ON CERTAIN THEOREMS OF MEAN VALUE FOR ANALYTIC FUNCTIONS OF A COMPLEX VARIABLE

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On account of the fundamental importance of the theorems of mean value for real functions of a real variable an investigation of the validity of the corresponding formulas for analytic functions of a complex variable may prove of interest, even if the results obtained are of no great value in themselves.

Darboux has established theorems closely analogous to those expressed by the formulas:

$$\left.\begin{array}{ll}(1) & f(x)-f(a)=(x-a)\,f'[a+\theta\,(x-a)] \\ (2) & \displaystyle\int_a^x f(y)\,\phi(y)\,dy=f\,[a+\theta\,(x-a)]\int_a^x\phi\,(y)dy\end{array}\right\}\left(\left|\theta-\tfrac{1}{2}\right|<\tfrac{1}{2}\right)$$

but with the essential difference that in the second members of the above equations there appears a factor λ whose absolute value is less than 1.* So far as I have been able to ascertain, no investigation of (1), (2), and allied formulas for real functions of a real variable has been published which considers their validity without change of form when x, y, θ and a take on complex values, and f and ϕ are analytic functions of their arguments.

We may regard (1) and (2) as equations defining θ as an implicit function of x. When f and ϕ are real functions of a real variable subject to certain restrictions, at least one of the branches of this implicit function takes on a real value between 0 and 1 for each value of x in a given interval. The question arises whether equations (1) and (2) have still a solution θ of which at least one branch is always to be found in a certain limited region, when f and ϕ are functions of a complex variable analytic in a region B which contains the point a. We shall prove that there exists a neighborhood of a within which formulas (1) and (2), as well as others similar to these, remain true. In the case of real functions of a real variable something can be said as to the extent of this

* *Journal de Mathématiques*, ser. 3, vol. 2 (1876), pp. 291, 294. Cf. Stolz, *Differential- und Integralrechnung*, part II, pp. 92–95. A recent publication by Brunn — *Beziehungen des Du Bois-Reymondschen Mittelwertsatzes zur Ovaltheorie* — gives some interesting theorems which are not, however, entirely in terms of functions of a complex variable.

region; in the present paper, where only analytic functions of a complex variable are considered, no positive results of this kind have been obtained.

1. The Mean Value Theorem for Derivatives. The following theorem corresponds to formula (1):

THEOREM I. *Let a be an interior point of a region B throughout which $f(z)$ is analytic. Then there exists a neighborhood A of a such that for any value of z in A the equation*

$$F(\theta, z) \equiv f(z) - f(a) - (z-a)\, f'[a + \theta(z-a)] = 0$$

is satisfied by a value of θ which verifies the inequality

$$|\theta - \tfrac{1}{2}| < \tfrac{1}{2}.$$

To prove this theorem we expand $F(\theta, z)$ in a series proceeding according to powers of $z-a$. We thus obtain a development

$$(3) \qquad F(\theta, z) = \sum_{k=2}^{\infty} \left[\frac{1}{k!} - \frac{\theta^{k-1}}{(k-1)!}\right] f^{(k)}(a)(z-a)^k,$$

which converges for all values of z within a circle C whose center is a, provided θ satisfies the inequality $|\theta - \frac{1}{2}| < \frac{1}{2}$.

If $f(z)$ has the form $\alpha z + \beta$, the above theorem is evidently true since $F(\theta, z)$ then vanishes identically. In all other cases, at least one of the successive derivatives of $f(z)$, beginning with the second, will not vanish at a. Accordingly we have

$$f''(a) = f'''(a) = \cdots = f^{(n-1)}(a) = 0, \quad f^{(n)}(a) \neq 0.$$

We now make use of an auxiliary function $\Phi(\theta, z)$, defined as follows:

$$(4) \qquad \begin{aligned} \Phi(\theta, z) &= \frac{F(\theta, z)}{(z-a)^n}, \quad z \neq a, \\ \Phi(\theta, a) &= \left(\frac{1}{n!} - \frac{\theta^{n-1}}{(n-1)!}\right) f^{(n)}(a). \end{aligned}$$

For this function we have

$$\Phi\left(\sqrt[n-1]{\frac{1}{n}}, a\right) = 0,$$

$$\frac{\partial}{\partial\theta}\Phi\left(\sqrt[n-1]{\frac{1}{n}}, a\right) \neq 0,$$

where $\sqrt[n-1]{\frac{1}{n}}$ is the real solution, lying between 0 and 1, of the equation

$$\theta^{n-1} = \frac{1}{n}.$$

Accordingly, by a well known existence theorem for implicit functions,* there is a neighborhood A of a such that for each point z of A the equation

$$\Phi(\theta, z) = 0$$

has a solution θ interior to the circle defined by the inequality

$$|\theta - \tfrac{1}{2}| < \tfrac{1}{2}.$$

But for $z \neq a$, $F(\theta, z)$ vanishes in the same points as $\Phi(\theta, z)$, while $F(\theta, a)$ is identically zero. Our theorem is thus established.

This method of demonstration also shows that, if we except the case where $f(z)$ has the form $\alpha z + \beta$, each branch of the implicit function θ is analytic wnen z is in the neighborhood of a, and approaches one of the $(n-1)$th roots of $\frac{1}{n}$ when z approaches a.

2. Restrictions on Theorem I, and a Related Theorem. It should be noted carefully that we have proved only the *existence* of a region A in which the above theorem holds,—nothing has been said as to its size. In fact the determination of such a region for a given function and a given point a is usually a difficult problem. We can easily construct elementary functions such that, *no matter how small* $|b - a|$ *may be, when* b *has once been fixed there is no value of* θ *for which* $F(\theta, b)$ *vanishes*, and again functions for which *no value of* θ *satisfying the equation* $F(\theta, b) = 0$ *lies inside the circle defined by the inequality*

$$|\theta - \tfrac{1}{2}| < \tfrac{1}{2}.$$

The function e^{ζ}, where

$$\zeta = 2\pi i \frac{z-a}{b-a},$$

* For a clear statement of this theorem and references see *Encyclopädie der mathematischen Wissenschaften*, II B 1, p. 103.

is an example of the former sort, while an illustration under the latter case is furnished by the polynomial

$$\left(\frac{z-a}{b-a}-\gamma\right)^m,$$

where m is an integer greater than 3, and γ is the solution

$$\gamma = \tfrac{1}{2}\left(1 + i\cot\frac{\pi}{m}\right)$$

of the equation

$$(\gamma - 1)^m = \gamma^m.$$

For this function, $\theta = \gamma$ is the only solution of the equation $F(\theta, b) = 0$, and we have

$$|\gamma - \tfrac{1}{2}| = \tfrac{1}{2}\cot\frac{\pi}{m} \geqq \tfrac{1}{2}.$$

A closely related question is involved in the discussion of the equation

(5) $$F(\theta, z_1, z_2) \equiv f(z_1) - f(z_2) - (z_1 - z_2)\, f'[z_2 + \theta\,(z_1 - z_2)] = 0,$$

where both z_1 and z_2 are variable. If $f''(a) \neq 0$, we can make use of an auxiliary function defined by the equations

(6) $$\Phi(\theta, z_1, z_2) = \frac{F(\theta, z_1, z_2)}{(z_1 - z_2)^2}, \quad z_1 \neq z_2,$$

$$\Phi(\theta, z_1, z_1) = (\tfrac{1}{2} - \theta)\, f''(z_1).$$

This function vanishes at the point $(\frac{1}{2}, a, a)$ and is analytic in the neighborhood of that point, while

$$\frac{\partial}{\partial\theta}\Phi(\tfrac{1}{2}, a, a) \neq 0.$$

We can therefore state the following theorem :

THEOREM II. *Let a be an interior point of a region B throughout which $f(z)$ is analytic, while $f''(a) \neq 0$. Then there exists a neighborhood A of a such that for all values of z_1 and z_2 in A the equation*

(5) $$f(z_1) - f(z_2) - (z_1 - z_2) f'[z_2 + \theta\,(z_1 - z_2)] = 0$$

has a solution θ satisfying the inequality

$$|\theta - \tfrac{1}{2}| < \tfrac{1}{2}.$$

If $f''(a) = 0$, the auxiliary function Φ defined by (6) cannot be used, and in fact *the above theorem may be false.* For example, if $f(z) = z^m$, where m is an integer greater than 3, it is possible to find points z_1, z_2 in every neighborhood of the origin such that all solutions of (5) satisfy the relation

$$|\theta - \tfrac{1}{2}| \geqq \tfrac{1}{2}.$$

We have this result when $z_1 = e^{\frac{2i\pi}{m}} z_2$, since the only solution of (5) is then

$$\theta = \frac{1}{1 - e^{\frac{2i\pi}{m}}} = \gamma.$$

3. Formulas Involving Integrals. As a preliminary to the development of theorems more general than the preceding, we shall now consider the equation

$$(7) \quad F_1(\theta, z) \equiv \int_a^z f(w, z)\, \phi(w, z)\, dw - f(a, z) \int_a^{a+\theta(z-a)} \phi(w, z)\, dw = 0,$$

which is analogous to the formula of Bonnet's theorem.* The functions $f(w, z)$, $\phi(w, z)$ will be taken as analytic functions of both w and z for all values of those variables satisfying inequalities

$$|z - a| < h, \qquad |w - a| < k,$$

while $f(a, z)$ will be supposed not identically zero in the region defined by the former of the above inequalities.

For the functions $f(w, z)$, $\phi(w, z)$ we have developments

$$f(w, z) = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} a_{\mu\nu}(w - a)^{\mu}(z - a)^{\nu},$$

$$\phi(w, z) = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} \beta_{\mu\nu}(w - a)^{\mu}(z - a)^{\nu}.$$

By substituting these developments in (7), we obtain for $F_1(\theta, z)$ the expansion

$$(8) \qquad F_1(\theta, z) = \sum_{\lambda=0}^{\infty} P_{\lambda+1}(\theta)(z - a)^{\lambda+1},$$

* *Journal de Mathématiques*, vol. 14 (1849), p. 249.

where

$$(9)\quad P_{\lambda+1}(\theta) \equiv \sum_{\mu=0}^{\lambda} \frac{1}{\mu+1} \sum_{\sigma=0}^{\lambda-\mu}\Big[\sum_{\rho=0}^{\mu} a_{\rho\sigma}\beta_{\mu-\rho,\,\lambda-\mu-\sigma} - \theta^{\mu+1}a_{0\sigma}\beta_{\mu,\,\lambda-\mu-\sigma}\Big].$$

For any value $\theta = \theta_1$, and for any positive real number K, there exists a positive real number H such that the series (8) converges throughout the region defined by the inequalities

$$|\theta - \theta_1| < K, \qquad |z - a| < H.$$

The following theorem can now be established :

THEOREM III. *Let $f(w, z)$ and $\phi(w, z)$ be analytic functions of both w and z throughout a region defined by inequalities*

$$|z - a| < h, \qquad |w - a| < k,$$

and let λ_1 be the least value of λ for which the polynomial $P_{\lambda+1}(\theta)$ does not vanish identically. Then if T is any region in the θ-plane of which a root θ_1 of the equation $P_{\lambda_1+1}(\theta) = 0$ is an interior point, there exists a neighborhood A of the point $z = a$ such that the equation (7) is satisfied by a value of θ in T for every value of z in A.

The auxiliary function to be used here is obvious. If θ_1 is not a multiple root of $P_{\lambda_1+1}(\theta) = 0$, so that

$$\frac{\partial}{\partial\theta} P_{\lambda_1+1}(\theta_1) \neq 0,$$

we use the same existence theorem for implicit functions as in the proof of theorem I. In the case of a multiple root we avail ourselves of the more general existence theorem which may be found, for example, in Picard's *Traité d'analyse*, vol. 2 (2d. edit.), p. 261.

From theorem III we can deduce formulas which correspond to familiar theorems of mean value. In the one which follows, an analogue of Bonnet's theorem, it is interesting to note that as z approaches a, θ approaches a point *on* the unit circle instead of a point in its interior as in theorems I, II, and V.

THEOREM IV. *Let $f(z)$ and $\phi(z)$ be analytic throughout a region B of which a is an interior point. Then there exists a neighborhood A of a such that for any value of z in A the equation*

$$(10)\qquad \int_a^z f(w)\phi(w)\,dw - f(a)\int^{a+\theta(z-a)}\phi(w)\,dw = 0$$

has a solution θ satisfying the inequality

$$|\theta - 1| < 1.$$

Equation (10) is a special case of (7) where f and ϕ are functions of but one variable, so that

$$\alpha_{\mu\nu} = \beta_{\mu\nu} = 0, \quad \nu \neq 0.$$

By hypothesis

$$f(a) = \alpha_{00} \neq 0,$$

hence, if $\beta_{\lambda_1 0}$ is the first non-vanishing coefficient $\beta_{\lambda 0}$, we have

$$P_{\lambda_1+1}(\theta) = \frac{\alpha_{00}\beta_{\lambda_1 0}}{\lambda_1 + 1}(1 - \theta^{\lambda_1+1}).$$

Since $\theta = 1$ is always a root of $P_{\lambda_1+1}(\theta) = 0$, theorem IV follows from theorem III, the special region T assigned in IV having been chosen in order to present a formula as closely analagous as possible to the ordinary Bonnet formula.

The two formulas of the following theorem are usually referred to as the first and second theorems of mean value for integrals:

THEOREM V. *Let $f(z)$ and $\phi(z)$ be analytic throughout a region B of which a is an interior point. Then for each of the equations*

$$(11) \qquad \int_a^z f(w)\phi(w)dw - f\Big(a + \theta(z-a)\Big)\int_a^z \phi(w)dw = 0,$$

$$(12) \quad \int_a^z f(w)\phi(w)dw - f(a)\int_a^{a+\theta(z-a)} \phi(w)dw - f(z)\int_{a+\theta(z-a)}^z \phi(w)dw = 0,$$

there exists a neighborhood A of a such that for any value of z in A the corresponding equation has a solution θ satisfying the inequality

$$|\theta - \tfrac{1}{2}| < \tfrac{1}{2}.$$

To deduce this result for formula (11) we make the following substitutions in equation (7):

$$f(w, z) = \int_w^z \phi_1(w)dw,$$

$$\phi(w, z) = f_1'(w).$$

Integration by parts now gives to (7) the form

$$F_1(\theta, z) \equiv \left[f_1(w) \int_w^z \phi_1(w)dw\right]_{w=a}^{w=z} + \int_a^z f_1(w)\phi_1(w)dw$$
$$- \Big[f_1[a + \theta(z - a)] - f_1(a)\Big]\int_a^z \phi_1(w)dw$$
$$\equiv \int_a^z f_1(w)\phi_1(w)\,dw - f_1[a + \theta(z-a)]\int_a^z \phi_1(w)dw = 0.$$

We have only to drop subscripts to obtain equation (11), and the part of our theorem which relates to this equation is demonstrated if we can show that a root of $P_{\lambda_1+1}(\theta) = 0$ lies in the region defined by the inequality

$$|\theta - \tfrac{1}{2}| < \tfrac{1}{2}.$$

The functions $f(w)$ and $\phi(w)$ will have developments

$$f(w) = \sum_{\nu=0}^{\infty} a_\nu (w - a)^\nu,$$
(13)
$$\phi(w) = \sum_{\nu=0}^{\infty} \beta_\nu (w - a)^\nu.$$

If $f(w)$ is a constant, or if $\phi(w) \equiv 0$, equation (11) is obviously an identity. In any other case let a_m and β_k be the first non-vanishing numbers of the respective sequences

$$a_1, a_2, \cdots,$$
$$\beta_0, \beta_1, \cdots.$$

We then have

$$\lambda_1 = m + k,$$
$$P_{\lambda_1+1}(\theta) = a_m\beta_k\left(\frac{1}{m+k+1} - \frac{\theta^m}{k+1}\right),$$

and this last equation always has a real root θ_1 satisfying the inequality

$$|\theta - \tfrac{1}{2}| < \tfrac{1}{2}.$$

Equation (12) may be derived from (7) by the substitution

$$f(w, z) = f(w) - f(z),$$
$$\phi(w, z) = \phi(w).$$

As in the discussion of equation (11), we note that if $f(w)$ is a constant, or if $\phi(w) \equiv 0$, (12) is an identity. In any other case a_m and β_k have the same meaning as before, when $f(w)$ and $\phi(w)$ have the developments (13). We find

$$\lambda_1 = m + k,$$

$$P_{\lambda_1+1}(\theta) = -\frac{a_m \beta_k}{k+1}\left[\frac{m}{m+k+1} - \theta^{k+1}\right],$$

so that in this case also $P_{\lambda_1+1}(\theta) = 0$ has a root between 0 and 1 on the axis of reals.

Cauchy's formula*

$$\frac{f(z) - f(a)}{\phi(z) - \phi(a)} = \frac{f'[a + \theta(z-a)]}{\phi'[a + \theta(z-a)]} \qquad |\theta - \tfrac{1}{2}| < \tfrac{1}{2},$$

and the various forms of Taylor's series with remainder are special cases of the first formula of theorem V. In particular, theorem I is a corollary of this part of V.

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* *Calcul differential*, p. 37.