\right) ,$$

and we have a contradiction to Lemma 6.11. The theorem is proved.

Remark 7.1. Suppose a Borel set $A \subseteqq \mathcal{O}(\varepsilon, d)$ satisfies $m(A) \geqq \alpha m\mathcal{O}((\varepsilon, d))$ for some $\alpha > 0$. It is not difficult to see there is a constant c, depending only upon ε, M, and the dimension, such that for some $w \in \mathcal{O}(\varepsilon, d)$ $m(A \cap B_{r_0R}(w)) \geqq c\alpha m(B_{r_0R})$. If $\mathcal{O}(\varepsilon, d)$ has type (R, k, Σ, t), then we can use w for the distinguished point, changing the type in the worst case to $(R, 2k, \Sigma, t)$. Thus by the results of Section 5 there is a number $\lambda = \lambda(\alpha, \varepsilon, M) > 0$ such that for every $x \in \mathcal{O}(\varepsilon, d)$ the μ_x probability of ever visiting A is at least λ.

If d_k is a sequence decreasing to 0, the event

$$E_\varepsilon = \{\omega \mid x_n(\omega) \in \bigcup_k \mathcal{O}(\varepsilon, d_k) \text{ infinitely often}\} ,$$

which is a $\mathcal{B}_I$ set, has positive probability if ε is small, as one can see by a modification of the case (b) argument above. Since $\mu_x(E_\varepsilon) = 0$ or 1, it must be that $\mu_x(E_\varepsilon) = 1$ for all x and small ε.

If $A \subseteqq \Omega$ is a Borel set, we define

$$D_P(A) = \limsup_{d \to 0} \frac{m(A \cap \mathcal{O}(0, d))}{m(\mathcal{O}(0, d))}$$

where $\mathcal{O}(0, d)$ is $\mathcal{O}(\varepsilon, d)$ with $\varepsilon = 0$. If $D_P(A) > 0$, we say A has *positive upper density* at P.

From the definition of $\mathcal{O}(0, d)$,

$$\mathcal{O}(0, d) = \{(z, t) \mid \|z - z_0\| < d, b(z) < t \leqq (M + 1)d + t_0\}$$

we see that

$$d^n m(B_{n-1}) \leqq m(\mathcal{O}(0, d)) \leqq (2M + 1)d^n m(B_{n-1}) .$$

The region $\mathcal{O}(0, d) \cap \mathcal{O}(\varepsilon, d)^c$ has measure at most $\varepsilon d^n m(B_{n-1})$, and therefore

$$m(\mathcal{O}(\varepsilon, d)) \geqq (1 - \varepsilon)m(\mathcal{O}(0, d)) .$$

Thus, if A is a set such that $m(A \cap \mathcal{O}(0, d)) \geqq \alpha m(\mathcal{O}(0, d))$, and if $\varepsilon < \alpha$,

$$(7.4) \qquad \begin{aligned} m(A \cap \mathcal{O}(\varepsilon, d)) &\geqq (\alpha - \varepsilon) m(\mathcal{O}(0, d)) \\ &> (\alpha - \varepsilon) m(\mathcal{O}(\varepsilon, d)) \,. \end{aligned}$$

If $\varepsilon < \alpha/2$, the number on the right in (7.4) is at least $(\alpha/2) m(\mathcal{O}(\varepsilon, d))$.

Suppose A is such that $D_P(A) \geqq 2\alpha > 0$. There exists a sequence $d_k \to 0$ such that $m(A \cap \mathcal{O}(0, d_k)) > \alpha m(\mathcal{O}(0, d_k))$. We fix $0 < \varepsilon < \alpha/2$ small enough that $\mu_x(E_\varepsilon) = 1$, where E_ε is associated to $\{d_k\}$, as above.

By Remark 7.1 there is a number $\lambda > 0$ such that if $x \in \mathcal{O}(\varepsilon, d_k)$, the μ_x probability of visiting A starting from x is at least λ. In fact, the arguments leading to Lemma 5.3 can be modified slightly to show the probability of visiting A before leaving the ε_k neighborhood of $\mathcal{O}(\varepsilon, d_k)$, $\varepsilon_k = (1/2) \inf d(z, \partial\Omega)$, $z \in \mathcal{O}(\varepsilon, d_k)$, is at least λ. Define stopping times $\tau_1 < \tau_2 < \cdots$ as follows: τ_1 is the time of the first visit to $\bigcup \mathcal{O}(\varepsilon, d_k)$. Given $\tau_1, \cdots, \tau_n$, $\tau_{n+1} > \tau_n$ is the time of the first visit to $\bigcup_{l \neq k} \mathcal{O}(\varepsilon, d_k)$, $x_{\tau_n} \in \mathcal{O}(\varepsilon, d_l)$. If the d_k's decrease rapidly, each event $E_n = \{\omega \mid x_m(\omega) \in A,$ some $m,$ $\tau_n(\omega) \leqq m < \tau_{n+1}(\omega)\}$ will have probability at least λ. Thus $\bigcap_{l=1}^{\infty} \bigcup_{n=l}^{\infty} E_n = E_\infty$ has measure at least λ. Since $E_\infty \in \mathcal{B}_I$, $\mu_x(E_\infty) = 1$.

THEOREM 7.2. *If $A \subseteqq \Omega$ is a Borel set which has positive upper density at $P \in \partial\Omega$, and if g is a kernel function at P, μ_x^g almost every sample path visits A infinitely often.*

Remark 7.3. Of course Theorem 7.2 implies $\mu_x(E_\varepsilon) = 1$ for all $\varepsilon > 0$, E_ε as above.

Boundary Values 7.4. We continue to suppose μ_x is μ_x^g for a kernel function g at $P \in \partial\Omega$. Let h be a function on Ω which satisfies Harnack's inequality, and suppose $h(x_n)$ converges for μ_x almost every sample path. Theorem 2.7 implies the limit is constant, say $\lim h(x_n) = c$ a.e., and the constant is the same for all starting points x.

Given $\varepsilon > 0$, define

$$C_\varepsilon = \{(z, t) \mid t > b(z) + \varepsilon \,\|z - z_0\|\} \cap U \,.$$

We claim C_ε cannot contain a sequence x_n such that $\lim h(x_n) = c_0$ exists, but $c \neq c_0$. For there would exist by Harnack's inequality an $s > 0$ such that if $\Lambda_n = B_{sr_n}(x_n)$, $r_n = d(x_n, \partial\Omega)$, then $|h(x) - c| \geqq (1/2)|c - c_0|$ on Λ_n, n large. If $A = \bigcup_n \Lambda_n$, then A has positive upper density at P, and therefore must be visited by μ_x almost all sample paths. This is a contradiction, and we have

THEOREM 7.5. *Let h be a positive function obeying Harnack's inequality, and suppose $\lim_n h(x_n)$ exists a.e. μ_x^g, g a kernel function at P. Then h has a nontangential boundary value at P.*

Remark 7.6. Theorem 7.5 is a generalization of Fatou's theorem. For let h be a positive harmonic function on Ω. If $f \equiv 1$ then $h(x_n)$ converges μ_x^f a.e.. Thus, for (harmonic measure) almost all $P \in \partial\Omega$, $\lim_n h(x_n)$ exists a.e. μ_x^g, g a kernel function at P. By Theorem 7.5 h has a nontangential boundary value at almost all $P = \partial\Omega$. (Harmonic measure.) (See [8] for a probabilistic treatment of the boundary value problem for superharmonic functions.)

Remark 7.7. Hunt and Wheeden [16] prove that nontangential boundary values at almost all points of $\partial\Omega$ imply fine boundary values at almost all points. Fine boundary values are obtained probabilistically by Doob [8]. It would be interesting to have Theorem 7.5 for fine boundary values. What is involved is a converse to Lemma 6.6.

Remark 7.8. Of course not the full strength of Harnack's inequality is needed in Theorem 7.5. Since there are no obvious applications, we see no point in complicating the statement.

8. Proof of Lemma 6.1

This is the generalization from 2 to n dimensions of Lemma 6.1 of [21]. The proof in [21] was "elementary", involving only the Poisson integral formula, Fubini's theorem, and plane geometry. For the general case we found it necessary to use an argument which was not elementary (but which did establish more; namely the phrase "function g on B" can be replaced by "function g on $B \cap D_2$"). The argument which appears below is elementary and has been communicated to us by Richard Hunt. Hunt's argument is simpler than the one in [21], although it bears at least some spiritual resemblance to it.

First we note that the applications of Lemma 6.1 require only that it be proved under the additional assumptions $0 < \alpha \leqq 1/2$ and $s \in B$, where s is the center of D_2. (In the applications $s \in \partial\Omega$ and B is contained in Ω.) Also, the ratios of the integrals appearing in Lemma 6.1, as well as the harmonicity of the integrands, are unaffected by translations and dilations. For this reason we may suppose always that B is the unit ball, D_2 is the ball of radius $R > 0$ centered at $s = (S, 0, \cdots, 0)$, $S \geqq \max(1, R)$, and D_1 is the ball of radius αR centered at s, $0 < \alpha \leqq 1/2$. If $B \cap D_1 = \varnothing$, then (6.1) is true with 1 replacing $\gamma(\alpha)$, and therefore we may suppose $0 \leqq S - R \leqq S - \alpha R \leqq 1$. Define D_3 to be the ball of radius $R/8$ centered at $s' = (S - (3/4)R, 0, \cdots, 0)$. Our assumptions imply $D_3 \subseteqq B \cap \Delta = B \cap D_1^c \cap D_2$.

By the Poisson integral formula and the Fubini theorem it will be enough to produce $\gamma(\alpha)$ with the desired properties for the functions $P(z, \cdot)$, $z \in \partial B$, where $P(z, x) = (1 - \|x\|^2)(\|z - x\|^{-n})$. To this end we consider separately the cases $\|z - s\| \leqq 2R$ and $\|z - s\| > 2R$.

Case 1. $\|z - s\| \leqq 2R$. Below C denotes a constant, possibly different on different lines, but which in all instances depends only on the dimension. Notice that $S \geqq 1$ implies $1 - (S - \alpha R) \leqq \alpha R$. If $B_{S-\alpha R}$ is the ball of radius $S - \alpha R$ ($\leqq 1$) centered at 0, then

$$\begin{aligned}\int_{B \cap D_1} P(z, x)dx &\leqq \int_B P(z, x)dx - \int_{B_{S-\alpha R}} P(z, x)dx \\ &\leqq C(1 - (S - \alpha R)) \leqq C\alpha R\,. \end{aligned} \tag{8.1}$$

Now $D_3 \subseteqq \Delta$ implies

$$\int_{B \cap \Delta} P(z, x)dx \geqq CR^n P(z, s')$$

and since $\|z - s'\| \leqq \|z - s\| + \|s - s'\| \leqq 3R$, the right side of this inequality is at least $C(1 - \|s'\|)$. Finally, $0 < \alpha \leqq 1/2$ and $0 \leqq S - \alpha R \leqq 1$ combine to imply $1 - \|s'\| = 1 - \big(S - (3/4)R\big) \geqq R/4$, and therefore

$$\int_{B \cap \Delta} P(z, x)dx \geqq CR\,. \tag{8.2}$$

Together, (8.1) and (8.2) tell us $\gamma(\alpha) = (1 + C\alpha)^{-1}$.

Case 2. $\|z - s\| > 2R$. This assumption implies $\|z - y\| > R$, $y \in D_2$, and therefore for x, $y \in D_2$, $\|z - x\| \leqq 3\|z - y\|$. ($\|z - x\| \leqq \|z - y\| + \|y - x\| \leqq \|z - y\| + 2R \leqq 3\|z - y\|$.) Therefore, if $x \in B \cap D_2$,

$$C^{-1}\frac{1 - \|x\|}{\|z - s'\|^n} \leqq P(z, x) \leqq C\frac{1 - \|x\|}{\|z - s'\|^n}\,. \tag{8.3}$$

The measure of D_1 is $C(\alpha R)^n$, and $1 - (S - \alpha R) \leqq \alpha R$, and therefore by (8.3)

$$\begin{aligned}\int_{B \cap D_1} P(z, x)dx &\leqq C\frac{1 - (S - \alpha R)}{\|z - s'\|^n}(\alpha R)^n \\ &\leqq C\|z - s'\|^{-n}(\alpha R)^{n+1}\,. \end{aligned} \tag{8.4}$$

On the other hand the mean value theorem together with (8.3) and the earlier observation $1 - \|s'\| \geqq R/4$ tell us

$$\begin{aligned}\int_{B \cap \Delta} P(z, x)dx &\geqq \int_{D_3} P(z, x)dx \\ &= CR^n P(z, s') \\ &\geqq CR^{n+1}\|z - s'\|^{-n}\,. \end{aligned} \tag{8.5}$$

Finally, (8.4) and (8.5) combine to give $\gamma(\alpha) = (1 + C\alpha^{n+1})^{-1}$, and the lemma is proved.

9. Proof of Theorem 2

We begin with the statement of a lemma whose proof will be deferred until later.

LEMMA 9.1. *For each integer* $n \geqq 1$ *there exists an absolute constant* $L = L_n$ *with the following property. If* $Q_1, Q_2, \cdots, Q_m$ *is a collection of balls contained in the unit ball of* $\mathbf{R}^n$, *with each containing* O *and if* $S = \partial(\bigcup_{i=1}^m Q_i)$, *then*

$$|S| \leqq L_n \tag{9.1}$$

where $|\cdot|$ *denotes* $(n-1)$ *dimensional surface area.*

Proof of the Theorem. 9.2. Let $E \subseteqq B_n$ be measurable, and let $\alpha, \beta > 0$ be numbers such that $m(E) > \beta$ and $\|\alpha(\cdot, E)\|_\infty < \alpha$. By the regularity of Lebesgue measure and the monotonicity of $\alpha(x, E)$ in E, we may and shall assume that E is compact and $\alpha(x, E) < \alpha$ for every $x \in E$. This being so there exist balls $Q_1, \cdots, Q_m$ in B_n whose union covers E, and such that $m(Q_i \cap E) < \alpha m(Q_i)$, $1 \leqq i \leqq m$.

If Q is a ball and $\lambda > 0$ a real number, we use λQ to denote the concentric ball of radius λR, $R =$ radius Q.

Let Q_1^* be one of $Q_1, \cdots, Q_m$ whose radius, R_1, is maximal. If W_1 is the union of the Q_i's which intersect $(1/2)Q_1^*$, and if $T_1 = (5/2)Q_1^*$ (possibly $T_1 \nsubseteq B_n$), then $Q_1^* \subseteqq W_1 \subseteqq T_1$. Of the balls which remain, if any, let Q_2^* have the maximal radius R_2. We associate W_2 and T_2 to Q_2^* as above, this time using only balls which do not enter into the definition of W_1. Continuing in like fashion we exhaust the original collection of balls forming a sequence (Q_i^*, R_i, W_i, T_i), $i = 1, \cdots, l$, such that

(a) $Q_j^* \subseteqq T_j$ concentrically and $T_j = (5/2)Q_j^*$

(b) $(1/2)Q_i^* \cap (1/2)Q_j^* = \varnothing$, $i \neq j$

(c) $E \subseteqq \bigcup_{j=1}^l W_j$, and $W_j \subseteqq T_j$, all j.

Define β_j by $\beta_j m(T_j) = m(E \cap W_j)$, $j = 1, \cdots, l$. Using the above properties we have

$$\begin{aligned}\beta < \sum_{j=1}^l m(E \cap W_j) &= \sum_{j=1}^l \beta_j m(T_j) \\ &= 5^n \sum_{j=1}^l \beta_j m\left(\frac{1}{2} Q_j^*\right) \leqq 5^n m(B_n) \max \beta_j \,.\end{aligned}$$

Setting $\omega_n = m(B_n)$, it follows there is a j with $\beta_j > 5^{-n}\omega_n^{-1}\beta$.

We now ignore all balls except those whose union forms W_j, j as selected above. A translation and dilation sends T_j to B_n, Q_j^* to $B_{1/5}(0)$, $E \cap W_j$ to a set which we again denote by E and which has measure at least $5^{-n}\beta$, and the balls comprising W_j to a collection which we denote by $Q_1, \cdots, Q_m$ (new

m) such that $Q_1 = B_{1/5}(0)$ and $Q_j \cap (1/2)Q_1 \neq \emptyset$, all j. Notice that $m(Q_j \cap E) < \alpha m(Q_j)$ remains in force. We have also

(9.2) $$m(Q_j) \geqq 20^{-n}\omega_n .$$

(9.3) $$\text{Radius } Q_j \geqq \delta,\ \ \delta = \frac{1}{20} .$$

(9.4) $$m(E \cap Q_j) < \alpha m(Q_j) \leqq \alpha\omega_n .$$

Fix any $\varepsilon > 0$, $\varepsilon < \delta/2$, to be determined later, and let π be a partition of $\mathbf{R}^n$ into cubes with sides parallel to the axes and side length $\varepsilon/n^{1/2}$. Each cube has diameter ε, and so if $x \in Q_j$, $d(x, \partial Q_j) \geqq \varepsilon$, the cube containing x will be contained in Q_j. Define $f = 1 - \varepsilon/\delta$. We let Q_j^* for each j be the ball concentric with Q_j and of radius $f\delta_j$, $\delta_j =$ radius Q_j. If $x \in Q_j^*$, then $d(x, \partial Q_j) \geqq (1 - f)\delta_j \geqq (1 - f)\delta = \varepsilon$, and the cube of π containing x will be contained in Q_j. Define $E_1 = E \cap (\bigcup_{j=1}^m Q_j^*)$. If $C_1, \cdots, C_s$ are the cubes of π which are contained in one of the Q_j's, then by the observation above, $E_1 \subseteqq \bigcup_{i=1}^s C_i$. It follows there exists C_i such that $m(E_1 \cap C_i) \geqq \big(m(E_1)/\omega_n\big)m(C_i)$, or if Q_j is the ball containing C_i,

(9.5) $$m(E_1 \cap Q_j) \geqq \frac{1}{\omega_n}\left(\frac{\varepsilon}{n^{1/2}}\right)^n m(E_1) .$$

Next, define $E_0 = E \cap E_1^c$. By definition,

$$E_0 \subseteqq (\textstyle\bigcup_{j=1}^m Q_j) \cap \big(\bigcap_{j=1}^m (Q_j^*)^c\big) .$$

If $x \in E_0$, let $i = i(x)$ be the smallest subscript such that $x \in Q_i$. This function is measurable, as is the following $q(\cdot, \cdot)$ on $E_0 \times B_n$:

$$q(x, y) = \begin{cases} m(Q_i)^{-1} & y \in Q_i^*,\ \ i = i(x) \\ 0 & y \notin Q_i^*,\ \ i = i(x) . \end{cases}$$

Since $m(Q_i^*) = f^n m(Q_i)$, we see that

$$\int_{E_0}\int_{B_n} q(x, y)dxdy \geqq f^n m(E_0) .$$

Therefore, there exists $y \in B_n$ such that

(9.6) $$\int_{E_0} q(x, y)dx \geqq \frac{f^n m(E_0)}{\omega_n} .$$

Now $m(Q_i) \geqq \delta^n\omega_n$ for all i, and therefore $q(x, y) \leqq \delta^{-n}\omega_n^{-1}$. Using (9.6) we see that a portion of E_0 measuring at least $f^n\delta^n m(E_0)$ is covered by balls which contain y. Let E_0^* be the portion of E_0 so covered. We have

(9.7) $$m(E_0^*) \geqq f^n\delta^n m(E_0) .$$

Discarding balls which do not contain y and renumbering, we have a collection $Q_1, \cdots, Q_m$ of balls containing y and covering E_0^* and also satisfying (9.3)–(9.4) above.

If Σ is a measurable subset of ∂Q_j^*, the volume of the set $\Sigma_0 \subseteqq (Q_j^*)^c \cap Q_j$ consisting of all points which project radially onto Σ is, since radius $Q_j^* \geqq f\delta_j$, at most

$$m(\Sigma_0) \leqq \frac{\delta_j^n(1-f^n)}{n}|\Sigma_1|$$

where Σ_1 is the radial projection of Σ onto the concentric sphere of radius 1. Since $|\Sigma| \geqq f^{n-1}\delta_j^{n-1}|\Sigma_1|$ we have

$$\begin{aligned} m(\Sigma_0) &\leqq \frac{\delta_j(1-f^n)}{nf^{n-1}}|\Sigma| \\ &\leqq 2^{n-1}(1-f)|\Sigma| \end{aligned} \tag{9.8}$$

because $f \geqq 1/2$ (recall that $\varepsilon < \delta/2$).

Now the measure of E_0^* is bounded by the measure of

$$\left(\bigcup_{j=1}^m Q_j\right) \cap \left(\bigcap_{j=1}^m (Q_j^*)^c\right).$$

Every point of this latter set is contained in a radial segment connecting a point x in some ∂Q_j^*, $x \in \partial(\bigcup_{j=1}^m Q_j^*)$, to ∂Q_j. Therefore by (9.8) and the fact $1 - f = \varepsilon/\delta$

$$m(E_0^*) \leqq 2^{n-1}\frac{\varepsilon}{\delta}|S|$$

where $S = \partial(\bigcup_{j=1}^m Q_j^*)$. By Lemma 9.1 and (9.7) we have

$$\begin{aligned} m(E_0) &\leqq f^{-n}\delta^{-n}m(E_0^*) \\ &\leqq 2^{2n-1}L_n\frac{\varepsilon}{\delta^{n+1}}. \end{aligned} \tag{9.9}$$

If $m(E_1) \geqq (1/2)m(E) \geqq (1/2)5^{-n}\beta$, then by (9.5)

$$\alpha\omega_n \geqq \frac{1}{\omega_n}\left(\frac{\varepsilon}{n^{1/2}}\right)^n\frac{1}{2}5^{-n}\beta$$

or

$$\alpha \geqq \Lambda_n\varepsilon^n\beta \tag{9.10}$$

Λ_n a dimensional constant. If $m(E_0) \geqq (1/2)m(E) \geqq (1/2)5^{-n}\beta$, then (9.9) tells us

$$\beta \leqq \Lambda_n'\varepsilon \tag{9.11}$$

for a dimensional constant Λ_n'. New set $\varepsilon = \alpha^u$, $u = (n+1)^{-1}$, assuming this number is less than $\delta/2 = (\lambda/4)\beta$. Then (9.10) and (9.11) coalesce into an estimate

$$\alpha \geqq \Lambda_n''\beta^{(n+1)}. \tag{9.12}$$

If the ε chosen above is larger than $\delta/2$, then $\alpha \geqq (1/40)^{n+1}$. It follows that $C_n > 0$ can be chosen depending only on the dimension so that in all cases $\alpha \geqq C_n m(E)^{(n+1)}$. Theorem 2 is proved, assuming Lemma 9.1.

Remark 9.3 Our first argument for Theorem 2 yielded the exponent $(n+1)^2$, but after seeing the aforementioned argument of Hunt's we realized it is more efficient, at the first stage, to consider the balls which intersect $(1/2)Q_1^*$, etc., instead of those which intersect Q_1^*.

9.4. *Proof of Lemma* 9.1. Let $B \subseteqq \mathbf{R}^n$ be a ball, and $w \in \partial B$. If l is a unit normal vector at w, and if v is a unit vector such that $|(v, l)| \geqq 1/n^{1/2}$, there is a dimensional constant $c > 0$ such that if $w + tv \in \partial B$, $t \neq 0$, then $|t| \geqq cr$, $r =$ radius B.

Let S, $Q_1, \cdots, Q_m$ be as in the statement of Lemma 9.1. For each $x \in B_{n-1}$ we consider those values of $t \geqq 0$ such that $(x, t) = w \in S \cap \partial Q_i$, and if $l(w)$ is the unit normal at w in ∂Q_i, $|(l, l_n)| \geqq 1/n^{1/2}$, $l_n = n^{\text{th}}$ standard basis vector. Let $N(x)$ be the cardinality of the set so defined. By considering the various coordinate planes $x_j = 0$ in like manner we see that if A is a bound for the integral of N over B_{n-1}, then $2n^{3/2} A$ is a bound for $|S|$.

Let c be as above. We order the points above x entering into $N(x)$ as $(x, t_1), (x, t_2), \cdots, (x, t_k)$, with $t_1 > t_2 > \cdots > t_k$. If j is odd, $j < k - 1$, there exists Q_r and t'_j such that (x, t_j) and $(x, t'_j) \in \partial Q_r$ with $t_j > t'_j \geqq t_{j+1} > t_{j+2}$. By definition of c, and the fact Q_r has radius at least $(1/2)(\|x\|^2 + t_j^2)^{1/2}$,

$$t_j - t_{j+2} > \frac{c}{2} \max(\|x\|, t_j) .$$

Simple computation shows the number of solutions with $t_j \geqq \|x\|$ is bounded by $d \log(1/\|x\|)$, d a dimensional constant. The number with $t_j \leqq \|x\|$ is bounded by $2/c$. Thus, the integral of N has a finite upper bound, and as remarked above, Lemma 9.1 is proved.

10. Remarks

If $n = 1$, say $\Omega = (0, 1)$, we have that every harmonic function is linear, hence bounded. Thus, only g processes for $g \equiv 1$ need be considered. It is easy to see the μ_x probability of convergence to 0 is $1 - x$, and to 1 is x. It is possible to give a direct proof in this case of Theorem 7.2. (That is, if A has positive upper density at 1, then x_n visits A infinitely often with probability 1, given that it converges to 1.) We omit the details. However, this result does imply Theorem 1 for $n = 1$.

RICE UNIVERSITY, HOUSTON, TEXAS

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