A LOCAL MEAN VALUE THEOREM FOR ANALYTIC FUNCTIONS

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The classical mean value theorem of differential calculus does not extend to the complex plane. The purpose of this note is to establish a local counterpart for analytic functions.

THEOREM. *If f is analytic in a domain containing z_0 then there is a neighborhood N of z_0 such that if z_1 is any point in this neighborhood then there exists a point z with*

$$\left| z - \tfrac{1}{2}(z_0 + z_1)\right| < \tfrac{1}{2}\left| z_1 - z_0\right| ,$$

such that $f(z_1) - f(z_0) = (z_1 - z_0)f'(z)$.

A slightly weaker version of this theorem has been proved by J. M. Robertson [**1**]. As a matter of fact, with the additional assumption that $f''(z_0) \neq 0$, Robertson's proof yields our theorem.

Proof. We may assume that f has the form

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + (z - z_0)^{k+1}h(z),$$

where $k \geqq 1$ is an integer and $h(z_0) \neq 0$.

We may also assume, without loss of generality, that throughout the domain of analyticity we have

$$\left| h(z)\right| \geqq \tfrac{1}{2}\left| h(z_0)\right| \quad \text{and} \quad \left| h'(z)\right| \leqq 1 .$$

It suffices to show that if the neighborhood $N = \{z ; | z - z_0| < r\}$ is chosen so that $0 < r \leqq | h(z_0)|/2(k + 2)$ and $z_1 \in N$, then the function

$$f'(z) - \frac{f(z_1) - f(z_0)}{z_1 - z_0}$$

has exactly one zero in the domain

$$D = \left\{z ; \left| z - \tfrac{1}{2}(z_0 + z_1)\right| < \tfrac{1}{2}\left| z_1 - z_0\right| , \quad \left|\arg \frac{z - z_0}{z_1 - z_0}\right| < \frac{\pi}{k}\right\} .$$

A direct computation shows that

$$f'(z) - \frac{f(z_1) - f(z_0)}{z_1 - z_0} = \Phi(z) + h(z_1)\psi(z),$$

where $\Phi(z) = (z - z_0)^{k+1}h'(z) + (k + 1)(z - z_0)^k(h(z) - h(z_1))$ and

$$\psi(z) = (k + 1)(z - z_0)^k - (z_1 - z_0)^k .$$

If $z \in \partial D$, the boundary of D, then

$$|\Phi(z)| \leqq |z - z_0|^{k+1} \; |h'(z)| + (k+1)|z - z_0|^k \left| \int_{z_1}^{z} h'(\zeta)d\zeta \right|$$
$$\leqq (k+2)|z_1 - z_0|^{k+1}.$$

If z is on the circular arc of ∂D, i.e., if $z = \frac{1}{2}(z_0 + z_1) + \frac{1}{2}(z_1 - z_0)e^{i2\theta}$, $|\theta| \leqq \pi/k$, then

$$|\psi(z)|^2/|z_1 - z_0|^{2k} = 1 + (k+1)((k+1)\cos^k\theta - 2\cos k\theta)\cos^k\theta.$$

Using the inequality

$$(k+1)\cos^k\theta - 2\cos k\theta \geqq 0 \text{ for } |\theta| \leqq \pi/k,\ k = 1,2,\cdots,$$

readily established by induction, we see that $|\psi(z)| \geqq |z_1 - z_0|^k$. If $k > 2$, then the boundary ∂D contains two line segments, namely $z = z_0 + t(z_1 - z_0)e^{\pm i\pi/k}$, $0 \leqq t \leqq \cos\pi/k$. On these line segments we have

$$|\psi(z)| = (1 + (k+1)t^k)|z_1 - z_0|^k \geqq |z_1 - z_0|^k.$$

We have shown that $|\psi(z)| \geqq |z_1 - z_0|^k$ on ∂D. Hence, for $z_1 \in N$ and $z \in \partial D$,

$$\left|\frac{\Phi(z)}{h(z_1)\psi(z)}\right| \leqq \frac{k+2}{|h(z_1)|}|z_1 - z_0| < \frac{|h(z_0)|}{2|h(z_1)|} \leqq 1.$$

By Rouche's theorem we conclude that the functions $\Phi + h(z_1)\psi$ and ψ have equally many zeros in D, namely one. This proves our theorem.

Reference

1. J. M. Robertson, A local mean value theorem for the complex plane, Proc. Edinburgh Math. Soc. (2) 16 (1968/69), 329–331.

A THEOREM ON SET INCLUSION IN METRIC SPACES

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Let A and B be subsets of a metric space (X,d). We shall show that under certain (essentially sharp) conditions, A will be contained in B if $\partial A \subset B$. This result has applications in the study of the stability properties of certain differential equations and to the variation of the spectrum of a Banach algebra element.

For any set A in a metric space (X,d), let A' denote the complement of A, $C(A)$ the closure of A, and ∂A the boundary of A.

THEOREM 1. *Suppose A and B are relatively compact (i.e. $C(A)$ and $C(B)$ are compact) subsets of a non-compact metric space (X,d) with B' connected. Then the condition $\partial A \subset B$ implies $A \subset B$.*