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More on a Mean Value Theorem Converse

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We note that  $\mathbf{M}$  is an unbounded operator on  $l_2$ . For example, using the Euclidean norm the sequence of unit vectors defined by

$$\mathbf{w}_m = \left\{ \{\mathbf{w}_m\}_n = \frac{1}{\sqrt{m}} \quad \text{for } n \leq m, \quad \text{and } = 0 \text{ for } n > m \right\}$$

transforms into the sequence  $\mathbf{M}\mathbf{w}_m$ , which diverges because

$$\|\mathbf{M}\mathbf{w}_m\|^2 = \frac{1}{m} \sum_{n=1}^m \left( \sum_{k=1}^n \frac{1}{k} \right)^2$$

grows faster than  $\ln(m!)/m \approx \ln m$ , as  $m$  increases.

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## More on a Mean Value Theorem Converse

**H. Fejzić and D. Rinne**

In a recent MONTHLY article Tong and Braza considered two possible versions of a converse to the Mean Value Theorem [2]. For  $c \in (a, b)$ , a continuous function  $f$  on  $[a, b]$  that is differentiable on  $(a, b)$  satisfies the

1. Weak Form at  $c$  if  $f'(c) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha}$  for some interval  $(\alpha, \beta) \subset (a, b)$ , and the
2. Strong Form at  $c$  if  $f'(c) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha}$  for some interval  $(\alpha, \beta) \subset (a, b)$  with  $c \in (\alpha, \beta)$ .

In [2] the authors give a function that fails the Weak Form (and so fails both forms) at all values in a countable closed set. Borwein and Wang provided a function that fails the Weak Form on a residual set (one whose complement is of first category) that is of Lebesgue measure zero [1].

We show that a differentiable function can fail the Weak Form on a set that is both residual and of relative measure arbitrarily close to 1 while the Strong Form must hold on some subset of positive Lebesgue measure. In the rest of this Note *measure* means Lebesgue measure, denoted by  $\lambda$ .

We consider  $[a, b] = [0, 1]$  and let  $Z$  be any measurable set in  $[0, 1]$  with  $\lambda(Z) < 1$ . Let  $E \subset [0, 1] \setminus Z$  be an  $F_\sigma$  set with  $\lambda(E) = \lambda([0, 1] \setminus Z) > 0$  and  $E$  having density 1 at each  $x \in E$  ( $\lim_{\epsilon \rightarrow 0} \lambda(E \cap (x - \epsilon, x + \epsilon)) / (2\epsilon) = 1$ ). Let  $g$  be an approximately continuous function (at each  $x$  the restriction of  $g$  to some subset with density 1 at  $x$  is continuous at  $x$ ) such that:

1.  $0 < g(x) \leq 1$  for  $x \in E$ , and
  2.  $g(x) = 0$  for  $x \notin E$ .
- (1)

A construction of such functions can be found in Zahorski [3]. Since  $g$  is bounded

and approximately continuous it is the derivative of its integral  $f(x) = \int_0^x g(t) dt$ . Therefore  $f' \equiv 0$  on  $Z$ . We can pick  $Z$  to be dense in  $[0, 1]$  and of measure arbitrarily close to 1 with  $E$  having positive measure in every subinterval of  $[0, 1]$ . Then  $f$  is strictly increasing and thus has no difference quotient equal to zero. Hence  $f$  fails the Weak Form at every point of  $\{x|f'(x) = 0\}$  and thus at every point of  $Z$ . Since  $\{x|f'(x) = 0\}$  is a dense  $G_\delta$  (it's the complement of the  $F_\sigma$  set  $E$ ),  $f$  fails the Weak Form on a residual set.

However, the following theorem shows that a differentiable function cannot fail the Weak Form almost everywhere.

**Theorem 1.** *If  $f$  is a continuous function on  $[a, b]$  that is differentiable on  $(a, b)$ , then  $f$  satisfies the Strong Form on a subset of  $[a, b]$  that has positive measure in every subinterval.*

*Proof:* Let  $[\alpha, \beta] \subset [a, b]$ . We may assume that  $f$  is not linear on any subinterval of  $[\alpha, \beta]$  since it would then obviously satisfy the Strong Form there. Let

$$h(x) = \begin{cases} \frac{f(x) - f(\alpha)}{x - \alpha} & \text{for } \alpha < x \leq \beta \\ f'(\alpha) & \text{for } x = \alpha \end{cases}$$

Then  $h$  is continuous on  $[\alpha, \beta]$  and  $h([\alpha, \beta])$  is some nondegenerate interval  $[r, s]$ . Since  $h$  can have only countably many local extrema we can pick  $u \in (\alpha, \beta)$  so that  $h(u)$  is not a local extremum. Let  $c$  be a point in  $(\alpha, u)$  with  $f'(c) = h(u)$ . Using  $p = (c + u)/2$  we see that  $f'(c)$  is in the interior of  $h([\alpha, \beta])$ . Call this interior  $I$ . Let  $g$  be the restriction of  $f$  to the interval  $[\alpha, p]$ . Then  $G = (g')^{-1}(I) \neq \emptyset$  since it contains  $c$  and thus  $\lambda(G) > 0$  by the Denjoy-Clarkson Property (the inverse image under a derivative of an open interval is either empty or of positive measure). For each  $x \in G$ , there is a  $y \in [p, \beta]$  with  $f'(x) = g'(x) = h(y) = (f(y) - f(\alpha))/(y - \alpha)$ . Since  $\alpha < x < y$ ,  $f$  satisfies the Strong Form at  $x$ . ■

As a final comment, we point out that a differentiable function can fail the Strong Form on a set of positive measure and still satisfy the Weak Form on all of  $(a, b)$ . As an example we can simply extend our function  $g$  in (1) to the interval  $[0, 4]$  as follows: Let

$$G(x) = \begin{cases} g(x) & 0 \leq x \leq 1 \\ -g(1)(x - 2) & 1 < x \leq 2 \\ 0 & 2 < x \leq 3 \\ (x - 3) & 3 < x \leq 4 \end{cases}$$

and set  $F(x) = \int_0^x G(t) dt$ . Then  $F$  still fails the Strong Form on the set  $Z$  above but satisfies the Weak Form on  $(0, 4)$ . This is because  $0 \leq G = F' < 1$  on  $(0, 4)$  while the difference quotients for  $F$  inside the interval  $(2, 4)$  assume all values in  $[0, 1)$ .

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