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THE WORD PROBLEM[1]

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In Parts I, II and III the word problem for groups is shown unsolvable. In Part IV, certain stronger results are obtained and related questions discussed. Simplifications are made in the argument of [2].[2] The Turing Machine with two tapes used in [2] has no counterpart in our present account; thus we decrease both the number and complexity of the defining relations required of a group presentation to show it has an unsolvable word problem.[3]

Introduction

RESULT a. *There is exhibited a group given by a finite number of generators and a finite number of defining relations and having an unsolvable word problem.*[4]

To obtain Result a we use the fact that in [20] Post has exhibited a Thue system and shown a problem about the words of this system unsolvable (our Lemma 1 below), but a detailed knowledge of [20] is not required of the reader. In all other respects the argument is self-

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2 Numbers in square brackets [ ] refer to the references given at the end of this paper.

3 In [1] and [2], Parts I-IV, a generalization of the word problem, called the quasi-Magnus problem (*Can an element be written in terms of positive powers of certain generators?*) was shown unsolvable. Parts V and VI of [2] amend the argument so as to yield the word problem result. (The pertinent portions of the revised text of [2] are Parts I, II, Diagrams $\mathcal{E}$ and $\mathcal{F}$ occurring on page 256 of Part III, Parts V and VI). Below we relate [1] and [2] to the present article.

We have proceeded independently of [19] and Novikov's argument is unfortunately still essentially unknown to us. In [14] Markov vouches for the essential correctness of Novikov's proof, describing it as based on [24]. A translation of [19] by K. A. Hirsch is to appear in the American Mathematical Society series. Through J. L. Britton we do know that Novikov uses a certain result of Malcev used by us and that the symmetric argument of [2], Part V. but having no counterpart in the present account, corresponds to a technique of Novikov. At the British Mathematical Colloquium, Nottingham, September 1957, Britton announced a new proof of the unsolvability of the word problem based to some extent on Novikov's proof. (*Added in proof*: Cf. footnote 47, page 263.)

4 The concept of an unsolvable problem is discussed near the end of this Introduction. Here in the Introduction a kind of general knowledge of the subject matter is assumed; terms are used which are systematically defined later in the paper.

contained and consists only of a primitive kind of combinatorial reasoning about how words can cancel. Our hope is that this demonstration (pp. 213-249 below) is presented in such a way that each lemma and auxiliary theorem follows from the earlier ones almost as a triviality.

The argument is made somewhat longer and considerably more abstract than necessary because these combinatorial results are shown for finite presentations of groups in general rather than for certain specific presentations; but this generality seems desirable from several points of view — in particular, it tends to lay bare the motivation.

It will also be shown that in the argument of Result a the Thue system of [20] can easily be replaced by an arbitrary Thue system with unsolvable word problem. In the terminology of Post, Result b asserts that for any Thue system, $\mathfrak{T}$, we can explicitly display, in terms of $\mathfrak{T}$, a finite presentation of a group, $\varphi(\mathfrak{T})$, such that the word problem for $\mathfrak{T}$ is reducible to that for $\varphi(\mathfrak{T})$.[5]

RESULT b. *There is explicitly given a recursive mapping, φ, from the set of Thue systems into the set of presentations of groups. The generators and defining relations of $\varphi(\mathfrak{T})$ are explicitly given in terms of those of $\mathfrak{T}$. The equality of the arbitrary words* **A** *and* **B** *in the Thue system $\mathfrak{T}$ is equivalent to the equality of certain words — explicitly specified in terms of* **A** *and* **B** *— in the group presentation $\varphi(\mathfrak{T})$. Thus if $\mathfrak{T}$ has an unsolvable word problem so also has $\varphi(\mathfrak{T})$.*

In [10] Magnus has shown that any finite presentation of a group consisting of one non-trivial defining relation has a solvable word problem. The presentation of Result a with an unsolvable word problem has a fantastic number of defining relations, the number being closely connected with the number of operations of a Universal Turing Machine (see [23] or [9]). Natural questions to ask are these: What is the smallest number of defining relations which a finite presentation of a group with unsolvable word problem can have? How long and how "complicated" must these relations be? What form may they take? It is known, for example, that finite presentations of Abelian groups have solvable word problems; the number of generators needed for unsolvability is settled by the theorem of Higman, Neumann, and Neumann in [8] — or see [18] — that any denumerably generated group can be (recursively) embedded in a group on two generators and the same number of defining relations as the given group. The following result gives a program for producing, from Thue systems, manageable finite presentations of groups having unsolvable

[5] With the stipulation of explicitness dropped, Result b would follow directly from Result a — noting the Universal Turing Machine concept of [23] or [9].

word problems.[6]

RESULT c. *Let* $g_1, g_2, \cdots, g_N$ *and* $\mathbf{A}_1 = \mathbf{B}_1, \mathbf{A}_2 = \mathbf{B}_2, \cdots, \mathbf{A}_M = \mathbf{B}_M$ *be the generators and defining relations of an arbitrary Thue system* $\mathfrak{T}$. *Let* $\mathbf{P}$ *be any fixed word of* $\mathfrak{T}$. *Then* $\mathfrak{T}_\mathbf{P}$ *is the finite presentation of a group depending on* $\mathfrak{T}$ *and* $\mathbf{P}$ *described as follows*:

The $N + 9$ *generators of* $\mathfrak{T}_\mathbf{P}$:

$$g_1, g_2, \cdots, g_N; \quad q, t_1, t_2, k, a, b, c, d, e$$

The $6N + M + 13$ *non-trivial defining relations of* $\mathfrak{T}_\mathbf{P}$:

$$q\mathbf{A}_j = d^j e^j a c^j d^j q \mathbf{B}_j b^j c^j a e^j b^j, \quad j = 1, 2, \cdots, M$$

$$\left.\begin{aligned} qg_i &= g_i q \\ ag_i &= g_i a \\ cg_i &= g_i c \\ eg_i &= g_i e \end{aligned}\right\} i = 1, 2, \cdots, N, \text{ here and below}$$

$$g_i d = d^{M+1} a d^{M+1} g_i \qquad bg_i = g_i b^{M+1} a b^{M+1}$$

$$\left.\begin{aligned} t_u a &= a t_u \\ t_u c &= c t_u \\ t_u d &= d t_u \\ t_u e &= e t_u \end{aligned}\right\} u = 1, 2 \qquad \begin{aligned} ak &= ka \\ bk &= kb \\ ck &= kc \\ ek &= ke \end{aligned}$$

$$t_1 q \mathbf{P} k \mathbf{P}^{-1} q^{-1} t_1^{-1} = t_2 q \mathbf{P} k \mathbf{P}^{-1} q^{-1} t_2^{-1}$$

For any word $\mathbf{W}$ *of* $\mathfrak{T}$, $\mathbf{W}$ *equals* $\mathbf{P}$ *in* $\mathfrak{T}$ *if and only if*

$$t_1 \mathbf{W} k \mathbf{W}^{-1} t_1^{-1} \text{ equals } t_2 \mathbf{W} k \mathbf{W}^{-1} t_2^{-1}$$

in $\mathfrak{T}_\mathbf{P}$. *Thus if it is recursively unsolvable to determine for an arbitrary word* $\mathbf{W}$ *of* $\mathfrak{T}$ *whether or not* $\mathbf{W}$ *equals* $\mathbf{P}$ *in* $\mathfrak{T}$, *then the word problem for* $\mathfrak{T}_\mathbf{P}$ *is unsolvable.*

In [12] (or see [15], the review by Mostowski) Markov displayed a Thue system with thirty-three defining relations and an unsolvable word problem. A considerable improvement has been announced by Dana Scott in [22] assuming the unsolvability of the word problem for groups, i.e., Result a. While Scott does not assert this, it is easy to verify, using the ideas of Markov [12], that Scott has exhibited a Thue system $\mathfrak{T}$ with seven defining relations such that for a certain fixed word $\mathbf{P}$ of this system it is recursively unsolvable to determine for an arbitrary word $\mathbf{W}$ of $\mathfrak{T}$ whether or not $\mathbf{W}$ equals $\mathbf{P}$ in $\mathfrak{T}$. As pointed out by Hall in [7], every Thue system can be (recursively) embedded in a Thue system on two generators and the same number of defining relations as the given Thue

[6] Result b can also be used in this way but the consequences are not as sharp if number of defining relation is the primary criterion of comparsion.

system.[7] It thus follows from Result c (using the embedding result of [8] noted above) that one can exhibit a finite presentation of a group consisting of two generators and thirty-two defining relations and having an unsolvable word problem. As a method of producing "simple" finite presentations of groups with unsolvable word problems, however, Result c has certain a priori undesirable features. First, more is required of the Thue system concerned than its merely having an unsolvable word problem. Secondly, if a Thue system, $\mathfrak{T}$, does satisfy the antecedent of the last sentence of Result c the word **P** may be extremely long. Indeed, if one argues along the lines of Markov in [12] the length of **P** is closely related to the number of operations of a Universal Turing Machine; in particular, this is the situation as we verify the applicability of Result c to Scott's Thue system so that one of the thirty-two defining relations of the group presentation referred to above is astronomical in length.

This situation can be remedied by using Theorem XI of Part IV (page 251) which relates the question of words being equal in an arbitrary Thue system, $\mathfrak{T}$, to the question of words being equal to some fixed word, **P**, in another Thue system, $\mathfrak{T}_0$, which depends upon $\mathfrak{T}$. Where $\mathfrak{T}$ has N generators and M defining relations, $\mathfrak{T}_0$ has $2N + 2$ generators and $M + N^2 + 2N$ defining relations; the latter are short when the defining relations of $\mathfrak{T}$ are short, while **P** is only three generator occurrences long. Thus starting with a Thue system $\mathfrak{T}$ given by N generators and M defining relations and having an unsolvable word problem, we may first apply Theorem XI, embed the result in a two-generator Thue system, and then apply Result c, obtaining thereby a presentation of a group with eleven generators and $M + N^2 + 2N + 25$ non-trivial defining relations and having an unsolvable word problem. If $\mathfrak{T}$ is taken to be the two generator extension[7] of Scott's Thue system, the eleven-generator, forty-non-trivial-relation group presentation obtained can be explicitly written down in a few minutes time. To this group presentation the two-generator embedding result of Higman, Neumann, and Neumann [8] can, of course, finally be applied although the resulting forty non-trivial defining relations are then more complicatedin appearance.[8]

[7] But since Scott's Thue system has no defining relation of the form *a non-empty word equals the empty word* one may use a more simple correspondence to so embed Scott's system, viz., *the* i^{th} *generator of Scott's system corresponds to* ab^ia.

[8] In his dissertation [21] Michael Rabin has shown, using ideas of Markov in [13], that a very comprehensive class of group theoretic problems are unsolvable as a consequence of the word problem's unsolvability. If Rabin's construction is applied to the group given by the forty defining relations above described, Rabin's original result is sharpened in the sense that the group presentations evolved can be given explicitly.

In Section **35** an easy direct proof is given of the equivalence of Magnus' extended word problem [10] and the (ordinary) word problem. In Section **36** we show directly the unsolvability of the word problem for the finitely generated, infinitely related case, the argument being in effect an alternative proof of a well known result of B. H. Neumann ([17], Theorem 13) coupled with a well-known device of William Craig [6].

In principle, throughout the article, we identify the solvability of a problem with the existence of a Turing Machine to solve the problem; however, precisely what the technical definition of a Turing Machine is need not be brought into our discussion.[9] The reader will find that he follows all the arguments if he uses instead the intuitive notion of an *effective procedure* to solve a problem,[10] i.e., a uniform set of directions which when applied to any one of the questions constituting the problem, produces the correct answer after a finite number of steps, never at any stage of the process leaving the user in doubt as to what to do next.[11]

Post's very short and elegant argument in [20] is more intimately associated with Turing Machines, depending as it does on the unsolvability of a certain problem about some one Turing Machine.

Result a depends upon a logical equivalence called the "Main Theorem." The demonstration of this equivalence in one direction is very easy. In order to illustrate our point of view toward what is called a *proof* as quickly as possible, this easy argument is given in Part I, prior to the introduction of the many general ideas set forth in Part II. At the end of Part I the overall program for showing the Main Theorem in the non-trivial direction is stated. In Part III, then, this program is carried out using the methods developed in Part II. The basic concept of Part II is that of a *marker* (Section **8**); the central argument, that of Reduction D (on page 233) around which the entire demonstration has been built. In turn, the leading idea of Reduction D is given by Diagrams $\mathcal{E}$ and $\mathcal{F}$ (on page 240) so that these diagrams are the heart of the matter.

Questions about the solvability of problems do not enter at all into the demonstration of the Main Theorem but only into its use in conjunction with [20] to obtain Result a.

We briefly relate Result a to the arguments for the unsolvability of the word problem in [2]. First, the finite presentation of a group considered

[9] A Turing Machine may be roughly described as the most general computing machine possible. See [23] or [9].

[10] Correspondingly, then, the word "recursive" would be read "effective". It is *Church's Thesis* that every effective procedure is recursive. The same thesis is stated by Turing in [23], directly in terms of Turing Machines.

[11] The procedure need not be "practical".

here is much simpler in form. Secondly, the symmetric argument of Theorem III*, Case 2, [2], Part V, is replaced by the very simple Lemma 7 of [2], Part II. Speaking vaguely, the idea behind the change is that the inverses of the group generators relative to which a presentation is given can be taken as an anti-isomorphic copy of the generators. Since in any group $\mathbf{A}^{-1}$ equals $\mathbf{B}^{-1}$ is a consequence of $\mathbf{A}$ equals $\mathbf{B}$, $\mathbf{PAP}^{-1}$ equals $\mathbf{QAQ}^{-1}$ of $\mathbf{AP}^{-1}\mathbf{Q}$ equals $\mathbf{P}^{-1}\mathbf{QA}$—and vice versa—the connection between the old, and the new argument is seen from the well-known Tietze transformation theorems. To permit the change just explained, we generalize certain reduction processes of [2] by relativizing the notion of a word being positive to "positive in a certain set of generators".[12] Except for this generalization, and our now stating matters for finite presentations of groups in general—rather than this or that particular presentation as in [2]—all reductions are as given in [2].[13] While the argument for the original version of each reduction is valid, mutatis mutandis, for the generalized version, we have modified certain details and amplified the explanation in the present account. A reduction of Malcev was stated but not shown in [2], as Lemma 6 of Part II. We are indebted to A. H. Clifford for the very short demonstration of this reduction given in Part II of the present article.

Various forms of Post's Lemma II [20] are used throughout the paper. While cancellation semi-groups do not enter at all into the discussion, one crucial concept is drawn from [24]: that of the two-phase Turing Machine. The defining relation 2.10 of page 215 is a disguised form of the phase change operation of a two-phase Machine, the t_1 and t_2 being the last remnants of distinct first and second phase symbols in the sense of Turing. As we have viewed the matter, the extension of the argument for the unsolvability of the quasi-Magnus problem to that for the word problem is essentially a question of adjusting a demonstration concerned with an ordinary single-phase Turing Machine to fit a two-phase Machine.[14]

We proceed somewhat formally with the actual argument which is cast

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[12] In [2] the term *normal* was used instead of the perhaps-more-usual term *positive*.

[13] Indeed, excepting only in so far as the analysis of [2], Part IV, Section 10, differs from its more algebraic counterpart, [2], Part VI, Section 21, these are a subset of the reductions used for the quasi-Magnus problem in [1] and [2], but strung together now in a different order.

Lemma 19 of [2], Section 10, contains a minor slip. (There is no error however in the corresponding argument for the word problem.) Add "if Case 5 or 6" to the first sentence of Lemma 19. For the other cases of Theorem V′ replace references to this lemma by a reference to Lemma 5.1.

[14] For an explanation of the relations connecting the single-phase Turing Machine, the two-phase Machine, the quasi-Magnus problem, and the word problem see [3].

in terms of *finite presentations of* groups (or semi-groups), rather than in terms of the abstract algebraic structures themselves.[15]

PART I

1. *Basic definitions.* We assume as given a certain universe of objects called *symbols*. The letter $\mathfrak{Z}$ is to be a variable for finite sets of symbols. A *word on* $\mathfrak{Z}$ is any finite sequence of the symbols of $\mathfrak{Z}$. The empty word, 1, i.e., the null sequence, is not excluded. A *rule over* $\mathfrak{Z}$ is an ordered pair of words on $\mathfrak{Z}$. The letter $\mathfrak{U}$ is to be a variable for finite sets of rules over any $\mathfrak{Z}$. A *semi-Thue system* or a *substitutional system* — or simply a *system* — is a pair $(\mathfrak{Z}, \mathfrak{U})$ where $\mathfrak{U}$ is a set of rules over $\mathfrak{Z}$.[16]

In any system we use $\mathbf{a}, \mathbf{b}, \cdots$ as variables for symbols and $\mathbf{A}, \mathbf{B}, \cdots$ as variables for words; the rule whose first member is $\mathbf{A}$ and second $\mathbf{B}$ is written $\mathbf{A} \to \mathbf{B}$. The word $\mathbf{AB}$ is the word $\mathbf{A}$ followed by the word $\mathbf{B}$.

A *proof of* $\mathbf{A}/\mathbf{B}$ *in the system* $(\mathfrak{Z}, \mathfrak{U})$ is a finite sequence of words on $\mathfrak{Z}$, termed *steps*, say $\mathbf{C}_1, \mathbf{C}_2, \cdots, \mathbf{C}_n$ such that

(**1**.1) $\mathbf{C}_1$ is $\mathbf{A}$ and $\mathbf{C}_n$ is $\mathbf{B}$;

(**1**.2) Each $\mathbf{C}_\nu$ and $\mathbf{C}_{\nu+1}$, $\nu = 1, 2, \cdots, n-1$, have form $\mathbf{PDQ}$ and $\mathbf{PEQ}$ where $\mathbf{D} \to \mathbf{E}$ is a rule of $\mathfrak{U}$ for some words $\mathbf{P}$ and $\mathbf{Q}$, possibly empty.

We use $\mathbf{A} \vdash_{\mathfrak{Z},\mathfrak{U}} \mathbf{B}$ to assert that there exists a proof of $\mathbf{A}/\mathbf{B}$ in the system $(\mathfrak{Z}, \mathfrak{U})$. The *word problem for the system* $(\mathfrak{Z}, \mathfrak{U})$ is the problem of determining for an arbitrary (ordered) pair of words over $\mathfrak{Z}$, $\mathbf{A}$ and $\mathbf{B}$, whether or not $\mathbf{A} \vdash_{\mathfrak{Z},\mathfrak{U}} \mathbf{B}$.

The system $(\mathfrak{Z}, \mathfrak{U})$ is a *Thue system* or a *finite presentation of a semi-group* if $\mathbf{A} \to \mathbf{B}$ is a rule of $\mathfrak{U}$ whenever $\mathbf{B} \to \mathbf{A}$ is a rule of $\mathfrak{U}$. The Thue system $(\mathfrak{Z}, \mathfrak{U})$ is a *finite presentation of a group* if for every symbol $\mathbf{a}$ of $\mathfrak{Z}$ there is a symbol $\mathbf{b}$ of $\mathfrak{Z}$ such that $1 \to \mathbf{ba}$ and $\mathbf{ba} \to 1$ are rules of $\mathfrak{U}$.

As is well-known, if the system $(\mathfrak{Z}, \mathfrak{U})$ is a Thue system then the relation $\vdash_{\mathfrak{Z},\mathfrak{U}}$ is an equivalence relation on the totality of words over $\mathfrak{Z}$ and

[15] Result a was presented on a United Kingdom lecture tour under the auspices of the Fulbright Inter-foundation Lectureship Program in May 1957; Results b, c and related material were announced at the British Mathematical Colloquium, Nottingham, September 1957. But these accounts, while dealing with the same group presentations as here, used the old symmetric argument of [2].

For a preliminary reading of this lengthy paper we are indebted to Horst Kiesow. Our realization that the symmetric argument of [2], Part V was unnecessary arose obliquely out of discussions with Graham Higman regarding the theorem by him, B. H. Neumann, and Hanna Neumann mentioned in Section **37** of Part IV. Certain suggestions of Hans Hermes regarding the marker convention have been incorporated into Section **8**.

[16] The synonymous term of [1] and [2] is "formal deductive system". Certain obvious changes in the notation for symbols in that account have also been made here.

the equivalence classes defined by this relation form a semi-group[17] under the (compatible) operation of juxtaposing representatives; if $(\mathfrak{Z}, \mathfrak{U})$ is a finite presentation of a group, then this semi-group of equivalence classes is a group. The semi-group (or group) is said to be the semi-group (or group) presented by $(\mathfrak{Z}, \mathfrak{U})$. All this is essentially in agreement with the usual mathematical terminology.[18,19]

Although we do not discuss the matter in detail, it should be pointed out in passing that having a solvable word problem is correctly regarded as a property of finitely presented semi-groups or groups rather than their presentations in the following precise sense: if $(\mathfrak{Z}, \mathfrak{U})$ presents the same semi-group (or group) as $(\mathfrak{Z}', \mathfrak{U}')$ then the solvability of the word problem for $(\mathfrak{Z}, \mathfrak{U})$ implies the solvability of the word problem for $(\mathfrak{Z}', \mathfrak{U}')$.[20]

2. *Exhibition of the finite presentation of a group with unsolvable word problem.* We now frequently write $\mathfrak{T}$ for the system $(\mathfrak{Z}, \mathfrak{U})$, $\mathfrak{T}'$ for the system $(\mathfrak{Z}', \mathfrak{U}')$, and $\mathfrak{T}_i$, $i = 1, 2, \cdots$, for the system $(\mathfrak{Z}_i, \mathfrak{U}_i)$; $\vdash_{\mathfrak{T}_i}$ is abbreviated $\vdash_i$. In connection with Thue systems we now write $\mathbf{A} \leftrightarrow \mathbf{B}$ for $\mathbf{B} \rightarrow \mathbf{A}$, $\mathbf{A} \rightarrow \mathbf{B}$. The rules of the Thue system are then specified by a table of expressions of form $\mathbf{A} \leftrightarrow \mathbf{B}$ called the *rule couples* of the system. We say that $\mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{B} \rightarrow \mathbf{A}$ are *converses of each other*.

The system $\mathfrak{T}_1$ given below can be taken to be any Thue system having the form stipulated. The system $\mathfrak{T}_2$, which depends on $\mathfrak{T}_1$, is a finite presentation of a group. As will be explained, for suitably chosen $\mathfrak{T}_1$, e.g., the Thue system of Post [20] with unsolvable word problem, the word problem is unsolvable for the resulting system $\mathfrak{T}_2$.

$\mathfrak{T}_1$

$\mathfrak{Z}_1$: $s_1, s_2, \cdots, s_M;\ q_1, q_2, \cdots, q_N, q;$

$\mathfrak{U}_1$: $\Sigma_1 \leftrightarrow \Gamma_1,\ \Sigma_2 \leftrightarrow \Gamma_2, \cdots, \Sigma_P \leftrightarrow \Gamma_P$ where each Σ_ι and Γ_ι, $\iota = 1, 2, \cdots, P$, is of the form $\Delta q_\alpha \Pi$, Δ and Π being words on $s_1, s_2, \cdots, s_M$ and q_α being $q_1, q_2, \cdots, q_N$, or q.

17 A semi-group is a set of objects, S, with an associated binary law of composition defined for any ordered pair of elements of S and which is associative.

18 Various definitions for *finite presentation of a group* suggest themselves, but for our purposes they are all equivalent to the one given. Defining a finite presentation of a group to be a Thue system, such that the semi-group presented is a group, is inadequate, for it is a result of Markov [13] that there is no recursive method to determine for an arbitrary Thue system whether or not it has this property.

19 Of course $\mathbf{A} \rightarrow \mathbf{B}$ and $\mathbf{B} \rightarrow \mathbf{A}$ are one defining relation in the usual terminology.

20 This follows from the fact that all the pairs of equal words of a presentation can be recursively enumerated with a consequent recursive (but in general completely impractical) procedure to find the isomorphism between two presentations known to be isomorphic.

$\mathfrak{T}_2$

$\mathfrak{Z}_2$: All symbols of $\mathfrak{Z}_1$;

$t_1, t_2, k, x, y, l_\iota, r_\iota$, $\iota = 1, 2, \cdots, P$;

Each of the above symbols with a bar superimposed.

$\mathfrak{U}_2$: Where $\iota = 1, 2, \cdots, P$, $\alpha = 1, 2$, and $\beta = 1, 2, \cdots, M$, the rule couples 2.1 through 2.9 are rules of $\mathfrak{U}_2$:

2.1 $\Sigma_\iota \leftrightarrow l_\iota \Gamma_\iota r_\iota$

2.2	$s_\beta l_\iota \leftrightarrow yl_\iota ys_\beta$	2.6	$r_\iota s_\beta \leftrightarrow s_\beta xr_\iota x$
2.3	$s_\beta y \leftrightarrow yys_\beta$	2.7	$xs_\beta \leftrightarrow s_\beta xx$
2.4	$t_\alpha l_\iota \leftrightarrow l_\iota t_\alpha$	2.8	$r_\iota k \leftrightarrow kr_\iota$
2.5	$t_\alpha y \leftrightarrow yt_\alpha$	2.9	$xk \leftrightarrow kx$

2.10 $k\bar{q}\bar{t}_1t_2q \leftrightarrow \bar{q}\bar{t}_1t_2qk$

Where **a** is any symbol of $\mathfrak{Z}_2$ without a bar, the rules 2.11 and 2.12 are rules of $\mathfrak{U}_2$:

2.11 $\bar{\mathbf{a}}\mathbf{a} \leftrightarrow 1$

2.12 $\mathbf{a}\bar{\mathbf{a}} \leftrightarrow 1$

3. *Statement of the key theorem.* The following remarks are illustrated by the tables for $\mathfrak{T}_1$ and $\mathfrak{T}_2$. We now assume that there are two distinct kinds of symbols, *unbarred* and *barred*. Unbarred symbols are lightface lower-case Latin italic letters possibly with a subscript from some well-defined set added. A barred symbol is an unbarred symbol with a bar added. According as **a** is an unbarred or barred symbol, the symbol $\bar{\mathbf{a}}$ is the symbol **a** with the bar added or removed; $\bar{1}$ is to be 1 and $\overline{\mathbf{Ba}}$ is to be $\bar{\mathbf{a}}\bar{\mathbf{B}}$. Thus $\bar{\mathbf{A}}$ is well-defined for any word **A**. Note $\bar{\bar{\mathbf{A}}}$ is **A**. We now further require of any finite presentation of a group $(\mathfrak{Z}, \mathfrak{U})$ that if **a** is a symbol of $\mathfrak{Z}$ then so is $\bar{\mathbf{a}}$ and $\bar{\mathbf{a}}\mathbf{a} \leftrightarrow 1$ are rules of $\mathfrak{U}$. We use $\alpha, \beta, \cdots$ as variables for all kinds of subscripts on symbols — the blank subscript included. The letters Δ and Π are to be variables for words on $s_1, s_2, \cdots, s_M$; and Σ and Γ are variables for words on $\mathfrak{Z}_1$ of the form $\Delta q_\alpha \Pi$, — words which we shall call *special words*.

We now take the following result from Post [20].[21]

LEMMA 1. *For a certain choice of $\mathfrak{T}_1$ it is recursively unsolvable to determine for an arbitrary special word Σ on $\mathfrak{Z}_1$, whether or not $\Sigma \vdash_1 q$.*

[21] To the Thue system given by Post in [20] we must add the symbol q and the rule couple $hq_{N+2}h \leftrightarrow q$ to obtain Lemma 1 in the form in which we have stated it. Alternatively, we can interpret q to be the word $hq_{N+2}h$, for the proof of the Main Theorem is valid if q is interpreted as an arbitrary special word. This reinterpretation of notation is actually used in the argument for Result c in Part IV.

Main Theorem. FOR ANY CHOICE OF $\mathfrak{T}_1$ AND FOR ANY SPECIAL WORD Σ ON $\mathfrak{Z}_1$, $\Sigma \vdash_1 q$ IF AND ONLY IF $t_1\Sigma k\bar{\Sigma}\bar{t}_1 \vdash_2 t_2\Sigma k\bar{\Sigma}\bar{t}_2$.

Now suppose that the word problem for $\mathfrak{T}_2$ were solvable when $\mathfrak{T}_1$ is taken to be the Thue system of Post [20]. Then, in particular, it would be possible to determine for any two words of the form $t_1\Sigma k\bar{\Sigma}\bar{t}_1$ and $t_2\Sigma k\bar{\Sigma}\bar{t}_2$ whether or not $t_1\Sigma k\bar{\Sigma}\bar{t}_1 \vdash_2 t_2\Sigma k\bar{\Sigma}\bar{t}_2$. From the Main Theorem, then, it would follow immediately that the problem of determining for an arbitrary special word Σ on $\mathfrak{Z}_1$ whether or not $\Sigma \vdash_1 q$, would be solvable. This would contradict Lemma 1.

Thus the Main Theorem implies the unsolvability of the word problem for a particular finitely presented group. We now undertake its demonstration.

4. *Some elementary results.* We first review, in terms of our formal terminology, certain familiar facts so as to illustrate our point of view towards what we have called a "proof". We now often write "group presentation" for "finite presentation of a group". For any group presentation $(\mathfrak{Z}, \mathfrak{U})$ the sequences of rules of $\mathfrak{U}$, $ins(\mathbf{A}\bar{\mathbf{A}})$ and $del(\mathbf{A}\bar{\mathbf{A}})$ are defined for any word $\mathbf{A}$ as follows. The sequence $ins(1\bar{1})$ is to be the empty sequence of rules; $ins(\mathbf{B}\mathbf{a}\bar{\mathbf{a}}\bar{\mathbf{B}})$ is the sequence consisting of $ins(\mathbf{B}\bar{\mathbf{B}})$ followed by $1 \to \mathbf{a}\bar{\mathbf{a}}$. The sequence $del(\mathbf{A}\bar{\mathbf{A}})$ is the sequence obtained from $ins(\mathbf{A}\bar{\mathbf{A}})$ by replacing each rule by its converse and reversing order. Obviously, and as the notation is meant to suggest, $ins(\mathbf{A}\bar{\mathbf{A}})$ effects a proof of $1/\mathbf{A}\bar{\mathbf{A}}$ in any group presentation, and $del(\mathbf{A}\bar{\mathbf{A}})$ of $\mathbf{A}\bar{\mathbf{A}}/1$.[22]

LEMMA 2. *For any semi-Thue system* $\mathfrak{T}$, *if* $\mathbf{A} \to \mathbf{B}$ *is a rule of* $\mathfrak{U}$, *then* $\mathbf{A} \vdash_{\mathfrak{T}} \mathbf{B}$.

LEMMA 3. *For any Thue system* $\mathfrak{T}$:

(3.1) $\mathbf{A} \vdash_{\mathfrak{T}} \mathbf{A}$.

(3.2) $\mathbf{A} \vdash_{\mathfrak{T}} \mathbf{B}$ *implies* $\mathbf{B} \vdash_{\mathfrak{T}} \mathbf{A}$.

(3.3) $\mathbf{A} \vdash_{\mathfrak{T}} \mathbf{B}$ *and* $\mathbf{B} \vdash_{\mathfrak{T}} \mathbf{C}$ *implies* $\mathbf{A} \vdash_{\mathfrak{T}} \mathbf{C}$.

(3.4) $\mathbf{A} \vdash_{\mathfrak{T}} \mathbf{B}$ *and* $\mathbf{C} \vdash_{\mathfrak{T}} \mathbf{D}$ *implies* $\mathbf{AC} \vdash_{\mathfrak{T}} \mathbf{BD}$.

The two-step proof $\mathbf{A}$, $\mathbf{B}$ shows Lemma 2. The one-step proof consisting of $\mathbf{A}$ alone shows Lemma 3.1. Reversing the steps in a given proof of $\mathbf{A}/\mathbf{B}$, yields a proof of $\mathbf{B}/\mathbf{A}$ showing Lemma 3.2. A proof of $\mathbf{A}/\mathbf{B}$ followed by a proof of $\mathbf{B}/\mathbf{C}$ (with the first step omitted) is a proof of $\mathbf{A}/\mathbf{C}$ thus showing Lemma 3.3. The demonstration of Lemma 3.4 is also trivial. Usually Lemma 3 is used without comment.

[22] But at the moment $ins(\mathbf{A}\bar{\mathbf{A}})$ and $del(\mathbf{A}\bar{\mathbf{A}})$ are simply sequences of rules depending on $\mathbf{A}$ with no implication that they are to be applied in a certain way intended.

LEMMA 4. *For any group presentation* $\mathfrak{T}$:

(4.1) $1 \vdash_{\mathfrak{T}} \mathbf{A\bar{A}}$.

(4.2) $\mathbf{A} \vdash_{\mathfrak{T}} \mathbf{B}$ *implies* $\mathbf{\bar{A}} \vdash_{\mathfrak{T}} \mathbf{\bar{B}}$.

(4.3) $\mathbf{CP} \vdash_{\mathfrak{T}} \mathbf{QC}$ *implies* $\mathbf{C\bar{P}} \vdash_{\mathfrak{T}} \mathbf{\bar{Q}C}$.

(4.4) $\mathbf{ABC} \vdash_{\mathfrak{T}} \mathbf{D}$ *implies* $\mathbf{B} \vdash_{\mathfrak{T}} \mathbf{\bar{A}D\bar{C}}$.

(4.5) $\mathbf{A\bar{P}Q} \vdash_{\mathfrak{T}} \mathbf{\bar{P}QA}$ *implies* $\mathbf{PA\bar{P}} \vdash_{\mathfrak{T}} \mathbf{QA\bar{Q}}$.

As we noted before, $ins(\mathbf{A\bar{A}})$ effects a proof of $1/\mathbf{A\bar{A}}$ in $\mathfrak{T}$ so that Lemma 4.1 is clear. Assuming $\mathbf{A} \vdash_{\mathfrak{T}} \mathbf{B}$ and using Lemmas 4.1 and 3, $\mathbf{\bar{A}} \vdash_{\mathfrak{T}} \mathbf{\bar{A}B\bar{B}}$, $\mathbf{\bar{A}B\bar{B}} \vdash_{\mathfrak{T}} \mathbf{\bar{A}A\bar{B}}$, and $\mathbf{\bar{A}A\bar{B}} \vdash_{\mathfrak{T}} \mathbf{\bar{B}}$. Hence Lemma 4.2 by Lemma 3.3. Lemmas 4.3, 4.4 and 4.5 are almost as trivial.[23]

5. *Proof of the Main Theorem in the trivial direction.* We use Ξ as a variable for words on y, $\bar{y}$, or any l_α or $\bar{l}_\alpha$ of $\mathfrak{Z}_2$; Ω, for words on x, $\bar{x}$, or any r_α or $\bar{r}_\alpha$ of $\mathfrak{Z}_2$. Where $\mathbf{W}$ is any word, $|\mathbf{W}|$, *the length of* $\mathbf{W}$, is the number of symbol occurrences making up $\mathbf{W}$.

LEMMA 5. *For any* Ξ, Δ, *and* Ω:

(5.1) *There is a* Ξ' *such that* $\Delta\Xi \vdash_2 \Xi'\Delta$;

(5.2) *There is an* Ω' *such that* $\Omega\Delta \vdash_2 \Delta\Omega'$;

(5.3) $t_\nu \Xi \vdash_2 \Xi t_\nu$, $\nu = 1, 2$;

(5.4) $\Omega k \vdash_2 k\Omega$.

Clearly $s_\beta l_\alpha \vdash_2 y l_\alpha y s_\beta$ and $s_\beta y \vdash_2 y y s_\beta$ by the rules $\mathfrak{U}_{2.2}$ and $\mathfrak{U}_{2.3}$ and Lemma 2. Hence $s_\beta \bar{l}_\alpha \vdash_2 \bar{y} \bar{l}_\alpha \bar{y} s_\beta$ and $s_\beta \bar{y} \vdash_2 \bar{y}\bar{y} s_\beta$ by Lemma 4.3. Thus Lemma 5.1 follows by an induction on the length of $\Delta\Xi$. Noting the rules $\mathfrak{U}_{2.6}$ and $\mathfrak{U}_{2.7}$ for Lemma 5.2, the rules $\mathfrak{U}_{2.4}$ and $\mathfrak{U}_{2.5}$ for Lemma 5.3, the rules $\mathfrak{U}_{2.8}$ and $\mathfrak{U}_{2.9}$ for Lemma 5.4, these latter results also follow by induction on the length of the first step of the proof concerned, using Lemmas 2 and 4.3 for all of them.

THEOREM I. *If* $\Sigma \vdash_1 q$, *then there are* Ξ *and* Ω *such that* $\Sigma \vdash_2 \Xi q \Omega$.

The following lemma implies this theorem:

(I†) *If* $\Sigma \vdash_1 \Sigma'$ *of* n *steps, then there are* Ξ *and* Ω *such that* $\Sigma \vdash_2 \Xi\Sigma'\Omega$.

For $n = 1$, we note that $\Sigma \vdash_1 \Sigma$. If $n > 1$, let Σ' be the result of applying the rule $\Gamma \to \Gamma'$ of $\mathfrak{U}_1$ to $\Delta\Gamma\Pi$, so that Σ' itself is $\Delta\Gamma'\Pi$. The induction hypothesis then asserts that there exist Ξ and Ω such that $\Sigma \vdash_2 \Xi\Delta\Gamma\Pi\Omega$. By a comparison of $\mathfrak{U}_1$ and $\mathfrak{U}_2$, $\mathfrak{U}_2$ contains either the rule $\Gamma \to \Xi'\Gamma'\Omega'$ or the rule $\Gamma' \to \Xi'\Gamma\Omega'$ for some Ξ' and Ω' since $\mathfrak{U}_1$ contains the rule $\Gamma \to \Gamma'$. In either case $\Gamma \vdash_2 \Xi''\Gamma'\Omega''$ for some Ξ'' and Ω'' by Lemma 4.4. Hence $\Sigma \vdash_2 \Xi\Delta\Xi''\Gamma'\Omega''\Pi\Omega$. But for some Ξ''' and Ω''', $\Delta\Xi'' \vdash_2 \Xi'''\Delta$ and $\Omega''\Pi \vdash_2 \Pi\Omega'''$ by Lemmas 5.1 and 5.2. This shows (I†).

[23] Lemma 37, a generalization of Lemma 4.5, is shown below on page 253.

The demonstration of the next theorem will serve to introduce certain notations. In connection with any system H, K, V, T, and U are variables for sequences of operation rules,—the empty sequence not being excluded. The sequence KV is the sequence K followed by the sequence V. We use $H(\mathbf{A}/\mathbf{B})$ for a proof $\mathbf{A}/\mathbf{B}$ effected by means of H. The following notation, as well as certain obvious elaborations of it, is called a *diagram*:

$$\begin{array}{ll} \mathbf{A} & \\ & H \\ \mathbf{B} & \end{array}$$

The above diagram stands simply for the proof $H(\mathbf{A}/\mathbf{B})$. The following diagram stands for the proof $HH'(\mathbf{A}/\mathbf{C})$ where this proof consists of $H(\mathbf{A}/\mathbf{B})$ followed by $H'(\mathbf{B}/\mathbf{C})$ with the last step of $H(\mathbf{A}/\mathbf{B})$ taken as the first of $H'(\mathbf{B}/\mathbf{C})$:

$$\begin{array}{ll} \mathbf{A} & \\ & H \\ \mathbf{B} & \\ & H' \\ \mathbf{C} & \end{array}$$

We must always, of course, specify in what system the proofs discussed occur. We may write "the proof H" for "$H(\mathbf{A}/\mathbf{B})$" if no confusion can arise.

THEOREM II. *If* $\Sigma \vdash_2 \Xi q \Omega$ *for some* Ξ *and* Ω, *then* $t_1 \Sigma k \overline{\Sigma} \overline{t}_1 \vdash_2 t_2 \Sigma k \overline{\Sigma} \overline{t}_2$.

Suppose given an $H(\Sigma/\Xi q \Omega)$ in $\mathfrak{T}_2$ for some Ξ and Ω. By Lemma 3.2 there is an $H'(\Xi q \Omega/\Sigma)$ in $\mathfrak{T}_2$. Further by Lemmas 5.3 and 3.2 there are proofs $V_1(t_1\Xi/\Xi t_1)$ and $V_2(\Xi t_2/t_2\Xi)$ in $\mathfrak{T}_2$; by Lemmas 5.4 and 3.2 there are proofs $K(\Omega k/k\Omega)$ and $K'(k\Omega/\Omega k)$ in $\mathfrak{T}_2$. Hence by Lemma 4.2 there are proofs $\overline{H}(\overline{\Sigma}/\overline{\Omega}\overline{q}\overline{\Xi})$, $\overline{H}'(\overline{\Omega}\overline{q}\overline{\Xi}/\overline{\Sigma})$, $\overline{V}_1(\overline{\Xi}\overline{t}_1/\overline{t}_1\overline{\Xi})$, $\overline{V}_2(\overline{t}_2\overline{\Xi}/\overline{\Xi}\overline{t}_2)$ in $\mathfrak{T}_2$. By Lemma 4.1, $\Omega\overline{\Omega} \vdash_2 1$ and $1 \vdash_2 \Omega\overline{\Omega}$, effected by means of $del(\Omega\overline{\Omega})$ and $ins(\Omega\overline{\Omega})$ respectively. In view of $\mathfrak{U}_{2.10}$, Lemmas 2 and 4.5, there is a $U(t_1 q k \overline{q} \overline{t}_1/t_2 q k \overline{q} \overline{t}_2)$ in $\mathfrak{T}_2$. Containing the proofs just described, the following diagram (which continues on the next page) then gives a proof of $t_1 \Sigma k \overline{\Sigma} \overline{t}_1/t_2 \Sigma k \overline{\Sigma} \overline{t}_2$ in $\mathfrak{T}_2$:

$$\begin{array}{ll} t_1 \Sigma k \overline{\Sigma} \overline{t}_1 & \\ & H\overline{H} \\ t_1 \Xi q \Omega k \overline{\Omega} \overline{q} \overline{\Xi} \overline{t}_1 & \\ & K\ del(\Omega\overline{\Omega}) \\ t_1 \Xi q k \overline{q} \overline{\Xi} \overline{t}_1 & \\ & V_1 \overline{V}_1 \\ \Xi t_1 q k \overline{q} \overline{t}_1 \overline{\Xi} & \end{array}$$

	U
$\Xi t_2 q k \bar{q} \bar{t}_2 \bar{\Xi}$	
	$V_2 \bar{V}_2$
$t_2 \Xi q k \bar{q} \bar{\Xi} \bar{t}_2$	
	$ins(\Omega\bar{\Omega})\ K'$
$t_2 \Xi q \Omega k \bar{\Omega} \bar{q} \bar{\Xi} \bar{t}_2$	
	$H' \bar{H}'$
$t_2 \Sigma k \bar{\Sigma} \bar{t}_2$	

6. *Program for showing the Main Theorem in the non-trivial direction.* We first stipulate three additional group presentations.

$\mathfrak{T}_3$

$\mathfrak{Z}_3$: All symbols of $\mathfrak{Z}_2$.
$\mathfrak{U}_3$: All the operation rules of $\mathfrak{U}_2$ except that each rule couple $\mathbf{a}k \leftrightarrow k\mathbf{a}$ of $\mathfrak{U}_{2.8}$ and $\mathfrak{U}_{2.9}$ is replaced by $\mathbf{a}k \leftrightarrow k$ and $\mathfrak{U}_{2.10}$ is replaced by $t_1 qk \leftrightarrow t_2 qk$.

$\mathfrak{T}_4$

$\mathfrak{Z}_4$: All symbols of $\mathfrak{Z}_3$.
$\mathfrak{U}_4$: All the operation rules of $\mathfrak{U}_3$ except that each rule couple $t_\nu \mathbf{a} \leftrightarrow \mathbf{a} t_\nu$ of $\mathfrak{U}_{2.4}$ and $\mathfrak{U}_{2.5}$ is replaced by $t_\nu \mathbf{a} \leftrightarrow t_\nu$ and $t_1 qk \leftrightarrow t_2 qk$ is excluded.

$\mathfrak{T}_5$

$\mathfrak{Z}_5$: All symbols of $\mathfrak{Z}_4$ except k, $\bar{k}$, t , $\bar{t}$, $\nu = 1, 2$.
$\mathfrak{U}_5$: All rules of $\mathfrak{U}_4$ not containing occurrences of k, $\bar{k}$, t_ν, $\bar{t}_\nu$, $\nu = 1, 2$.

Now let $\mathbf{A} \vdash_{iq} \mathbf{B}$, $i = 1, 2, \cdots 5$, mean $\mathbf{A} \vdash_i \mathbf{B}$ with no occurrences of q, $\bar{q}_1, \cdots, \bar{q}_N$ in any step. Then the plan of the argument is to show that the first statement in each brace implies the second:

$t_1 \Sigma k \bar{\Sigma} \bar{t}_1 \vdash_2 t_2 \Sigma k \bar{\Sigma} \bar{t}_2$	
	THEOREM III
$t_1 \Sigma k \bar{\Sigma} \bar{t}_1 \vdash_{2k} t_2 \Sigma k \bar{\Sigma} \bar{t}_2$	
	THEOREM IV
$t_1 \Sigma k \vdash_{3k} t_2 \Sigma k$	
	THEOREM V
$t_1 \Sigma k \vdash_{3kt} t_2 \Sigma k$	
	THEOREM VI
$\Sigma \vdash_5 \Xi q \Omega$ for some Ξ and Ω	
$\Sigma \vdash_5 \Xi q \Omega$	
	THEOREM VII
$\Sigma \vdash_{5q} \Xi q \Omega$	
	THEOREM VIII
$\Sigma \vdash_{5qs} \Xi q \Omega$	
	THEOREM IX
$\Sigma \vdash_1 q$	

It will be seen later in Part III that Theorems IV, VI, and IX have obvious demonstrations. (We shall use $\mathfrak{T}_4$ in showing Theorem VI.) Theorems III, V, VII, and VIII are non-trivial; to handle these matters we turn to the development of general methods.

PART II

Roughly speaking, the general results developed in Part II for group presentations of a certain form show that applications of the rules $1 \leftrightarrow \bar{\mathbf{a}}\mathbf{a}$ are useless under certain circumstances; thus our reduction processes are irrelevant modifications in the sense of Post. A particular technique used is to consider mappings of sequences of operation rules of a given group presentation into the words of a free group. For the most part the reader may ignore the specific structure of the earlier systems $\mathfrak{T}_1, \cdots, \mathfrak{T}_5$ in studying the theorems demonstrated in Part II but for purposes of illustration, we do relate here in Part II, rather than later, certain of the definitions given to the systems of Part I.

7. *Certain definitions.* In connection with the arbitrary system $\mathfrak{T}$ we make the following definitions, letting $\mathfrak{Z}'$ be any subset of the symbols of $\mathfrak{Z}$. Any symbol of $\mathfrak{Z}'$ is a *$\mathfrak{Z}'$-symbol.* A word containing no occurrences of $\mathfrak{Z}'$-symbols is *$\mathfrak{Z}'$-free.* Any rule $\mathbf{A} \to \mathbf{B}$ of $\mathfrak{U}$, $\mathbf{A}$ and $\mathbf{B}$ $\mathfrak{Z}'$-free, is *$\mathfrak{Z}'$-free.* A proof in which all steps are $\mathfrak{Z}'$-free (and consequently using only $\mathfrak{Z}'$-free rules) is *$\mathfrak{Z}'$-free.*

We now use $g, p, \cdots$ as variables for lightface lower-case Latin italic letters, i.e., as variables to be replaced by such letters as k, t, q, and s in the applications of the general theorems of Part II. For each g the symbols $g_\alpha, g_\beta, \cdots, g_\pi$ of $\mathfrak{Z}$ make up the *unbarred letter set* g; the symbols $\bar{g}_\alpha, \bar{g}_\beta, \cdots, \bar{g}_\pi$ of $\mathfrak{Z}$ the *barred letter set* $\bar{g}$. (Either g or $\bar{g}$ may be vacuous.) Clearly any symbol of $\mathfrak{Z}$ belongs to exactly one letter set.[24] The set $\hat{g}$ is the union of the sets g and $\bar{g}$; $\hat{g}_\alpha$ is a variable replaceable by g_α, $\bar{g}_\alpha$. The rules $1 \to g_\alpha \bar{g}_\alpha$ and $1 \to \bar{g}_\alpha g_x$ are *$\bar{g}$-insertion* rules — *right* and *left* respectively; their converses are *right* and *left* *$\bar{g}$-deletion rules.* Any insertion or deletion rule is *trivial.* Thus the $\bar{t}$-symbols of $\mathfrak{Z}_2$ are $\bar{t}_1$, $\bar{t}_2$, and the $\hat{k}$-symbols are k, $\bar{k}$. The rule $s_1\bar{s}_1 \to 1$ is a right $\bar{s}$-deletion rule of $\mathfrak{U}_2$. A rule of $\mathfrak{U}_{2.1}$ or $\mathfrak{U}_{2.3}$ is $\hat{k}$-free.

8. *The marker convention.* We now take the following point of view. The symbol occurrences making up a step of any proof under consideration and in the arbitrary system $\mathfrak{T}$ are a row of physical objects called

[24] While these definitions depend on the accidental form of notation, it is convenient to put matters in this way. In general we use barred letters only in connection with finite presentations of groups.

markers.[25] The application of the rule $\mathbf{A} \rightarrow \mathbf{B}$ of $\mathfrak{U}$ to step $\mathbf{C}_k$, of form **PAQ**, to obtain step $\mathbf{C}_{k+1}$, of form **PBQ**, is the operation O of replacing the row of markers **A** in $\mathbf{C}_k$ by a new row of markers which spell out **B** and which do not occur in any step prior to $\mathbf{C}_{k+1}$. The row of the markers making up **P** in $\mathbf{C}_k$ are not affected by O and thus occur undisturbed in $\mathbf{C}_{k+1}$. The same is true for the markers making up **Q** in $\mathbf{C}_k$. We call this point of view the *basic convention* and, subject to an emendation to be given in a moment, it is always to be taken. So as to have notations which depend upon O, the four rows of markers $\mathbf{C}_k$, $\mathbf{C}_{k+1}$, **A** in $\mathbf{C}_k$, and **B** in $\mathbf{C}_{k+1}$ will be respectively called [O, *the premiss of* O; O], *the conclusion of* O; {O, *the argument of* O; and O}, *the value of* O. As is convenient, we modify the basic convention by stipulating that if O is the application of any one of a certain specified set of non-trivial rules then not all the markers in {O are removed from the proof by O and so continue to occur in O}.

This is done by giving a proviso called a *g-qualification* for each non-vacuous unbarred letter set g of $\mathfrak{Z}$ in the following way. First we designate some set of rules of $\mathfrak{U}$ each of which is of the form

$$(\mathbf{8}g)\ \mathbf{M}g_\alpha\mathbf{D} \rightarrow \mathbf{E}g_\beta\mathbf{F}, \qquad \mathbf{M}, \mathbf{D}, \mathbf{E}, \mathbf{F}\ \hat{g}\text{-free}$$

as *g-shift* rules. Secondly, we amend the basic convention thus: If O is the application of a g-shift rule then the occurrence of the g-symbol in O} is the same marker as the occurrence of the g-symbol in {O.[26]

A particular *marker convention*, i.e., a precise declaration of what convention is to apply to some proof being analyzed, is completely determined by specifying the g-shifts for every non-vacuous unbarred letter set g of $\mathfrak{Z}$. In the *null g-qualification* no rules are called g-shift rules; in the *universal g-qualification* every rule of form $\mathbf{8}g$ is called a g-shift rule. Note that a rule may be both a g-shift rule and a p-shift rule where g and p are different non-vacuous letter sets; this cannot lead to a contradiction of our physical interpretation, however, since g and p are disjoint.

In connection with proofs in $\mathfrak{T}_1, \cdots, \mathfrak{T}_5$ we always enforce the basic convention along with the universal k-qualification, universal t-qualification, universal q-qualification and the s-qualification wherein the rules of $\mathfrak{U}_{2.2}$, $\mathfrak{U}_{2.3}$, $\mathfrak{U}_{2.6}$, and $\mathfrak{U}_{2.7}$ are designated s-shifts. For all other unbarred letter sets the qualifications are null. It should be clear to the reader

[25] Cf. [2], Part I, p. 235. At the end of this section these physical definitions are replaced by formal ones.

[26] In other words, we regard a g-shift, *in so far as it affects $\hat{g}$-markers*, as the operation of merely changing the subscript on a g-marker.

that the universal k-qualification is a natural point of view to demonstrate Theorem III, the universal t-qualification to demonstrate Theorem V, and the universal q-qualification to demonstrate Theorem VII.

In any system under any marker convention, then, a particular marker has at most one occurrence in any step, and the steps in which it occurs are consecutive; the marker carries a bar either in every such step or in none, — and is called accordingly a *barred* or *unbarred* marker. All occurrences of a barred marker are occurrences of the same barred symbol; if this symbol is a $\bar{g}$-symbol the marker is a *$\bar{g}$-marker*. All occurrences of an unbarred marker are occurrences of unbarred symbols belonging to the same letter set; if these symbols are g-symbols the marker is a *g-marker*. Both g- and $\bar{g}$-markers are *$\hat{g}$-markers*.

We shall say that a marker which occurs in $\{$O but not in O$\}$ *leaves the proof via* O; in O$\}$, but not in $\{$O, that it *enters the proof via* O. If [O is **P**$\{$O**Q**, then O *occurs*, or *is performed*, *right of* **P** or any marker in P. If a marker occurs in $\{$O, O is said to be *performed on* that marker.

Terms defined for rules will be used for an operation if applicable to the rule applied by the operation; e.g., we call an operation a g-shift or a $\bar{g}$-insertion.

To implement the marker convention we shall use non-negative integers as first superscripts on symbol occurrences making up a proof. A particular marker is to have the same superscript at all occurrences. Distinct markers carry distinct superscripts except that a barred and unbarred marker which enter the proof via the same insertion are assigned the same superscript. We use g^i and $\bar{g}^i$ respectively to denote the g-marker and $\bar{g}$-marker assigned i as superscript; but frequently $\bar{g}^i_\alpha$ instead of $\bar{g}^i$ is used where α is the subscript carried by $\bar{g}^i$ at all occurrences. The notation $\hat{g}^i$ is a variable for g^i and $\bar{g}^i$. Our notation for a marker occurrence is the obvious one: the corresponding symbol occurrence plus the superscript carried by the marker.

Markers entering the proof via left insertions are called *left* markers; via right insertions, *right* markers. The addition of L or R as a second superscript to a marker indicates its being left or right. But notations for markers with only a single superscript, e.g., $\mathbf{a}^i$, $\bar{g}^i_\alpha$, are meant to be ambiguous in that the marker concerned may have entered the proof via a left or right insertion, may have entered the proof via a non-trivial operation, or may have occurred in the first step of the proof.

A g^i-shift is a g-shift such that g^i occurs in the argument and value thereof. It is convenient to write *ins*(O$\}$) for O where O is an insertion, i.e., $ins(g^{iR}_\alpha \bar{g}^{iR}_\alpha)$ is that application of the rule $1 \rightarrow g_\alpha \bar{g}_\alpha$ via which g^i_α and

$\bar{g}^i_\alpha$ enter the proof. Similarly, we write *del*({O) for O where O is a deletion, i.e., $del(\bar{g}^j_\alpha g^i_\alpha)$ is that application of $\bar{g}_\alpha g_\alpha \rightarrow 1$ via which $\bar{g}^j_\alpha$ and g^i_α leave the proof.

A logically complete notation and terminology would now distinguish between words as symbol sequences and sequences of operation rules on the one hand, and words as marker sequences (i.e., as part of a step of a proof) and sequences of applications of rules (i.e., operation sequences) on the other. For brevity, however, we do not always make this distinction. We use $\mathbf{A}, \bar{\mathbf{A}}, \cdots$ for words in both senses and $H, \cdots, ins(\mathbf{A}\bar{\mathbf{A}})$, $del\,(\mathbf{A}\bar{\mathbf{A}})$ for operation sequences as well as rule sequences. Either the distinction will be non-essential, or the sense intended made clear contextually, — frequently by use of the words "operation sequence H" or "step **C**". As is convenient, in the notation "$H(\mathbf{A}/\mathbf{B})$" the H is now always to mean the operation sequence H.

Lastly, it seems best to point out that our physical definition of "marker" and related definitions can be replaced by quite formal ones. A *marker convention*, C, for the system $\mathfrak{T}$ is a collection of pairs $(g, \mathfrak{U}_g)$, g any non-vacuous unbarred letter set of $\mathfrak{Z}$, $\mathfrak{U}_g$ any subset of the rules of $\mathfrak{U}$ of form $\mathbf{8}g$, exactly one pair of form $(g, \mathfrak{U}_g)$ being in C for each non-vacuous g of $\mathfrak{Z}$. The rules of $\mathfrak{U}_g$ are *g-shifts* (of C).

Let (i, k) be the i^{th} symbol occurrence of the k^{th} step of some given proof of $\mathfrak{T}$. Then the relation $\sim_C$ is to be the narrowest equivalence relation on the symbol occurrences of this given proof, satisfying the following conditions, where the $k + 1^{\text{st}}$ step of form **PBQ** results from the k^{th} step of form **PAQ** by an application of $\mathbf{A} \rightarrow \mathbf{B}$:

$$\begin{array}{lr} (i, k) \sim_C (i, k+1), & 1 \leqq i \leqq |\mathbf{P}| \\ (|\mathbf{PA}| + j, k) \sim_C (|\mathbf{PB}| + j, k+1), & 1 \leqq j \leqq |\mathbf{Q}| \\ (|\mathbf{PM}| + 1, k) \sim_C (|\mathbf{PE}| + 1, k+1) & \text{where } \mathbf{A} \rightarrow \mathbf{B} \text{ is the } g\text{-shift } \mathbf{8}g \text{ of } C. \end{array}$$

A *marker* is an equivalence class of symbol occurrences defined by $\sim_C$. An *occurrence of a marker* is a symbol occurrence which is a member of the marker.[27]

9. *Application of a result of Malcev.* Where $\hat{k}$ is k or $\bar{k}$ according as $\hat{g}^k$ is g^k or $\bar{g}^k$, we now use $H^{(\hat{k}\mathrm{L})}$ for the sub-sequence of operations of the operation sequence H which are performed left of the marker $\hat{g}^k$, $H^{(\hat{k}\mathrm{R})}$ the sub-sequence performed right.

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[27] It has been pointed out to us by Hermes that with the point of view just given rules of the form $\mathbf{A} \rightarrow \mathbf{A}$ must be discarded from systems considered. This of course causes no difficulty.

LEMMA $6^{\ddagger}$.[28] *For any* $H(\mathbf{A}\hat{g}^k\mathbf{C}/\mathbf{D}\hat{g}^k\mathbf{E})$ *in any* $\mathfrak{T}$ *such that no operation of* H *is applied to* $\hat{g}^k$, $H^{(\hat{k}\mathrm{L})}(\mathbf{A}/\mathbf{D})$ *and* $H^{(\hat{k}\mathrm{L})}H^{(\hat{k}R)}(\mathbf{A}\hat{g}^k\mathbf{C}/\mathbf{D}\hat{g}^k\mathbf{E})$ *are valid proofs in* $\mathfrak{T}$.

The lemma is obvious by an induction on the number of operations of H. Lemma 6 is implicit in our later use of diagrams like that used in Section **5** of Part I.

$\mathfrak{T}$ is *g-positive* means every non-trivial rule of $\mathfrak{U}$ is $\bar{g}$-free.

The essential property of g-positive group presentations is this: for any proof in these systems, under any marker convention, $\bar{g}$-markers enter only via $\bar{g}$-insertions, leave only via $\bar{g}$-deletions; thus for each $\bar{g}$-marker occurring in a proof whose first and last step are $\bar{g}$-free, there are a $\bar{g}$-insertion and a $\bar{g}$-deletion via which the marker enters and leaves.

$Cond_1(\mathfrak{T}; g)$ means $\mathfrak{T}$ is a g-positive group presentation for which some fixed marker convention has been stipulated.

LEMMA 7. *Suppose* $Cond_1(\mathfrak{T}; g)$. *Where* **A** *and* **B** *are* $\bar{g}$-*free words, any proof* $H(\mathbf{A}/\mathbf{B})$ *in* $\mathfrak{T}$ *with no* $\bar{g}$-*deletions has no* $\bar{g}$-*insertions and is* $\bar{g}$-*free.*

Any $\bar{g}$-marker occurring in $H(\mathbf{A}/\mathbf{B})$ must remain in the proof since no $\bar{g}$-deletions are performed and $Cond_1(\mathfrak{T}; g)$. This contradicts the fact that **B** is $\bar{g}$-free.

DEFINITION OF g-MALCEV. Where $Cond_1(\mathfrak{T}; g)$, a proof $H(\mathbf{A}/\mathbf{B})$ of $\mathfrak{T}$ and its operation sequence H are said to be *g-normal in the sense of Malcev*, or simply *g-malcev* if:

(**9**.1) **A** and **B** are $\bar{g}$-free;

(**9**.2) For each operation, O, of H, $\lfloor$O is of form **D**$\{$O**E** where **D** contains no right $\bar{g}$-markers and **E** contains no left $\bar{g}$-markers;

(**9**.3) H contains no $\bar{g}$-deletion of form $del(g^e_\alpha \bar{g}^{f\mathrm{L}}_\alpha)$ or $del(\bar{g}^{f\mathrm{R}}_\alpha g^e_\alpha)$.

We note that the definition just given is independent of the particular marker convention in force, i.e., if a proof is g-malcev under one convention then it is also g-malcev under any other.

We now let $\mathrm{M}(H^u, \mathbf{C} \to \mathbf{D})$ be the number of applications of the rule $\mathbf{C} \to \mathbf{D}$ in the operation sequence H^u.

Reduction A[29] (Malcev. This proof by A. H. Clifford.) *Suppose*

[28] We frequently omit definitions, lemmas, and demonstrations of special cases which can be obtained from those given by interchanging notions of left and right. With reference to the system $\mathfrak{T}_2$, l and y are to be interchanged with r and x, respectively. Statements are marked with‡ to indicate that this convention is pertinent thereto.

[29] Reduction A as stated is redundant in that A2 implies that if the proof H is $\bar{p}$-free so also is the proof H^{+}. This arrangement makes for ease in checking applications in Part III where it is at only one point (Theorem VIII″, Section **25**) that A2 and the corresponding clauses of later reductions are needed. In all other applications these clauses can thus be ignored.

$Cond_1(\mathfrak{T}; g)$, $Cond_1(\mathfrak{T}; p)$, and **A** *and* **B** *are $\bar{g}$-free words over $\mathfrak{Z}$. If there is a $\bar{p}$-free $H(\mathbf{A}/\mathbf{B})$ in $\mathfrak{T}$ then there is a $\bar{p}$-free $H^+(\mathbf{A}/\mathbf{B})$ in $\mathfrak{T}$ such that*

(A1) *$H^+(\mathbf{A}/\mathbf{B})$ is g-malcev;*

(A2) $\mathrm{M}(H^+, \mathbf{C} \to \mathbf{D}) \leqq \mathrm{M}(H, \mathbf{C} \to \mathbf{D})$ *for any* $\mathbf{C} \to \mathbf{D}$ *of* $\mathfrak{U}$.

Reduction A and the lemmas used in demonstrating this theorem are typical of many of the lemmas and theorems to follow, in which we are given a proof $H(\mathbf{A}/\mathbf{B})$ and wish to find a proof $H^+(\mathbf{A}/\mathbf{B})$ with certain properties. We take the point of view that the desired $H^+(\mathbf{A}/\mathbf{B})$ is constructed from the pattern of markers making up $H(\mathbf{A}/\mathbf{B})$ by making certain alterations in that array, i.e., by rearranging markers, discarding markers, and adding new markers. Notation will be suggestive of this motivating idea; the superscript notation will serve to indicate the identification of certain markers in $H^+(\mathbf{A}/\mathbf{B})$ with certain markers in $H(\mathbf{A}/\mathbf{B})$ where such identification is intended.

We let V_g^u be the number of $\bar{g}$-deletions of H^u of form $del(g_\alpha^c \bar{g}_\alpha^{f\mathrm{L}})$ or $del(\bar{g}_\alpha^{f\mathrm{R}} g_\alpha^c)$.

LEMMA 8$^+$.[30] *Suppose $Cond_1(\mathfrak{T}; g)$, $Cond_1(\mathfrak{T}; p)$ and* **A** *and* **B** *are $\bar{g}$-free words over $\mathfrak{Z}$. If there is a $\bar{p}$-free $H(\mathbf{A}/\mathbf{B})$ in $\mathfrak{T}$ containing the $\bar{g}$-deletion $\mathrm{O} = del(\bar{g}_\alpha^{k\mathrm{R}} g_\alpha^t)$ then there is a $\bar{p}$-free $H'(\mathbf{A}/\mathbf{B})$ in $\mathfrak{T}$ such that*

(8.1) $\mathrm{V}'_g = \mathrm{V}_g - 1$;

(8.2) $\mathrm{M}(H', \mathbf{C} \to \mathbf{D}) \leqq \mathrm{M}(H, \mathbf{C} \to \mathbf{D})$ *for any* $\mathbf{C} \to \mathbf{D}$ *of* $\mathfrak{U}$.

In discussing the various diagrams used we shall use Diagram $\mathscr{X}|\mathscr{Y}$ to designate the diagram obtained by substituting Diagram $\mathscr{Y}$ for that part of Diagram $\mathscr{X}$ enclosed in brackets. We abbreviate "taken to be a representation of" by "*rep*". We use $\mathscr{X}|\mathscr{Y}||H^u(\mathbf{P}/\mathbf{Q})$ for the proof represented by Diagram $\mathscr{X}|\mathscr{Y}$ when Diagram $\mathscr{X}$ is *rep* $H^u(\mathbf{P}/\mathbf{Q})$. All diagrams used in Part II are at the end of Part II. In view of Lemma 6 the desired $H'(\mathbf{A}/\mathbf{B})$ of Lemma 8 is $\mathscr{A}|\mathscr{B}||H(\mathbf{A}/\mathbf{B})$, as is clear by inspection.[31]

For any proof $H^u(\mathbf{A}/\mathbf{B})$ of $\mathfrak{T}$, **A** and **B** $\bar{g}$-free, $Cond_1(\mathfrak{T}; g)$ we shall call a left (right) $\bar{g}$-marker *violated* if an operation of H^u occurs left (right) of this marker, and the operation so performed is said to *violate* the marker; P_g^u is the number of violated $\bar{g}$-markers of $H^u(\mathbf{A}/\mathbf{B})$.

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[30] It is this lemma, not the full Reduction A, which is really essential to our argument. In a demonstration of the unsolvability of the word problem for groups based upon that for cancellation semi-groups Reduction A might well be used in a more fundamental way.

[31] In comparing Diagrams $\mathscr{A}$ and $\mathscr{A}|\mathscr{B}$ note two points which are essential to the argument for Lemma 8. (1) *Where $\bar{g}^\alpha$ is any $\bar{g}$-marker occurring in the proof H', $\bar{g}^\alpha$ occurs in the proof H and is a left or right barred marker in both proofs.* (2) *If $\bar{g}^\alpha$ leaves the proof H' via a left (right) $\bar{g}$-deletion, then $\bar{g}^\alpha$ leaves the proof H via a left (right) $\bar{g}$-deletion.* Similarly, in comparing Diagrams $\mathscr{C}$ and $\mathscr{C}|\mathscr{D}$ used in the demonstration of Lemma 9 it is essential to the argument for that lemma to note that points 1 and 2 hold

LEMMA 9. *Suppose* $Cond_1(\mathfrak{T}; g)$, $Cond_1(\mathfrak{T}; p)$ *and* **A** *and* **B** *are* $\bar{g}$-*free words over* $\mathfrak{Z}$. *If there is a* $\bar{p}$-*free* $H^0(\mathbf{A}/\mathbf{B})$ *in* $\mathfrak{T}$ *such that* $V_g^0 = 0$ *and containing* $O^0 = del(g_\alpha^t \bar{g}_\alpha^{kR})$ *where* $\bar{g}_\alpha^{kR}$ *is a violated* $\bar{g}$-*marker then there is a* $\bar{p}$-*free* $H^*(\mathbf{A}/\mathbf{B})$ *in* $\mathfrak{T}$ *such that*

(9.1) $V_g^* = 0$;

(9.2) $P_g^* \leqq P_g^0 - 1$;

(9.3) $M(H^*, \mathbf{C} \to \mathbf{D}) = M(H^0, \mathbf{C} \to \mathbf{D})$ *for any* $\mathbf{C} \to \mathbf{D}$ *of* $\mathfrak{U}$.

The desired $H^*(\mathbf{A}/\mathbf{B})$ is $\mathcal{C}|\mathcal{D}||H(\mathbf{A}/\mathbf{B})$. The $\bar{p}$-freeness, Lemmas 9.1 and 9.3 are immediate by inspection[31] of Diagrams $\mathcal{C}$ and $\mathcal{C}|\mathcal{D}$. The marker $\bar{g}_\alpha^{kR}$ is not violated in $H^*(\mathbf{A}/\mathbf{B})$. Thus for the demonstration of Lemma 9.2 it is sufficient[31] to show that (9†) *if* $\bar{g}^u$, $u \neq k$, *is a violated marker in* $H^*(\mathbf{A}/\mathbf{B})$ *then there is an operation of* H^0, *say* O_u^0, *which violates* $\bar{g}^u$ *in* $H^0(\mathbf{A}/\mathbf{B})$.

Where $\bar{g}^u$ is any violated marker of $H^*(\mathbf{A}/\mathbf{B})$, we distinguish two cases according as $\bar{g}^u$ is (Case a) or is not (Case b) violated by $ins(g_\alpha^{kR}\bar{g}_\alpha^{kR})$ or $del(g_\alpha^t\bar{g}_\alpha^{kR})$ in $H^*(\mathbf{A}/\mathbf{B})$. (Throughout the following argument Diagram $\mathcal{C}$ is *rep* $H^0(\mathbf{A}/\mathbf{B})$ and Diagram $\mathcal{C}|\mathcal{D}$ is *rep* $H^*(\mathbf{A}/\mathbf{B})$). Suppose Case a. If $\bar{g}^u$ is $\bar{g}^{uL}$ then $\bar{g}^{uL}$ occurs in **F** of Diagram $\mathcal{C}|\mathcal{D}$, so that the desired O_u^0 is $del(g_\alpha^t\bar{g}_\alpha^{kR})$. If $\bar{g}^u$ is $\bar{g}^{uR}$, then $\bar{g}^{uR}$ occurs in **C** of Diagram $\mathcal{C}|\mathcal{D}$ if violated by $ins(g_\alpha^{kR}\bar{g}_\alpha^{kR})$, and in **E** of Diagram $\mathcal{C}|\mathcal{D}$ if violated by $del(g_\alpha^t\bar{g}_\alpha^{kR})$; and accordingly O_u^0 is $ins(g_\alpha^{kR}\bar{g}_\alpha^{kR})$ or is $del(g_\alpha^t\bar{g}_\alpha^{kR})$. Suppose Case b. Then $\bar{g}^u$ is violated by some operation, say O_u^*, of the subsequence K_1, $K_3^{(\bar{k}R)}$, $K_3^{(\bar{k}L)}$, or K_5 of H^*. Since Case b, if $\bar{g}^u$ is $\bar{g}^{uL}$ then $\bar{g}^{uL}$ does not occur in **F** of Diagram $\mathcal{C}|\mathcal{D}$ and if $\bar{g}^u$ is $\bar{g}^{uR}$ then $\bar{g}^{uR}$ does not occur in **C** or **E** of this diagram. But a left barred marker occurring in **C** of Diagram $\mathcal{C}|\mathcal{D}$ cannot be violated by an operation of $K_3^{(\bar{k}R)}$, and a right barred marker occurring in **F** of this diagram cannot be violated by an operation of $K_3^{(\bar{k}L)}$. Thus $\bar{g}^u$ and O_u^* are a marker occurring and operation used in some *one* of the following subproofs of $H^*(\mathbf{A}/\mathbf{B})$: $K_1(\mathbf{A}/\mathbf{CD})$, $K_3^{(\bar{k}R)}(\mathbf{D}/\mathbf{F})$, $K_3^{(\bar{k}L)}(\mathbf{C}g_\alpha^{kR}/\mathbf{E}g_\alpha^t)$, $K_5(\mathbf{EF}/\mathbf{B})$. It is immediate then by inspection of Diagram $\mathcal{C}$ that the

with H^* and H^0 substituted for H' and H respectively. In later constructions of a desired proof from a given proof, the correspondents of points 1 and 2 will not hold in general. (We are indebted to the referee for pointing out the necessity of the foregoing remarks which were added at his suggestion.)

If K is an operation sequence specified in one of the given diagrams, markers occurring in that portion of $[K$ (of $K])$ not underscored (not overscored) also occur in $K]$ (in $[K)$, i.e., are markers which we know—perhaps as a result of the fact that a proof is g-malcev for some g—are not affected by the operations of K. In a diagram obtained by substitution these under- and over-scorings are not always correctly placed. (These scorings were suggested for [1] by Roger Lyndon).

operation O_u^* also violates $\bar{g}^u$ in $H^0(\mathbf{A}/\mathbf{B})$, i.e., O_u^* itself is the desired O_u^0. This shows (9†), hence Lemma 9.

Suppose that we are given an $H(\mathbf{A}/\mathbf{B})$ in $\mathfrak{T}$ as described in Reduction A. Then by induction on V_g and using Lemma 8 there is a $\bar{p}$-free $H^0(\mathbf{A}/\mathbf{B})$ in $\mathfrak{T}$ such that $V_g^0 = 0$ and $M(H^0, \mathbf{C} \rightarrow \mathbf{D}) \leqq M(H, \mathbf{C} \rightarrow \mathbf{D})$ for all $\mathbf{C} \rightarrow \mathbf{D}$ of $\mathfrak{U}$. The existence of a $\bar{p}$-free $H^+(\mathbf{A}/\mathbf{B})$ in $\mathfrak{T}$ such that A1 and A2 now follow by induction on P_g^0 using Lemma 9^+.

10. *The relative positions maintained by certain markers; definition of g-stability.* We first shall make precise the concept of non-commutability of markers.

DEFINITION OF $g \otimes p$. For any $\mathfrak{T}$ with any marker convention, $g \otimes p$ (*in* $\mathfrak{T}$) means $\mathfrak{U}$ has no g-shift p-shift rules of this form or its converse:

$$(\mathbf{10}gp)\ \mathbf{A}g_\alpha\mathbf{B}p_\beta\mathbf{C} \rightarrow \mathbf{D}p_\gamma\mathbf{E}g_\delta\mathbf{F}\,.$$

E.g., $g \otimes p$ in $\mathfrak{T}_1, \cdots, \mathfrak{T}_5$ where g and p are *any* unbarred letter sets of those systems. Note that for any $\mathfrak{T}$ and g under any marker convention $g \otimes g$ in $\mathfrak{T}$,— as is clear directly from the definition of a g-shift.

$Cond_2(\mathfrak{T}; g, p)$ means $Cond_1(\mathfrak{T}; g)$, $Cond_1(\mathfrak{T}; p)$, g and p are disjoint, and $g \otimes p$ in $\mathfrak{T}$

DEFINITION[32] OF $\hat{g}^u < \hat{p}^v$. For any proof in any system, $\hat{g}^u < \hat{p}^v$ *in step* $\mathbf{C}$ means that both $\hat{g}^u$ and $\hat{p}^v$ occur in $\mathbf{C}$ with $\hat{g}^u$ occurring left of $\hat{p}^v$.

Note that both $Cond_2$ and $\otimes$ depend upon the marker convention in force.

LEMMA 10. *Suppose either* (i) $\hat{g}^s$ *is* $\bar{g}^s$ *or* (ii) $g \otimes p$ *in* $\mathfrak{T}$. *Then for any proof in* $\mathfrak{T}$ *there do not exist steps* $\mathbf{C}_u$ *and* $\mathbf{C}_v$ *such that* $\hat{g}^s < \hat{p}^t$ *in* $\mathbf{C}_u$ *and* $\hat{p}^t < \hat{g}^s$ *in* $\mathbf{C}_v$.

Clearly a counterexample to this lemma would mean that in some proof of $\mathfrak{T}$ there is used an operation O such that $\hat{g}^s < \hat{p}^t$ in [O and $\hat{p}^t < \hat{g}^s$ in O] for certain markers $\hat{g}^s$ and $\hat{p}^t$. But such an O much be of form $\mathbf{10}gp$.

DEFINITION OF g-STABLE. For any $\mathfrak{T}$ under any marker convention the g-shifts, $\bar{g}$-insertions, $\bar{g}$-deletions, and $\hat{g}$-free rules of $\mathfrak{U}$ are called *g-stable*. A proof with only g-stable operations is *g-stable*; if every rule of $\mathfrak{U}$ is g-stable then $\mathfrak{T}$ itself is called *g-stable*.

Note that a rule's being g-stable depends on the marker convention in force. The essential property of g-stable proofs is this: g-markers enter only by $\bar{g}$-insertions and leave only by $\bar{g}$-deletions. (Cf., proofs in g-positive group presentations).

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[32] $\hat{g}_\alpha^u < \hat{p}_\alpha^v$ or $\mathbf{a}^u < \mathbf{b}^v$ are variations of this notation with the obvious meaning.

$Cond_3(\mathfrak{T}; g)$ means $Cond_1(\mathfrak{T}; g)$ and $\mathfrak{T}$ is g-stable.

Note that $Cond_3(\mathfrak{T}; g)$ determines the form of the rules of $\mathfrak{U}$ in that every non-trivial rule of $\mathfrak{U}$ not $\hat{g}$-free must be of form **8**g. Moreover, $Cond_3(\mathfrak{T}; g)$ implies that the universal g-qualification applies.

LEMMA 11. *Suppose* $Cond_1(\mathfrak{T}; g)$. *Any* $\bar{g}$-*free* g-*stable proof* $H(\mathbf{A}/\mathbf{B})$ *in* $\mathfrak{T}$ *has in each step the same* g-*markers, viz., those occurring in the first step.*

Obvious; since no g-markers enter or leave the proof $H(\mathbf{A}/\mathbf{B})$.

We now show two easy consequences of Lemma 10, namely, Lemmas 12 and 13.

LEMMA 12. *Suppose* $Cond_1(\mathfrak{T}; g)$. *In any* g-*stable* $K(g^j_\gamma\mathbf{E}/\mathbf{F}g^j_\alpha)$ *of* $\mathfrak{T}$ *containing no* $\bar{g}$-*deletions,* g^j *is the right-most* $\hat{g}$-*marker occurring in each step. Thus* $\mathbf{E}$ *is* $\hat{g}$-*free and no* $\bar{g}$-*insertions of* K *are performed right of* g^j.

Suppose that for some $\hat{g}^s$, $s \neq j$, (i) $g^j < \hat{g}^s$ in $\mathbf{C}$, some step of the proof K of the lemma. Since the proof K contains no $\bar{g}$-deletions and is g-stable, $\hat{g}^s$ must also occur in $\mathbf{C}'$, any step following $\mathbf{C}$. But $g \otimes g$, so that, by Lemma 10, not $\hat{g}^s < g^j$ in $\mathbf{C}'$. Hence i in $\mathbf{C}'$. But in fact not i in $\mathbf{F}g^j_\alpha$, i.e., in the last step of the proof K, so that not i in any step $\mathbf{C}$, as the lemma asserts.

$Cond_4(\mathfrak{T}; g, p)$ means (**10**.1) $Cond_3(\mathfrak{T}; p)$

(**10**.2) $Cond_2(\mathfrak{T}; g, p)$

(**10**.3) Any rule of $\mathfrak{U}$ which is not a p-shift is g-stable.

If $Cond_4(\mathfrak{T}; g, p)$, then the essential property of proofs in $\mathfrak{T}$ is this: g-markers enter only by $\bar{g}$-insertions and p-shifts and leave only by $\bar{g}$-deletions and p-shifts. It should be noted that if $\mathfrak{Z}$ contains no $\hat{p}$-symbols then $Cond_4(\mathfrak{T}; g, p)$ means simply $Cond_3(\mathfrak{T}; g)$ and that $Cond_3(\mathfrak{T}; g)$ implies **10**.3. Clearly $Cond_3(\mathfrak{T}_2; k)$, $Cond_3(\mathfrak{T}_i; g)$, $i=3, 4, 5$, $g=k, t, q$. Thus $Cond_4(\mathfrak{T}_3; t, k)$, $Cond_4(\mathfrak{T}_5; s, q)$ as is pertinent to Theorems V and VIII.

LEMMA 13. *Suppose* $Cond_4(\mathfrak{T}; g, p)$. *Any* $\bar{p}$-*free* $K(g^j_\gamma\mathbf{E}/\mathbf{F}g^j_\alpha)$ *of* $\mathfrak{T}$ *is both* $\hat{p}$-*free and* g-*stable.*

By Lemma 11 each step of the proof K contains the same p-markers. The argument now parallels that for Lemma 12. Using Lemma 10 and the fact that $g \otimes p$, no p-marker ever appears right of g^j since none appears right of g^j in the last step; none ever appears left of g^j since none appears left of g^j in the first. Thus the proof K is $\hat{p}$-free. Being $\hat{p}$-free, hence having no p-shifts, the proof K is g-stable by **10**.3 of the definition of $Cond_4$.

11. *Tabulation of the kinds of first-occurring $\bar{g}$-deletions possible.* For any $\mathfrak{T}$ and g such that $Cond_1(\mathfrak{T}; g)$ we now let O_g^u be the first $\bar{g}$-deletion of H^u.

LEMMA 14. *Suppose $Cond_4(\mathfrak{T}; g, p)$ and let $H(\mathbf{A}/\mathbf{B})$ be any g-malcev $\bar{p}$-free proof of $\mathfrak{T}$ containing $\bar{g}$-deletions.*

(14.1) *The following are the six exclusive and exhaustive possibilities for $\{O_g$. Here $\bar{g}_\alpha^i$ is the $\bar{g}$-marker leaving the proof via O_g. In cases 5g and 6g, g^u is a g-marker which has entered the proof via some p-shift.*

1g. $g_\alpha^{jL}\bar{g}_\alpha^{iR}$, $i \neq j$	3g. $\bar{g}_\alpha^{iL}g_\alpha^{jR}$, $i \neq j$
2g. $g_\alpha^{iR}\bar{g}_\alpha^{iR}$	4g. $\bar{g}_\alpha^{iL}g_\alpha^{iL}$
5g. $g_\alpha^{u}\bar{g}_\alpha^{iR}$	6g. $\bar{g}_\alpha^{iL}g_\alpha^{u}$

(14.2) *If Case* 1g *holds then Diagram* $\mathcal{E}$ *may be rep* $H(\mathbf{A}/\mathbf{D})$ *with full generality*;+[33]

(14.3) *If $Cond_3(\mathfrak{T}; g)$ then Cases* 5g *and* 6g *are excluded.*

The cases given in the second column of Lemma 14.1 are the duals of those in the first column. It is convenient to begin by showing four preliminary sublemmas, (14†) to (14††††). First, we note in general that (14†) *if*+ $O_g = del(g_\alpha^x\bar{g}_\alpha^{iR})$, *then not* $\bar{g}_\alpha^{iR} < g^x$ *in any step of the proof H.* Since $g^x < \bar{g}_\alpha^{iR}$ in $[O_g$, (14†) is immediate by Lemma 10.[34] Secondly, we show in general that (14††) *if*+ $O_g = del(g_\alpha^x\bar{g}_\alpha^{iR})$, *then not* (i) $g^x < p^a < \bar{g}_\alpha^{iR}$ *for any* p^a *in any step of the proof H.* For suppose there is a step **C** and marker p^a such that i in **C**. Then p^a occurs in $[O_g$, by Lemma 11 since the proof H is $\bar{p}$-free and p-stable. Consequently i in $[O_g$ by Lemma 10 since $p \otimes g$. But in fact not i in $[O_g$, so that (14††). Thirdly, we note in general that (14†††) *if*+ $O_g = del(g_\alpha^x\bar{g}_\alpha^{iR})$, *then not* (ii) $g^x < \hat{g}^y < \bar{g}^{iR}$ *for any* $\hat{g}^y$ *in any step of the proof H.* For suppose there is a step **C** and marker $\hat{g}^y$ such that ii in **C**. Then, by Lemma 10, since $g \otimes g$, ii also holds in any step following **C** and preceding $O_g]$ in which $\hat{g}^y$ occurs. Suppose there is an operation, O, preceding O_g, via which $\hat{g}^y$ leaves the proof. By the definition of O_g, O is not a $\bar{g}$-deletion so that by **10**.3 of the definition of $Cond_4$, O is a p-shift. But since i in [O, O is a p-shift applied between g^x and $\bar{g}^{iR}$ thus contradicting (14††). Thus there is no such O and $\hat{g}^y$ occurs in $[O_g$. Consequently ii in $[O_g$. But in fact not ii in $[O_g$, so that (14†††).

[33] If H is g-stable then **D** of Diagram $\mathcal{E}$ contains the marker g^{iR} when that diagram is *rep* the proof *H*.

[34] Alternatively by the fact that the proof H is g-malcev.

Fourthly, we note in general that (14††††) *if*$^{+}$ $O_g = del(g^x_\alpha \bar{g}^{iR}_\alpha)$, *then* g^x *does not occur in* [$ins(g^{iR}_\alpha \bar{g}^{iR}_\alpha)$. For according as $ins(g^{iR}_\alpha \bar{g}^{iR}_\alpha)$ is performed left or right of g^x, so either $\bar{g}^{iR}_\alpha < g^x$ or $g^x < g^{iR} < \bar{g}^{iR}_\alpha$ in $ins(g^{iR}_\alpha \bar{g}^{iR}_\alpha)$], thus contradicting either (14†) or (14†††).

We now show Lemma 14.1. Again recall the essential property of proofs in $\mathfrak{T}$ where $Cond_4(\mathfrak{T}; g, p)$. Suppose $O_g = del(g^{jR}_\alpha \bar{g}^{iR}_\alpha)$, $i \neq j$. If $ins(g^{jR}_\alpha \bar{g}^{jR}_\alpha)$ follows $ins(g^{iR}_\alpha \bar{g}^{iR}_\alpha)$ then either (14†††) or (14†) is contradicted according as $ins(g^{jR}_\alpha \bar{g}^{jR}_\alpha)$ is performed left or right of $\bar{g}^{iR}_\alpha$. This shows $O_g \neq del(g^{jR}_\alpha \bar{g}^{iR}_\alpha)$, $i \neq j$. By (14††††), $O_g \neq del(g^f_\alpha \bar{g}^{iR}_\alpha)$, where g^f occurs in the first step of the proof H. Clearly $O_g \neq del(\bar{g}^{iR}_\alpha g^x_\alpha)$ since H is g-malcev. The arguments^{+} that $O_g \neq del(\bar{g}^{iL}_\alpha g^{jL}_\alpha)$, $i \neq j$ or $del(\bar{g}^{iL}_\alpha g^f_\alpha)$ where g^f occurs in the first step or $del(g^x_\alpha \bar{g}^{iL}_\alpha)$ are the duals of the cases excluded above. This shows Lemma 14.1 which merely lists the combinatorially remaining possibilities.

Now assume Case 1g. By (14††††), $ins(g^{iR}_\alpha \bar{g}^{iR}_\alpha)$ precedes $ins(\bar{g}^{jL}_\alpha g^{jL}_\alpha)$ in H. By (14†), $ins(\bar{g}^{jL}_\alpha g^{jL}_\alpha)$ is performed left of $\bar{g}^{iR}_\alpha$. Since H is g-malcev no operations are performed right of $\bar{g}^{iR}_\alpha$ or left of $\bar{g}^{jL}_\alpha$. For these reasons Diagram $\mathcal{E}$ may be *rep* $H(\mathbf{A}/\mathbf{B})$ with full generality. This shows Lemma 14.2.

Lemma 14.3 is trivial since if $\mathfrak{T}$ is g-stable then no g-markers enter the proof via p-shifts.

12. *Reduction process for the case wherein* $O_g = del(g^{iR}_\alpha \bar{g}^{iR}_\alpha)$.

DEFINITION OF g-TRANSLATION RULE AND OF g^i-TRANSLATION. For any $\mathfrak{T}$ under any marker convention a g-shift rule of the form

$$(12g) \quad \mathbf{A} g_\eta \mathbf{B} \to \mathbf{C} g_\theta \mathbf{D}$$

where **AB** is **CD** is called a *g-translation rule*; a *g^i-translation* is an application of a g-translation rule to the marker g^i.

Reduction B. *Suppose* $Cond_1(\mathfrak{T}; g)$ *and* $Cond_1(\mathfrak{T}; p)$. *If there is a $\bar{p}$-free* $H(\mathbf{A}/\mathbf{B})$ *in* $\mathfrak{T}$ *such that* $O_g = del(g^{iR}_\alpha \bar{g}^{iR}_\alpha)$ *and the only operations of* H *between* $ins(g^{iR}_\alpha \bar{g}^{iR}_\alpha)$ *and* O_g *applied to* g^{iR} *are* g^{iR}*-translations, then there is a $\bar{p}$-free* $H^{+}(\mathbf{A}/\mathbf{B})$ *in* $\mathfrak{T}$ *such that*

(B1) $N^{+}_g = N_g - 1$;

(B2) $M(H^{+}, \mathbf{C} \to \mathbf{D}) \leq M(H, \mathbf{C} \to \mathbf{D})$ *for all* $\mathbf{C} \to \mathbf{D}$ *of* $\mathfrak{U}$.

Let $\mathbf{C}_n$, $n = 1, 2, \cdots, N$, be the n^{th} step of $H(\mathbf{A}/\mathbf{B})$ and $\mathbf{C}^{+}_n$ the result of erasing $\bar{g}^{iR}_\alpha$ and g^{iR}_β for any β from $\mathbf{C}_n$. Then if O_n, the n^{th} operation of H, is $ins(g^{iR}_\alpha \bar{g}^{iR}_\alpha)$, a g^{iR}-translation, or O_g, $\mathbf{C}^{+}_{n+1}$ is $\mathbf{C}^{+}_n$; otherwise $\mathbf{C}^{+}_{n+1}$ follows from $\mathbf{C}^{+}_n$ by O_n. But $\mathbf{C}^{+}_1$ is $\mathbf{C}_1$, i.e., **A**; and $\mathbf{C}^{+}_N$ is $\mathbf{C}_N$, i.e., **B**. Thus $\mathbf{C}^{+}_1$, $\mathbf{C}^{+}_2$, $\cdots$ $\mathbf{C}^{+}_N$ is, with repetitious steps omitted, the desired $H^{+}(\mathbf{A}/\mathbf{B})$.

[35] But Reduction D is more general than Reduction B. Sections **13** and **14** could be given prior to Section **12** for the notion of g-translation is not used.

13. *Interchanging the order of certain operations.* In this section and Section **14** we demonstrate a sequence of results needed to show an analogue of Reduction B of the preceding section for the case wherein $\mathrm{O}_g = del(g_\alpha^{jL}\bar{g}_\alpha^{jR})$, i.e., Reduction D below.[35] This new and crucial case is more involved, however. The central ideas might be described as follows. Let Diagram $\mathcal{E}$ be *rep* a given proof falling under Case 1g of Lemma 14.1. We wish to assert that $\mathcal{E}|\mathcal{F}||H(\mathbf{A}/\mathbf{B})$ has $\mathrm{N}_g - 1$ $\bar{g}$-deletions, but this assertion is not in general valid since $\hat{g}$-markers may occur in $\mathbf{F}$ of Diagram $\mathcal{E}$ when that diagram is *rep* $H(\mathbf{A}/\mathbf{B})$. The argument of this present section is that $\bar{g}$-insertions and g^t-shifts, $t \neq j$, of K_5 performed left of g^j can be postponed until after O_g, thus making $\mathbf{F}$ $\hat{g}$-free.

As an extension of the definition of [O and O] we now use [H for the premiss of the first operation of an operation sequence H under discussion and H] for the conclusion of the last.

LEMMA 15.[36] *Suppose* $Cond_1(\mathfrak{T}; g)$ *and* $Cond_1(\mathfrak{T}; p)$. *If there is a $\hat{p}$-free* $K(\mathbf{M}g_\gamma^j\mathbf{E}/\mathbf{F}g_\alpha^j\mathbf{P})$ *in* $\mathfrak{T}$, *then there is a $\hat{p}$-free* $TU(\mathbf{M}g_\gamma^j\mathbf{E}/\mathbf{F}g_\alpha^j\mathbf{P})$ *in* $\mathfrak{T}$ *such that*

(15.1) *Every operation of T performed left of g^j is $\hat{g}$-free*;

(15.2) *U contains neither g^j-shifts nor operations right of g^j*;

(15.3) *If* $\mathbf{C} \to \mathbf{D}$ *is any rule of* $\mathfrak{U}$ *other than a $\hat{g}$-free $\hat{p}$-free insertion or deletion,* $\mathrm{M}(TU, \mathbf{C} \to \mathbf{D}) = \mathrm{M}(K, \mathbf{C} \to \mathbf{D})$.

A finite sequence of proofs in $\mathfrak{T}$, $\{K^\nu(\mathbf{M}g_\gamma^j\mathbf{E}/\mathbf{F}g_\alpha^j\mathbf{P})\}_{\nu=1}^N$, for some N, will be recursively defined. Let the first term of this sequence be the proof $K(\mathbf{M}g_\gamma^j\mathbf{E}/\mathbf{F}g_\alpha^j\mathbf{P})$ assumed in the lemma. An operation of K^ν, $\nu = 1, 2, \cdots, N$, is *regular* if it is either a g^j-shift or an operation performed right of g^j; *irregular*, if it is an operation performed left of g^j and is not $\hat{g}$-free. Let O_R^ν be the first regular operation of K^ν preceded by an irregular operation of K^ν. If O_R^ν does not exist K^ν is K^N, i.e., the sequence of proofs terminates with the ν^{th} proof; if O_R^ν does exist, let O_I^ν be the last irregular operation of K^ν preceding O_R^ν. If O_R^ν is a g^j-shift then the $\nu + 1^{\text{st}}$ proof is $\mathcal{G}|\mathcal{H}||K^\nu$. If O_R^ν is an operation right of g^j let K^ν be $V_1\mathrm{O}_I^\nu V_3^\nu \mathrm{O}_R^\nu V_5^\nu$. Here V_3^ν contains neither g^j-shifts nor operations right of g^j by the definition of O_R^ν. Take $K^{\nu+1}$ to be $V_1^\nu \mathrm{O}_R^\nu \mathrm{O}_I^\nu V_3^\nu V_5^\nu$; by Lemma 6—identifying g^j with the $\hat{g}^k$ of that lemma—$K^{\nu+1}$ effects a proof of $\mathbf{M}g_\gamma^j\mathbf{E}/\mathbf{F}g_\alpha^j\mathbf{P}$. It is obvious inductively that the proof $K^{\nu+1}$ is $\hat{p}$-free and $\mathrm{M}(K^{\nu+1}, \mathbf{C} \to \mathbf{D}) = \mathrm{M}(K^\nu, \mathbf{C} \to \mathbf{D})$ for $\mathbf{C} \to \mathbf{D}$ not a $\hat{g}$-free $\hat{p}$-free insertion or deletion since these are the only kind of new operations added in the passage from K_ν to $K^{\nu+1}$. Clearly K^N exists. We write K^N in the form TU where the first

[36] The present arrangement of the argument seems a slight improvement over [2], Part II, page 496, to which it corresponds.

operation of U is the first irregular operation of K^N. This shows the lemma.

As will now be seen, we use only that special case of Lemma 15 wherein the words $\mathbf{M}$ and $\mathbf{P}$ are 1 and the proof K is g-stable; in this situation the proof K of Lemma 15 resembles — comparing first and last steps — the proofs of Lemmas 12 and 13. The leading idea behind Lemmas 15 and 16 is that the word $\mathbf{F}'$ as constructed by these lemmas, and as described in Lemma 16, is $\hat{g}$-free.

LEMMA 16. *Suppose* $Cond_1(\mathfrak{T}; g)$ *and* $Cond_1(\mathfrak{T}; p)$. *For any $\hat{p}$-free g-stable* $TU(g^j_\gamma\mathbf{E}/\mathbf{F}g^j_\alpha)$ *in* $\mathfrak{T}$ *with no $\bar{g}$-deletions such that* 15.1 *and* 15.2 *of Lemma* 15 *hold, there is a $\hat{p}$-free $\hat{g}$-free word on* $\mathfrak{Z}$, $\mathbf{F}'$, *such that*

(16.1) *T effects a $\hat{g}$-free proof of* $g^j_\gamma\mathbf{E}/\mathbf{F}'g^j_\alpha$ *in* $\mathfrak{T}$; *the only $\hat{g}$-marker occurring in this proof is* g^j;

(16.2) *U effects a $\hat{p}$-free proof of* $\mathbf{F}'/\mathbf{F}$ *in* $\mathfrak{T}$ *having no $\bar{g}$-deletions.*

We suppose given a proof $TU(g^j_\gamma\mathbf{E}/\mathbf{F}g^j_\alpha)$ as described. Consider first the subproof, $U(T]/\mathbf{F}g^j_\alpha)$. Since 15.2 of Lemma 15 holds and $U]$ is $\mathbf{F}g^j_\alpha$, $T]$ is $\mathbf{F}'g^j_\alpha$ for some word, $\mathbf{F}'$, on $\mathfrak{Z}$ and U effects a proof of $\mathbf{F}'/\mathbf{F}$ in $\mathfrak{T}$. Consider the subproof $T(g^j_\gamma\mathbf{E}/\mathbf{F}'g^j_\alpha)$ of the given proof $TU(g^j_\gamma\mathbf{E}/\mathbf{F}g^j_\alpha)$. The former is g-stable since the latter is. It is immediate that g^j is the only $\hat{g}$-marker occurring in $T(g^j_\gamma\mathbf{E}/\mathbf{F}'g^j_\alpha)$. For $\mathbf{E}$ is $\hat{g}$-free by Lemma 12, no $\bar{g}$-insertion of T is performed right of g^j by Lemma 12, and no $\bar{g}$-insertion of T is performed left of g^j since 15.2 of Lemma 15 holds. (Thus, we have made the crucial point that the word $\mathbf{F}'$ is $\hat{g}$-free.) The remaining points to be noted for Lemma 16 are completely trivial: the $\hat{p}$-freeness of both $T(g^j_\gamma\mathbf{E}/\mathbf{F}'g^j_\alpha)$ and $U(\mathbf{F}'/\mathbf{F})$ follows from that of the given $TU(g^j_\gamma\mathbf{E}/\mathbf{F}g^j_\alpha)$; the proof $U(\mathbf{F}'/\mathbf{F})$ has no $\bar{g}$-deletions since the given $TU(g^j_\gamma\mathbf{E}/\mathbf{F}g^j_\alpha)$ has none.

Reduction C. *Suppose* $Cond_4(\mathfrak{T}; g, p)$. *If there is a $\bar{p}$-free* $K(g^j_\gamma\mathbf{E}/\mathbf{F}g^j_\alpha)$ *in* $\mathfrak{T}$ *with no $\bar{g}$-deletions then for a certain word on* $\mathfrak{Z}$, $\mathbf{F}'$, *there are in* $\mathfrak{T}$ *proofs* $T(g^j_\gamma\mathbf{E}/\mathbf{F}'g^j_\alpha)$ *and* $U(\mathbf{F}'/\mathbf{F})$ *such that*

(C.1) $T(g^j_\alpha\mathbf{E}/\mathbf{F}'g^j_\alpha)$ *is $\hat{p}$-free and g^j is the only $\hat{g}$-marker occurring in this proof; thus* $\mathbf{E}$ *and* $\mathbf{F}'$ *are $\hat{p}$-free, $\hat{g}$-free;*

(C.2) $U(\mathbf{F}'/\mathbf{F})$ *is $\hat{p}$-free and has no $\bar{g}$-deletions;*

(C.3) *If* $\mathbf{C} \rightarrow \mathbf{D}$ *is any rule of* $\mathfrak{U}$ *other than a $\hat{g}$-free $\hat{p}$-free insertion or deletion,* $\mathrm{M}(TU, \mathbf{C} \rightarrow \mathbf{D}) = \mathrm{M}(K, \mathbf{C} \rightarrow \mathbf{D})$.

Suppose $Cond_4(\mathfrak{T}; g, p)$ and that we are given a $\bar{p}$-free $K(g^j_\gamma\mathbf{E}/\mathbf{F}g^j_\alpha)$ in $\mathfrak{T}$ with no $\bar{g}$-deletions. By Lemma 13 this proof is both $\hat{p}$-free and g-stable. Next applying Lemma 15 to this same proof, it follows that there is a $\hat{p}$-free proof in $\mathfrak{T}$, $TU(g^j_\gamma\mathbf{E}/\mathbf{F}g^j_\alpha)$ such that 15.1, 15.2, and 15.3 of Lemma 15 hold. Since 15.3, the proof $TU(g^j_\gamma \mathbf{E}/\mathbf{F}g^j_\alpha)$ is g-stable and without $\bar{g}$-deletions since these properties hold for $K(g^j_\gamma \mathbf{E}/\mathbf{F}g^j_\alpha)$. Thus, we may apply Lemma 16 to the proof $TU(g^j_\gamma \mathbf{E}/\mathbf{F}g^j_\alpha)$ directly obtaining Reduction C.

14. *Reduction process for the case wherein* $\mathrm{O}_g = del(g^{jL}_\alpha \bar{g}^{iR}_\alpha)$, $i \neq j$. This is the central idea of the entire paper.

Reduction D. *Suppose* $Cond_4(\mathfrak{T}; g, p)$. *If there is a* g*-malcev* $\bar{p}$*-free* $H(\mathbf{A}/\mathbf{B})$ *in* $\mathfrak{T}$ *such that* $\mathrm{O}_g = del(g^{jL}_\alpha \bar{g}^{iR}_\alpha)$, $i \neq j$, *then there is a* $\bar{p}$*-free* $H^+(\mathbf{A}/\mathbf{B})$ *in* $\mathfrak{T}$ *such that*

(D.1) $\mathrm{N}^+_g = \mathrm{N}_g - 1$;

(D.2) $\mathrm{M}(H^+, \mathbf{C} \to \mathbf{D}) \leqq \mathrm{M}(H, \mathbf{C} \to \mathbf{D})$ *for any* $\mathbf{C} \to \mathbf{D}$ *of* $\mathfrak{U}$ *except a* $\hat{g}$*-free* $\hat{p}$*-free insertion or deletion.*

The following lemma is a precise statement asserting that for our purposes we may assume without loss of generality that no $\hat{g}$-markers occur in $\mathbf{F}$ of Diagram $\mathcal{E}$ when that diagram is *rep* a proof whose first $\bar{g}$-deletion is $del(g^{jL}_\alpha \bar{g}^{iR}_\alpha)$, $i \neq j$.

LEMMA 17. *Suppose* $Cond_4(\mathfrak{T}; g, p)$ *and that there is an* $H(\mathbf{A}/\mathbf{B})$ *in* $\mathfrak{T}$ *as described in Reduction* D. *Then there is a* $\bar{p}$*-free* $H^0(\mathbf{A}/\mathbf{B})$ *in* $\mathfrak{T}$ *such that*

(17.1) $\mathrm{O}^0_g = del(g^{jL}_\alpha \bar{g}^{iR}_\alpha)$, $i \neq j$, *and Diagram* $\mathcal{E}$ *may be rep* $H^0(\mathbf{A}/\mathbf{B})$;

(17.2) $\mathrm{N}^0_g = \mathrm{N}_g$;

(17.3) *When Diagram* $\mathcal{E}$ *is rep* $H^0(\mathbf{A}/\mathbf{B})$ *the proof* $K_5(g^j_\gamma \mathbf{E}/\mathbf{F}g^j_\alpha)$ *is* $\hat{p}$*-free; moreover,* g^j *is the only* $\hat{g}$*-marker occurring in this proof so that the words* $\mathbf{E}$ *and* $\mathbf{F}$ *are* $\hat{p}$*-free* $\hat{g}$*-free*;

(17.4) $\mathrm{M}(H^0, \mathbf{C} \to \mathbf{D}) = \mathrm{M}(H, \mathbf{C} \to \mathbf{D})$ *for any* $\mathbf{C} \to \mathbf{D}$ *of* $\mathfrak{U}$ *except a* $\hat{g}$*-free* $\hat{p}$*-free insertion or deletion.*

By Lemma 14.2, Diagrams $\mathcal{E}$ may be *rep* the g-malcev $\bar{p}$-free $H(\mathbf{A}/\mathbf{B})$ of Reduction D. Identify the subproof $K_5(g^j_\gamma \mathbf{E}/\mathbf{F}g^j_\alpha)$ then given by this diagram — this proof has no $\bar{g}$-deletions by the definition of O_g —with the proof $K(g^j_\gamma \mathbf{E}/\mathbf{F}g^j_\alpha)$ of Reduction C. Then, by Reduction C, there are proofs $T(g^j_\gamma \mathbf{E}/\mathbf{F}'g^j_\alpha)$ and $U(\mathbf{F}'/\mathbf{F})$ in $\mathfrak{T}$ as described therein. The desired $H^0(\mathbf{A}/\mathbf{B})$ is obtained from $H(\mathbf{A}/\mathbf{B})$ by replacing $K_5(g^j_\gamma \mathbf{E}/\mathbf{F}g^j_\alpha)$ by $T(g^j_\gamma \mathbf{E}/\mathbf{F}'g^j_\alpha)$ and $K_7(\mathbf{FP}/g^k_\gamma \mathbf{G})$ by $UK_7(\mathbf{F}'\mathbf{P}/g^k_\gamma \mathbf{G})$.

LEMMA 18. *Suppose* $Cond_4(\mathfrak{T}; g, p)$. *If there is a* $\bar{p}$*-free* $H^0(\mathbf{A}/\mathbf{B})$ *satisfying* 17.1, 17.3 *of Lemma* 17, *then there is a* $\bar{p}$*-free* $H^+(\mathbf{A}/\mathbf{B})$ *such that* $\mathrm{N}^+_g = \mathrm{N}^j_g - 1$ *and* $\mathrm{M}(H^+, \mathbf{C} \to \mathbf{D}) \leqq \mathrm{M}(H^0, \mathbf{C} \to \mathbf{D})$ *for any* $\mathbf{C} \to \mathbf{D}$ *of* $\mathfrak{U}$ *except a* $\hat{g}$*-free* $\hat{p}$*-free insertion or deletion.*

Let Diagram $\mathcal{E}$ be *rep* the $H^0(\mathbf{A}/\mathbf{B})$ described. Then the words $\mathbf{E}$ and $\mathbf{F}$ of that diagram are $\hat{g}$-free and $\hat{p}$-free since 17.3 of Lemma 17 holds. The desired $H^+(\mathbf{A}/\mathbf{B})$ is $\mathcal{E}|\mathcal{F}||H^0$ since the operation sequences $ins(\overline{\mathbf{F}}\mathbf{F})$, *ins*($\mathbf{E\bar{E}}$), *del*($\mathbf{\bar{F}F}$), and *del*($\mathbf{E\bar{E}}$) thus consist solely of $\hat{g}$-free $\hat{p}$-free insertions and deletions.

Lemmas 17 and 18 imply Reduction D.

15. *g-translation group presentations.*

$\mathfrak{T}$ is a *g-translation group presentation*	means	$Cond_3(\mathfrak{T}; g)$ and every g-shift rule of $\mathfrak{U}$ is a g-translation.
$Cond_5(\mathfrak{T}; g, p)$	means	$Cond_4(\mathfrak{T}; g, p)$ and $\mathfrak{T}$ is a g-translation group presentation.

E.g., $Cond_5(\mathfrak{T}_2; k, p)$ where p is void, $Cond_5(\mathfrak{T}_3; t, k)$. Now $\mathbf{A} \vdash_{\mathfrak{T}_g} \mathbf{B}$ is to mean $\mathbf{A} \vdash_{\mathfrak{T}} \mathbf{B}$ which is $\bar{g}$-free.

Reduction E. *Suppose* $Cond_5(\mathfrak{T}; g, p)$. *Let* **A** *and* **B** *be* $\bar{g}$*-free* $\bar{p}$*-free words on* $\mathfrak{Z}$. *If* $\mathbf{A} \vdash_{\mathfrak{T}_p} \mathbf{B}$, *then* $\mathbf{A} \vdash_{\mathfrak{T}_{pg}} \mathbf{B}$.

We suppose given a $\bar{p}$-free $H(\mathbf{A}/\mathbf{B})$ of $\mathfrak{T}$ and show by induction on N_g that there is a $\bar{p}$-free $H'(\mathbf{A}/\mathbf{B})$ of $\mathfrak{T}$ such that $N'_g = 0$. If $N_g = 0$ then of course $H(\mathbf{A}/\mathbf{B})$ is the desired $H'(\mathbf{A}/\mathbf{B})$. Suppose $N_g > 0$. By Reduction A — noting the words **A** and **B** are $\bar{g}$-free — there is a g-malcev $\bar{p}$-free $H^M(\mathbf{A}/\mathbf{B})$ in $\mathfrak{T}$ such that $N_g^M \leqq N_g$. We shall show that

(E†) *if* $N_g^M > 0$, *then there is an* $H^*(\mathbf{A}/\mathbf{B})$ *in* $\mathfrak{T}$ *such that* $N_g^* < N_g^M$. As $\mathfrak{T}$ is g-stable an application of Lemmas 14.1 and 14.3 shows there are four possibilities for $\{O_g^M$, viz., $1g$, $2g$, $3g$, and $4g$ of Lemma 14.1.

If $1g$ then (E†) follows by Reduction D; if $2g$ then by Reduction B. Cases $3g$ and $4g$ are the duals+ of Cases $1g$ and $2g$ respectively. From (E†) the existence of the desired $H'(\mathbf{A}/\mathbf{B})$ is clear.

Since **B** is $\bar{g}$-free $H'(\mathbf{A}/\mathbf{B})$ is $\bar{g}$-free by Lemma 7.

16. *Redundant operation sequences.* Where $Cond_1(\mathfrak{T}; g)$, we now use Q_g^u for the number of g-shifts of operation sequence H^u, Q_{gi}^u for the number of g^i-shifts.

DEFINITION OF g^i-REDUNDANT. For any $\mathfrak{T}$ under any marker convention the operation sequence $K_1 O_E K_3 O_F K_5$ is called *g^i-redundant* (*in* O_E *and* O_F)—or simply *redundant*—if O_E and O_F are g^i-shifts which are converses of each other and K_3 contains no g^i-shifts.

Reduction F.+ *Suppose* $Cond_1(\mathfrak{T}; g)$ *and* $Cond_1(\mathfrak{T}; p)$. *If there is a* $\bar{p}$*-free* $H(\mathbf{A}/\mathbf{B})$ *in* $\mathfrak{T}$ *such that* H *is* g^i*-redundant in the* $\hat{p}$*-free operations* O_E *and* O_F, *then there is a* $\bar{p}$*-free* $H^{\diamond}(\mathbf{A}/\mathbf{B})$ *in* $\mathfrak{T}$ *such that*

(F1) $N_g^{\diamond} = N_g$;

(F2) $Q_g^{\diamond} < Q_g$;

(F3) $M(H^{\diamond}, \mathbf{C} \to \mathbf{D}) \leqq M(H, \mathbf{C} \to \mathbf{D})$ *for any* $\mathbf{C} \to \mathbf{D}$ *of* $\mathfrak{U}$ *except a* $\hat{g}$*-free* $\hat{p}$*-free insertion or deletion*;

(F4) *If* $N_g>0$ *and* $O_g=del(g_\alpha^{iR}\bar{g}_\alpha^{iR})$ *then* $O_g^\diamond=del(g_\alpha^{iR}\bar{g}_\alpha^{iR})$ *for a certain* i.

The desired $H^\diamond(\mathbf{A}/\mathbf{B})$ is $\mathcal{G}|\mathcal{G}||H$. The sequences $del(\mathbf{P}\overline{\mathbf{P}})$ and $del(\overline{\mathbf{Q}}\mathbf{Q})$ of $H^\diamond$ contain no $\bar{g}$-deletions by the definition of a g-shift and no $\bar{p}$-deletions since by hypothesis O_E and O_F are $\hat{p}$-free. (Trivially, then, $H^\diamond(\mathbf{A}/\mathbf{B})$ is $\bar{p}$-free by Lemma 7.) If $N_g>0$ and $O_g=del(g_\alpha^{iR}\bar{g}_\alpha^{iR})$ then $ins(g_\alpha^{iR}\bar{g}_\alpha^{iR})$ occurs in K_1 and O_g in K_4 so that F4 is clear.

17. $\mathfrak{Sig}$, *a mapping of operation rules of a group presentation into words of a free group.* For any finite group presentation a word is *reduced* if not of the form $\mathbf{B}\bar{\mathbf{a}}\mathbf{a}\mathbf{C}$. A finite group presentation $(\mathfrak{Z}, \mathfrak{U})$ is called *the free group on* $\mathfrak{Z}$ if $\mathfrak{U}$ consists precisely of the trivial rules; in this case we write $(\mathfrak{Z}, \varnothing)$ for $(\mathfrak{Z}, \mathfrak{U})$ and $\vdash^{\mathfrak{Z}}$ for $\vdash_{(\mathfrak{Z},\varnothing)}$.

LEMMA 19. *For any free group* $(\mathfrak{Z}, \varnothing)$ *if* $\mathbf{W}\vdash^{\mathfrak{Z}} 1$, *then* $\mathbf{W}\vdash^{\mathfrak{Z}} 1$ *without insertions.*

We omit the demonstration of this lemma because the lemma is so well-known; a direct demonstration by induction on the number of insertions in a given proof is easily supplied, however, using the marker convention and the technique of the diagrams. For the essential idea note Reduction B of this paper and diagrams page 577 [2], Part IV.

LEMMA 20. *For any free group* $(\mathfrak{Z}, \varnothing)$, *if* $\mathbf{W}$ *is reduced and* $\mathbf{W}\vdash^{\mathfrak{Z}} 1$ *then* $\mathbf{W}$ *is* 1.

Trivial, by the preceding lemma.

Let $(\mathfrak{Z}, \mathfrak{U})$ be any finite group presentation and $\mathfrak{Z}'$ a subset of $\mathfrak{Z}$ containing $\bar{\mathbf{a}}$ if $\mathbf{a}$. Let $\mathfrak{M}$ (or $\mathfrak{M}(\mathfrak{Z}')$) be any set of reduced words over $\mathfrak{Z}'$ not containing 1 and containing $\overline{\mathbf{A}}$ if $\mathbf{A}$. If a word $\mathbf{P}$ can be expressed in the form $\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_n$, $n=1, 2, \cdots$, where each $\mathbf{A}_i$ is a word of $\mathfrak{M}$, then $\mathbf{P}$ is an $\mathfrak{M}$-*product* and the sequence of words, $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_n$ is an $\mathfrak{M}$-*factorization of* $\mathbf{P}$. (An $\mathfrak{M}$-product may have more than one $\mathfrak{M}$-factorization.) An $\mathfrak{M}$-product, $\mathbf{P}$, is $\mathfrak{M}$-*reduced* if for any $\mathfrak{M}$-factorization of $\mathbf{P}$, say $\mathbf{A}_1, \mathbf{A}_2, \cdots \mathbf{A}_n$, no $\mathbf{A}_{i+1}$ is $\overline{\mathbf{A}}_i$.

The empty word also is an $\mathfrak{M}$-reduced $\mathfrak{M}$-product. If there is an $\mathfrak{M}$-reduced $\mathfrak{M}$-product, $\mathbf{P}$, $\mathbf{P}$ not 1, such that $\mathbf{P}\vdash^{\mathfrak{Z}'} 1$ we shall say, that $\mathfrak{M}$, or the set of $\mathfrak{M}$-products, is *dependent*, otherwise *independent*. (For independent $\mathfrak{M}$, the $\mathfrak{M}$-factorization of any $\mathfrak{M}$-product is unique.)

Let $\mathfrak{U}'$ be any subset of the non-trivial rules of $\mathfrak{U}$ such that $\mathbf{A}\to\mathbf{B}$ is a rule of $\mathfrak{U}'$ if $\mathbf{B}\to\mathbf{A}$ is. With regard to any proof in $(\mathfrak{Z}, \mathfrak{U})$, a $\mathfrak{U}'$-sequence is any sequence of operations each of which applies a rule of $\mathfrak{U}'$. Let $\mathfrak{Sig}$ (or $\mathfrak{Sig}$: $\mathfrak{U}'\longrightarrow\mathfrak{M}$) be any mapping of $\mathfrak{U}'$ onto $\mathfrak{M}$ satisfying the condition that $\mathfrak{Sig}(\mathbf{A}\to\mathbf{B})$ is $\overline{\mathfrak{Sig}(\mathbf{B}\to\mathbf{A})}$ for all $\mathbf{A}\to\mathbf{B}$ of $\mathfrak{U}'$. Suppose the mapping $\mathfrak{Sig}$ as initially given is now extended to a mapping of the set of $\mathfrak{U}'$-sequences onto the set of $\mathfrak{M}$-products in the following way. First, $\mathfrak{sig}\mathrm{O}$ is $\mathfrak{sig}(\mathbf{A}\rightarrow\mathbf{B})$ where O is an application of $\mathbf{A} \rightarrow \mathbf{B}$. Secondly, the word corresponding to the empty sequence is 1. Thirdly, $\mathfrak{sig}(K\mathrm{O})$ is $\mathfrak{sig}\mathrm{O}\mathfrak{sig}K$.

When the extension of $\mathfrak{sig}$ as just described is understood we shall say that $\mathfrak{sig}$ is a *right signature (for* $\mathfrak{U}'$ *in* $\mathfrak{M}(\mathfrak{Z}')$*)*. When we understand the third proviso in the construction of the extension for a right signature is replaced by the stipulation that $\mathfrak{sig}(K\mathrm{O})$ is $\mathfrak{sig}K\mathfrak{sig}\mathrm{O}$ we call $\mathfrak{sig}$ a *left signature (for* $\mathfrak{U}'$ *in* $\mathfrak{M}(\mathfrak{Z}')$*)*.

Now let $T_{(m,n)}$, $m < n$, be the operation sequence $\mathrm{O}_{(m+1)}\mathrm{O}_{(m+2)} \cdots \mathrm{O}_{(n)}$ and $T_{(m,m)}$ be the empty sequence of operations. Then $T'_{(m,n)}$ or $T''_{(m,n)}$ is to be any subsequence of $T_{(m,n)}$.

We shall say that $\mathfrak{sig}$, a right signature for $\mathfrak{U}'$ in $\mathfrak{M}(\mathfrak{Z}')$, and $\{\mathbf{V}_{(s)}\}_{s=m}^{n}$, any sequence of $n - m + 1$ words over $\mathfrak{Z}'$ *form a right signature complex for* the $\mathfrak{U}'$-subsequence $T'_{(m,n)}$ in $T_{(m,n)}$ if the following three conditions are satisfied:

(**17**.1) $\mathbf{V}_{(m)}$ is 1;

(**17**.2) Where $\mathrm{O}_{(s+1)}$ is an operation of $T'_{(m,n)}$, either $\mathbf{V}_{(s+1)}$ is $\mathfrak{sig}\mathrm{O}_{(s+1)}\mathbf{V}_{(s)}$ or $\mathbf{V}_{(s)}$ is $\mathfrak{sig}\mathrm{O}_{(s+1)}\mathbf{V}_{(s+1)}$;

(**17**.3) Where $\mathrm{O}_{(s+1)}$ is not an operation of $T'_{(m,n)}$, $\mathbf{V}_{(s+1)} \vdash^{\mathfrak{Z}'} \mathbf{V}_{(s)}$.

If in addition the following two conditions are satisfied we shall say the complex is *faithful*:

(**17**.4) $\mathfrak{M}(\mathfrak{Z}')$ is an independent set;

(**17**.5) $\mathfrak{sig}T'_{(m,n)}$ is $\mathfrak{M}$-reduced.

To obtain the definition of *left signature complex* replace "right", "$\mathfrak{sig}\mathrm{O}_{(s+1)}\mathbf{V}_{(s)}$", and "$\overline{\mathfrak{sig}\mathrm{O}_{(s+1)}}\mathbf{V}_{(s+1)}$", by "left", "$\mathbf{V}_{(s)}\mathfrak{sig}\mathrm{O}_{(s+1)}$", and "$\mathbf{V}_{(s+1)}\overline{\mathfrak{sig}\mathrm{O}_{(s+1)}}$" respectively.

LEMMA 21⁺. *For an arbitrary group presentation* $(\mathfrak{Z}, \mathfrak{U})$ *suppose* $\mathfrak{sig}\colon \mathfrak{U}' \longrightarrow \mathfrak{M}(\mathfrak{Z}')$ *and* $\{\mathbf{V}_{(u)}\}_{u=m}^{n}$ *form a right signature complex for the* $\mathfrak{U}'$*-subsequence* $T'_{(m,n)}$ *in* $T_{(m,n)}$. *Then*

(21.1) $\mathfrak{sig}T'_{(m,n)} \vdash^{\mathfrak{Z}'} \mathbf{V}_{(n)}$;

(21.2) *If the complex is faithful and* $\mathbf{V}_{(n)}$ *is* 1 *then* $T'_{(m,n)}$ *is empty.*

For $n = m$ Lemma 21.1 reads $1 \vdash^{\mathfrak{Z}'} 1$. Assume inductively then that

(21†) $\mathfrak{sig}T'_{(m,u)} \vdash^{\mathfrak{Z}'} \mathbf{V}_{(u)}$, $m \leqq u \leqq n$.

Suppose $\mathrm{O}_{(u+1)}$ is an operation of $T'_{(m,u)}$. Then it follows from **17**.2 that $\mathbf{V}_{(u+1)} \vdash^{\mathfrak{Z}'} \mathfrak{sig}\mathrm{O}_{(u+1)}\mathbf{V}_{(u)}$. Since $T'_{(m,u+1)}$ is $\mathrm{O}_{(u+1)}T_{(m,u)}$ the induction step follows from (21†). If $\mathrm{O}_{(u+1)}$ is not an operation of $T'_{(m,n)}$, $T'_{(m,u+1)}$ is $T'_{(m,u)}$, $\mathbf{V}_{(u+1)} \vdash^{\mathfrak{Z}'} \mathbf{V}_{(u)}$ by **17**.3 and the induction step follows by (21†). This shows Lemma 21.1.

As to Lemma 21.2, $\mathfrak{sig}T'_{(m,n)} \vdash^{\mathfrak{Z}'} 1$ follows from Lemma 21.1 and the assumption that $\mathbf{V}_{(n)}$ is 1. But $\mathfrak{Sig}\,T'_{(m,n)}$ is an $\mathfrak{M}$-product which is $\mathfrak{M}$-reduced by **17**.5 since the complex is faithful; further $\mathfrak{Sig}\,T'_{(m,n)}$ is 1 by **17**.4 and the definition of an independent set. Thus $T'_{(m,n)}$ is empty by the definition of $\mathfrak{Sig}$.

We now consider certain independent set of words in the system $\mathfrak{T}_2$. Let $\mathfrak{Z}_{rx}$ be the subset of $\mathfrak{Z}_2$ consisting of

the $\hat{r}$-symbols;
the $\hat{x}$-symbols.

Where r_α is any r-symbol let $\mathfrak{M}_{rx}$ consist of the following words on $\mathfrak{Z}_{rx}$ together with their inverses:

$$r_\alpha$$
$$xr_\alpha x$$
$$xx$$

$\mathfrak{Z}_{ly}$ and $\mathfrak{M}_{ly}$ have the dual definitions, i.e., $\mathfrak{Z}_{ly}$ consists of the $\hat{l}$- and $\hat{y}$-symbols of $\mathfrak{Z}_2$ and $\mathfrak{M}_{ly}$ of all words of the form l_α, $yl_\alpha y$, yy, together with their inverses.

Now let $\mathfrak{U}_{rx}$ be the rules $\mathfrak{U}_{2.1}$, $\mathfrak{U}_{2.6}$, and $\mathfrak{U}_{2.7}$ of $\mathfrak{U}_2$; $\mathfrak{U}_{ly}$ the rules $\mathfrak{U}_{2.1}$, $\mathfrak{U}_{2.2}$, and $\mathfrak{U}_{2.3}$ of $\mathfrak{U}_2$. The right signature of $\mathfrak{U}_{rx}$ onto $\mathfrak{M}_{rx}$, $\mathfrak{Sig}_{rx}$, is defined by the following table with the understanding that $\mathfrak{Sig}_{rx}(\mathbf{B}\to\mathbf{A})$ is $\overline{\mathfrak{Sig}_{rx}(\mathbf{A}\to\mathbf{B})}$.

$\mathbf{A} \to \mathbf{B}$	$\mathfrak{Sig}_{rx}(\mathbf{A} \to \mathbf{B})$.
$\Sigma \to l_\alpha \Gamma r_\alpha$	r_α
$r_\alpha s_\beta \to s_\beta x r_\alpha x$	$xr_\alpha x$
$xs_\beta \to s_\beta xx$	xx

The left signature of $\mathfrak{U}_{ly}$ on $\mathfrak{M}_{ly}$, $\mathfrak{Sig}_{ly}$, has the dual definition, i.e., $\mathfrak{Sig}_{ly}(\Sigma \to l_\alpha\Gamma r_\alpha)$ is l_α, $\mathfrak{Sig}_{ly}(s_\beta l_\alpha \to yl_\alpha ys_\beta)$ is $yl_\alpha y$, $\mathfrak{Sig}_{ly}(s_\beta y \to yys_\beta)$ is yy and $\mathfrak{Sig}_{ly}(\mathbf{B} \to \mathbf{A})$ is $\overline{\mathfrak{Sig}_{ly}(\mathbf{A} \to \mathbf{B})}$.

$\mathfrak{Z}_r$ is to be the set of $\hat{r}$-symbols, $\mathfrak{M}_r$ to be $\mathfrak{Z}_r$ itself, $\mathfrak{U}_r$ to be the rules $\mathfrak{U}_{2.1}$, $\mathfrak{Sig}_r(\Sigma\to l_\alpha\Gamma r_\alpha)$ to be r_α, and $\mathfrak{Sig}_r(l_\alpha\Gamma r_\alpha\to\Sigma)$ to be $\bar{r}_\alpha$. Thus, $\mathfrak{Sig}_r$, a right signature of $\mathfrak{U}_r$, i.e., the q-shift rules of $\mathfrak{U}_2$, onto $\mathfrak{M}_r$, is just a restriction of $\mathfrak{Sig}_{rx}$. We take the dual definitions for $\mathfrak{Z}_l$, $\mathfrak{M}_l$, $\mathfrak{U}_l$ and $\mathfrak{Sig}_l$, a left signature of $\mathfrak{U}_l$ (which is, of course, $\mathfrak{U}_r$) onto $\mathfrak{M}_l$.

$\mathfrak{Z}'_{ly}$ is to be the set of $\hat{l}$-symbols together with $y, \bar{y}$; $\mathfrak{M}'_{ly}$ to be $\mathfrak{Z}'_{ly}$ itself, $\mathfrak{U}'_{ly}$ to be the rules $\mathfrak{U}_{2.2}$ and $\mathfrak{U}_{2.3}$, $\mathfrak{Sig}'_{ly}(yl_\alpha ys_\beta \to s_\beta l_\alpha)$ is l_α, $\mathfrak{Sig}'_{ly}(yys_\beta \to s_\beta y)$ is y, and $\mathfrak{Sig}'_{ly}(\mathbf{B} \to \mathbf{A})$ is $\overline{\mathfrak{Sig}'_{ly}(\mathbf{A} \to \mathbf{B})}$. Then $\mathfrak{Sig}'_{ly}$ is a *right* signature of $\mathfrak{U}'_{ly}$ onto $\mathfrak{M}'_{ly}$. The dual definitions are taken for $\mathfrak{Z}'_{rx}$, $\mathfrak{M}'_{rx}$, $\mathfrak{U}'_{rx}$ (which is $\mathfrak{U}_{2.6}$ and $\mathfrak{U}_{2.7}$), and $\mathfrak{Sig}'_{rx}$, a left signature of $\mathfrak{U}'_{rx}$ onto $\mathfrak{M}'_{rx}$.

We now wish to show that $\mathfrak{M}_{rx}$, $\mathfrak{M}_r$, $\mathfrak{M}'_{ly}$ and their duals are independent sets. We now[+] write $\vdash^{rx}$, $\vdash^{r}$, $\vdash^{\prime ly}$ for $\vdash^{\mathfrak{Z}_{rx}}$, $\vdash^{\mathfrak{Z}_r}$, $\vdash^{\mathfrak{Z}'_{ly}}$ respectively.

LEMMA 22^+. *Let* $\mathbf{P}'$ *be any* $\mathfrak{M}_{rx}$*-reduced* $\mathfrak{M}_{rx}$*-product,* $\mathbf{A}_1, \mathbf{A}_2 \cdots \mathbf{A}_N$, $N = 0, 1, \cdots$, *consisting solely of* $\hat{x}$*-symbols. Then*

(22.1) $\mathbf{P}'$ *is reduced and in fact consists of an even number of occurrences of* x *alone or* $\bar{x}$ *alone;*

(22.2) *Where* $\mathbf{P}''$ *is* $x\mathbf{P}'\bar{x}$, $\bar{x}\mathbf{P}'x$, $\mathbf{P}'' \vdash^{rx} 1$ *implies* $\mathbf{P}'$ *is* 1;

(22.3) *Where* $\mathbf{P}''$ *is* $x\mathbf{P}'$, $\bar{x}\mathbf{P}'$, $\mathbf{P}'x$, $\mathbf{P}'\bar{x}$, *not* $\mathbf{P}'' \vdash^{rx} 1$.

Since $\mathbf{P}'$ is $\mathfrak{M}_{rx}$-reduced either every $\mathbf{A}_j$ is xx or every $\mathbf{A}_j$ is $\bar{x}\bar{x}$. This shows Lemma 22.1, from which Lemmas 22.2 and 22.3 follow trivially using Lemma 19.

LEMMA 23. $\mathfrak{M}_{rx}$, $\mathfrak{M}_{ly}$, $\mathfrak{M}'_{ly}$, $\mathfrak{M}'_{rx}$, $\mathfrak{M}_r$, $\mathfrak{M}_l$, *are independent sets.*

Let $\mathbf{P}$ be the $\mathfrak{M}_{rx}$-reduced $\mathfrak{M}_{rx}$-product $\mathbf{A}_1\mathbf{A}_2 \cdots \mathbf{A}_N$, $N = 0, 1, \cdots$ and suppose we are given an $H(\mathbf{P}/1)$ in the free group on $\mathfrak{Z}_{rx}$. We show that (23†) $\mathbf{P}$ *is* 1. By Lemma 19 we may assume without loss of generality that H consists entirely of deletions. Reverting here to the marker convention—with all qualifications null—and its superscript notation, suppose $\mathrm{O}_r = del(r^i_\varepsilon \bar{r}^j_\varepsilon)$ is the first $\bar{r}$-deletion of H where r^i_ε occurs in $\mathbf{A}_i$, $\bar{r}^j_\varepsilon$ in $\mathbf{A}_j$, $1 \leqq i < j \leqq N$.

Let $\mathbf{P}'$ be $\mathbf{A}_{i+1} \cdots \mathbf{A}_{j-1}$ and $\mathbf{P}''$ the word between r^i_α and $\bar{r}^j_\alpha$ in $\mathbf{P}$. By the definition of O_r and Lemma 10 both $\mathbf{P}'$ and $\mathbf{P}''$ consist solely of $\hat{x}$-markers. The following table gives $\mathbf{P}''$ in terms of $\mathbf{P}'$ for the various values of $\mathbf{A}_i$ and $\mathbf{A}_j$.

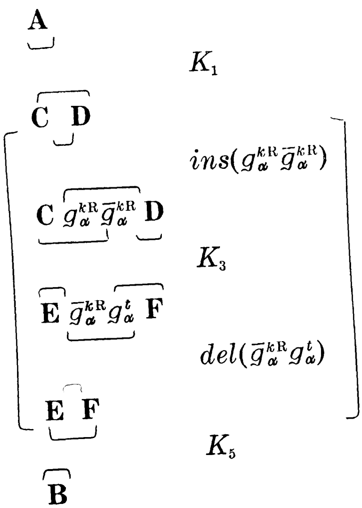
	$\mathbf{A}_i$	$\mathbf{A}_j$	$\mathbf{P}''$
(a)	r^i_ε	$\bar{r}^j_\varepsilon$	$\mathbf{P}'$
(b)	$x^s r^i_\varepsilon x^t$	$\bar{x}^u \bar{r}^j_\varepsilon \bar{x}^v$	$x^t \mathbf{P}' \bar{x}^u$
(c)	r^i_ε	$\bar{x}^u \bar{r}^j_\varepsilon \bar{x}^v$	$\mathbf{P}'\bar{x}^u$
(d)	$x^s r^i_\varepsilon x^t$	$\bar{r}^j_\varepsilon$	$x^t\mathbf{P}'$

The mere existence of O_r implies (23*) $\mathbf{P}'' \vdash^{rx} 1$. If a, then $\mathbf{P}'$, an $\mathfrak{M}_{rx}$-reduced $\mathfrak{M}_{rx}$-product consisting solely of $\hat{x}$-markers, is reduced by Lemma 22.1; hence is 1 by (23*) and Lemma 20. Thus $j = i + 1$, $\mathbf{A}_{i+1}$ is $\overline{\mathbf{A}_i}$ and $\mathbf{P}$ would not be $\mathfrak{M}_{rx}$-reduced. Thus a cannot occur. If b, then (23*) and Lemma 22.2 again imply the contradiction "$\mathbf{P}'$ is 1" as before. Thus b cannot occur. Clearly c or d cannot occur by (23*) and Lemma 22.3. Thus $\mathrm{O}_r \neq del(r^i_\varepsilon \bar{r}^j_\varepsilon)$, $i \neq j$. Interchanging barred and unbarred letters in the foregoing demonstration produces a valid argument that $\mathrm{O}_r \neq del(\bar{r}^i_\varepsilon r^j_\varepsilon)$, $i \neq j$.

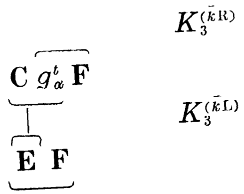
Thus O_r does not exist and $\mathbf{P}$ contains no $\hat{r}$-markers. But if $\mathbf{P}$ contains only $\hat{x}$-markers then Lemmas 22.1 and 20 suffice to show (23†).

This shows $\mathfrak{M}_{rx}$ is an independent set. The dual argument$^+$, using the dual of Lemma 22, shows $\mathfrak{M}_{ly}$ is an independent set. Where $\mathfrak{M}$ is $\mathfrak{M}'_{ly}$, $\mathfrak{M}'_{rx}$, $\mathfrak{M}_r$, or $\mathfrak{M}_l$ any $\mathfrak{M}$-product is $\mathfrak{M}$-reduced if and only if it is reduced. Thus these sets are independent by Lemma 20.

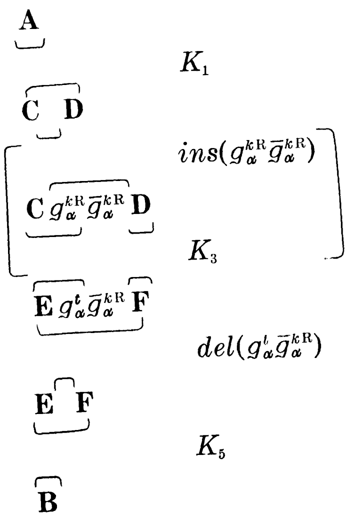
$\mathscr{A}$



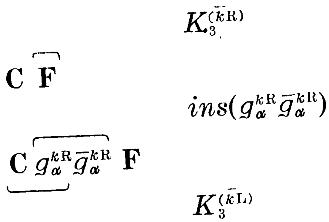
$\mathscr{B}$



$\mathscr{C}$



$\mathscr{D}$



$\mathscr{E}$

$\mathbf{A}$

K_1

$\mathbf{C}\ \mathbf{P}$

$ins(g_\alpha^{iR}\bar{g}_\alpha^{iR})$

$\mathbf{C}g_\alpha^{iR}\bar{g}_\alpha^{iR}\mathbf{P}$

K_3

$\mathbf{D}\ \mathbf{E}\bar{g}_\alpha^{iR}\mathbf{P}$

$ins(\bar{g}_\gamma^{jL}g_\gamma^{jL})$

$\mathbf{D}\bar{g}_\gamma^{jL}g_\gamma^{jL}\mathbf{E}\bar{g}_\alpha^{iR}\mathbf{P}$

K_5

$\mathbf{D}\bar{g}_\gamma^{jL}\mathbf{F}g_\alpha^{jL}\bar{g}_\alpha^{iR}\mathbf{P}$

$del(g_\alpha^{jL}\bar{g}_\alpha^{iR})$

$\mathbf{D}\bar{g}_\gamma^{jL}\mathbf{F}\ \mathbf{P}$

K_7

$\mathbf{D}\bar{g}_\gamma^{jL}g_\gamma^{k}\mathbf{G}$

$del(\bar{g}_\gamma^{jL}g_\gamma^{k})$

$\mathbf{D}\mathbf{G}$

K_9

$\mathbf{B}$

$\mathscr{F}$

$ins(\bar{\mathbf{F}}\mathbf{F})$

$\mathbf{D}\mathbf{E}\bar{g}_\alpha^{iR}\bar{\mathbf{F}}\mathbf{F}\mathbf{P}$

K_7

$\mathbf{D}\mathbf{E}\bar{g}_\alpha^{iR}\bar{\mathbf{F}}g_\gamma^{k}\ \mathbf{G}$

$ins(\mathbf{E}\bar{\mathbf{E}})$

$\mathbf{D}\mathbf{E}\bar{g}_\alpha^{iR}\bar{\mathbf{F}}g_\gamma^{k}\mathbf{E}\bar{\mathbf{E}}\mathbf{G}$

K_5

$\mathbf{D}\mathbf{E}\bar{g}_\alpha^{iR}\bar{\mathbf{F}}\mathbf{F}g_\alpha^{k}\bar{\mathbf{E}}\mathbf{G}$

$del(\bar{\mathbf{F}}\mathbf{F})$

$\mathbf{D}\mathbf{E}\bar{g}_\alpha^{iR}\ g_\alpha^{k}\bar{\mathbf{E}}\mathbf{G}$

$del(\bar{g}_\alpha^{iR}g_\alpha^{k})$

$\mathbf{D}\mathbf{E}\ \bar{\mathbf{E}}\mathbf{G}$

$del(\mathbf{E}\bar{\mathbf{E}})$

$\mathscr{G}$

$g_\gamma^{j}\mathbf{E}$

V_1^ν

$\mathbf{L}\mathbf{M}\mathbf{N}g_\varepsilon^{j}\mathbf{U}\mathbf{Z}$

O_I^ν

$\mathbf{L}\mathbf{M}'\mathbf{N}g_\varepsilon^{j}\mathbf{U}\mathbf{Z}$

V_3^ν

$\mathbf{R}\mathbf{V}g_\varepsilon^{j}\mathbf{U}\mathbf{Z}$

O_R^ν

$\mathbf{R}\mathbf{V}'g_{\varepsilon'}^{j}\mathbf{U}'\mathbf{Z}$

V_5^ν

$\mathbf{F}g_\alpha^{j}$

$\mathscr{H}$

$ins(\bar{\mathbf{V}}\mathbf{V})$

$\mathbf{L}\mathbf{M}\mathbf{N}\bar{\mathbf{V}}\mathbf{V}g_\varepsilon^{j}\mathbf{U}\mathbf{Z}$

O_R^ν

$\mathbf{L}\mathbf{M}\mathbf{N}\bar{\mathbf{V}}\mathbf{V}'g_{\varepsilon'}^{j}\mathbf{U}'\mathbf{Z}$

O_I^ν

$\mathbf{L}\mathbf{M}'\mathbf{N}\bar{\mathbf{V}}\mathbf{V}'g_{\varepsilon'}^{j}\mathbf{U}'\mathbf{Z}$

V_3^ν

$\mathbf{R}\mathbf{V}\bar{\mathbf{V}}\mathbf{V}'g_{\varepsilon'}^{j}\mathbf{U}'\mathbf{Z}$

$del(\mathbf{V}\bar{\mathbf{V}})$

$\mathcal{J}$

$\mathbf{A}$

K_1

$\mathbf{UC}g_\delta^{iL}\mathbf{DW}$

O_E

$\mathbf{UP}g_\omega^{iL}\mathbf{QW}$

K_3

$\mathbf{SP}g_\omega^{iL}\mathbf{QT}$

O_F

$\mathbf{SC}g_\delta^{iL}\mathbf{DT}$

K_4

$\mathbf{B}$

$\mathcal{J}$

$ins(\mathbf{P}\overline{\mathbf{P}})$
$ins(\overline{\mathbf{Q}}\mathbf{Q})$

$\mathbf{UP}\overline{\mathbf{P}}\mathbf{C}g_\delta^{iL}\mathbf{D}\overline{\mathbf{Q}}\mathbf{QW}$

K_3

$\mathbf{SP}\overline{\mathbf{P}}\mathbf{C}g_\delta^{iL}\mathbf{D}\overline{\mathbf{Q}}\mathbf{QT}$

$del(\mathbf{P}\overline{\mathbf{P}})$
$del(\overline{\mathbf{Q}}\mathbf{Q})$

PART III

The general methods of Part II are now used to carry out the program given on page 219 for showing the Main Theorem in the non-trivial direction.

18. *Demonstration of Theorem* III. Since $Cond_5(\mathfrak{T}_2; k, p)$ with p void and the words $t_\iota\Sigma k\bar{\Sigma}\bar{t}_\iota$, $\iota = 1, 2$, are $\bar{k}$-free, Theorem III is an instance of Reduction E.

19. *Demonstration of Theorem* IV. We assume given a $\bar{k}$-free proof $H(t_1\Sigma k\bar{\Sigma}\bar{t}_1/t_2\Sigma k\bar{\Sigma}\bar{t}_2)$ in $\mathfrak{T}_2$. By Lemma 11 exactly one $\hat{k}$-marker occurs in this proof and in every step thereof. We call this marker k^0 and let the i^{th} step of the proof H be $\mathbf{A}_i k^0 \mathbf{B}_i$, $i=1, 2, \cdots, N$. Then to show Theorem IV it clearly suffices, since $\mathbf{A}_1$ is $t_1\Sigma k^0$ and $\mathbf{A}_N$ is $t_2\Sigma k^0$, to show (IV†) $\mathbf{A}_i k^0 \vdash_{3k} \mathbf{A}_{i+1} k^0$ *for each* i. If O, the i^{th} operation of H, is the application of the $\hat{k}$-free rule $\mathbf{C} \rightarrow \mathbf{D}$ to markers in $\mathbf{A}_i$ then (IV†) follows since $\mathbf{C} \rightarrow \mathbf{D}$ is also a rule of $\mathfrak{U}_3$. If O is a $\hat{k}$-free rule applied in $\mathbf{B}_i$ then (IV†) since $\mathbf{A}_{i+1}$ is $\mathbf{A}_i$. If O applies the k-shift $\mathbf{a}k \rightarrow k\mathbf{a}$ of $\mathfrak{U}_{2.8}$ or $\mathfrak{U}_{2.9}$ let $\mathbf{A}_i$ be $\mathbf{A}'\mathbf{a}$ so that $\mathbf{A}_{i+1}$ is $\mathbf{A}'$; then (IV†) follows since $\mathbf{a}k \rightarrow k$ is a rule of $\mathfrak{U}_3$ and effects a proof of $\mathbf{A}'\mathbf{a}k/\mathbf{A}'k$ in $\mathfrak{T}_3$. If O applies $k\mathbf{a} \rightarrow \mathbf{a}k$ then (IV†) follows similarly since $k \rightarrow \mathbf{a}k$ is a rule of $\mathfrak{U}_3$. Now suppose that O is an application of $\bar{q}\bar{t}_1 t_2 qk \rightarrow k\bar{q}\bar{t}_1 t_2 q$ and let $\mathbf{A}_i$ be $\mathbf{A}'\bar{q}\bar{t}_1 t_2 q$ so that $\mathbf{A}_{i+1}$ is $\mathbf{A}'$. Then (*) $\mathbf{A}'\bar{q}\bar{t}_1 t_2 qk$, $\mathbf{A}'\bar{q}\bar{t}_1 t_1 qk$, $\mathbf{A}'\bar{q}qk$, $\mathbf{A}'k$ is a valid proof in $\mathfrak{T}_3$ since $t_2 qk \rightarrow t_1 qk$ is a rule of $\mathfrak{U}_3$. Supposing O applies $k\bar{q}\bar{t}_1 t_2 q \rightarrow \bar{q}\bar{t}_1 t_2 qk$ (IV†) follows similarly (using the steps (*) in reverse order as a proof) since $t_1 qk \rightarrow t_2 qk$ is a rule of $\mathfrak{U}_3$. This shows Theorem IV.

20. *Demonstration of Theorem* V. Since $Cond_5(\mathfrak{T}_3; t, k)$ and the words $t_\iota\Sigma k$, $\iota = 1, 2$, are $\bar{t}$-free and $\bar{k}$-free, Theorem V is an instance of Reduction E.

21. *Demonstration of Theorem* VI. Assume that we are given a $\bar{k}$-free $\bar{t}$-free $H(t_1\Sigma k/t_2\Sigma k)$ in $\mathfrak{T}_3$. We let k^0 and t^1 be the single k-marker and single t-marker occurring in the proof and occurring in each step. (Lemma 11.) Clearly H must contain at least one application of the rule of $\mathfrak{U}_3$ $t_1 qk \rightarrow t_2 qk$ for otherwise the last step of the proof could not be $t_2^1\Sigma k^0$. Then the premiss of the first such operation is $\mathbf{E}t_1^1 qk^0\mathbf{F}$ for some $\mathbf{E}$ and $\mathbf{F}$; so simply omitting all the steps that follow, gives a proof $H'(t_1^1\Sigma k^0/\mathbf{E}t_1^1 qk^0\mathbf{F})$ in $\mathfrak{T}_3$.

From the proof H' erase that part of each step left of t_1^1 and omit repetitious steps. Since $t_1\mathbf{M} \rightarrow t_1\mathbf{N}$ is a rule of $\mathfrak{U}_4$ if $\mathbf{G}t_1\mathbf{M} \rightarrow \mathbf{R}t_1\mathbf{N}$ is a rule of $\mathfrak{U}_3$ the result is a valid $\bar{k}$-free $\bar{t}$-free proof, $H''(t_1^1\Sigma k^0/t_1^1 qk^0\mathbf{F})$, in $\mathfrak{T}_4$. (The induction argument needed to show the trivial $t_1^1\mathbf{A}_i \vdash_{4kt} t_1^1\mathbf{A}_{i+1}$, where $\mathbf{B}_i t_1^1\mathbf{A}_i$ is the i^{th} step of the proof H' is the dual of (IV†) but without analogues of the last two cases considered there.)

From the proof H'' erase that part of each step right of k^0 and omit repetitious steps. Since every k-shift of $\mathfrak{U}_4$ is of form $\mathbf{M}k \rightarrow \mathbf{N}k$ the result is a valid $\bar{k}$-free $\bar{t}$-free proof, $H'''(t_1^1\Sigma k^0/t_1^1 qk^0)$, in $\mathfrak{T}_4$ whose i^{th} step may be written $t_1^1\mathbf{M}_i k^0$. (The trivial induction argument needed is a special case of the dual of the induction argument about H'' of the preceding paragraph.)

Clearly the following lemma, shown by induction on i, implies Theorem VI: (VI†) *For every* $\mathbf{M}_i$ *there are words* Ξ *and* Ω *such that* $\Sigma \vdash_5 \Xi\mathbf{M}_i\Omega$. For $i = 1$, we note $\Sigma \vdash_5 \Sigma$. To show the induction step it suffices to show (VI††) $\mathbf{M}_i \vdash_5 \Xi'\mathbf{M}_{i+1}\Omega'$ *for each* $\mathbf{M}_i$ *for some* Ξ' *and* Ω'. If O, the i^{th} operation of H''', is not a k- or t-shift, then $\mathbf{M}_i \vdash_5 \mathbf{M}_{i+1}$ since the rule applied is a rule of $\mathfrak{U}_5$ as well as $\mathfrak{U}_4$. Suppose O applies $t\mathbf{a} \rightarrow t$ (here $\mathbf{a}$ is y or an l-symbol) and $\mathbf{M}_i$ is $\mathbf{aM}$ so that $\mathbf{M}_{i+1}$ is $\mathbf{M}$. Thus (VI††) since $\mathbf{aM} \vdash_5 \mathbf{aM}$. Similarly, if O applies $t \rightarrow t\mathbf{a}$, $\mathbf{M}_i$ is $\mathbf{M}$, and $\mathbf{M}_{i+1}$ is $\mathbf{aM}$, then (VI††) since $\mathbf{M} \vdash_5 \bar{\mathbf{a}}\mathbf{aM}$ using $ins(\bar{\mathbf{a}}\mathbf{a})$. If O applies a k-shift (VI††) follows by the dual argument. This shows (VI††) in general, hence (VI†).

22. *Demonstration of Theorem* VII. Given an $H(\Sigma/\Xi q\Omega)$ in $\mathfrak{T}_5$ we show that (VII†) *if* $\mathrm{N}_q > 0$, *there is an* $H'(\Sigma/\Xi q\Omega)$ in $\mathfrak{T}_5$ *such that* $\mathrm{N}_q' < \mathrm{N}_q$. As Σ and $\Xi q\Omega$ are $\bar{q}$-free, by Reduction A we may assume without loss of generality that H is q-malcev in showing (VII†). As $\mathfrak{T}_5$ is q-stable by Lemmas 14.1 and 14.3 there are four possibilities for $\{\mathrm{O}_q$, viz., $1q$, $2q$, $3q$, and $4q$ of Lemma 14.1. If Case $1q$ holds then (VII†) follows as a special case of Reduction D taking $\mathfrak{T}$ to be $\mathfrak{T}_5$, g to be q, and p void.

23. *Demonstration of* (VII†) *if* $\mathrm{O}_q = del(q_\alpha^{i\mathrm{R}}\bar{q}^{i\mathrm{R}})$. In this case the stronger version of (VII†) obtained by dropping the assumption that H is q-malcev will be shown by an induction on Q_q. If $\mathrm{Q}_q = 0$ then $\mathrm{Q}_{qi} = 0$ so that (VII†) follows by Reduction B taking $\mathfrak{T}$ to be $\mathfrak{T}_5$, g to be q, and p to be empty. Suppose $\mathrm{Q}_q > 0$. If H is $q^{i\mathrm{R}}$-redundant then by Reduction F ignoring F3 there is an $H^\diamond(\Sigma/\Xi q\Omega)$ in $\mathfrak{T}_5$ such that $\mathrm{Q}_q^\diamond < \mathrm{Q}_q$. We shall show (VII††) *if* $\mathrm{O}_q = del(q_\alpha^{i\mathrm{R}}\bar{q}_\alpha^{i\mathrm{R}})$ *and* H *is not* $q^{i\mathrm{R}}$*-redundant* $\mathrm{Q}_{qi} = 0$. Thus (VII†) would again follow by Reduction B.

24. *Demonstration of* (VII††). For this case let $\mathrm{O}_{(0)}$ be $ins(q_\alpha^{i\mathrm{R}}\bar{q}_\alpha^{i\mathrm{R}})$, assume O_q is the $N + 1^{\text{st}}$ operation following $\mathrm{O}_{(0)}$ in H and let $\mathrm{O}_{(n)}$, $n =$

$1, 2, \cdots, N$ be the n^{th} operation following $O_{(0)}$. We write $O_{(n)}]$, $n = 1, 2, \cdots, N$, as

$$\mathbf{A}_{(n)} q^{i\text{R}}_{\beta_{(n)}} \mathbf{B}_{(n)} \bar{q}^{i\text{R}}_{\alpha} \mathbf{C}_{(n)}, \qquad \mathbf{B}_{(n)}\ \hat{q}\text{-free.}$$

This is completely general by the definition of O_q and Lemma 10. Let $\tilde{\mathbf{W}}$, where $\mathbf{W}$ is any word on $\mathfrak{Z}_5$, be $\mathbf{W}$ with all symbols except $\hat{r}$-symbols everywhere erased. Clearly $\tilde{\mathbf{W}}$ is $\tilde{\mathbf{U}}\tilde{\mathbf{V}}$ where $\mathbf{W}$ is $\mathbf{UV}$. Let $T'_{(0,N)}$ be the subsequence of $q^{i\text{R}}$-shifts of $T_{(0,N)}$.

LEMMA 24. *$\mathfrak{S}\text{ig}_r$ and $\{\tilde{\mathbf{B}}_{(n)}\}_{n=0}^{N}$ form a faithful right signature complex for $T'_{(0,N)}$ in $T_{(0,N)}$ and $\tilde{\mathbf{B}}_{(N)}$ is 1.*

The conditions **17**.1, **17**.2, **17**.3 are verified as follows. First, as to **17**.1, $\tilde{\mathbf{B}}_{(0)}$ is 1 since no $\hat{r}$-markers occur between $q^{i\text{R}}$ and $\bar{q}^{i\text{R}}_{\alpha}$ in $ins(q^{i\text{R}}_{\alpha}\bar{q}^{i\text{R}}_{\alpha})]$. Secondly, as to **17**.2, suppose $O_{(n+1)}$ is a $q^{i\text{R}}$-shift applying $\mathbf{P}q_{\beta}\mathbf{N} \to l_{\pi}\mathbf{P}'q_{\alpha}\mathbf{N}'r_{\pi}$ where of course $\mathbf{N}$ and $\mathbf{N}'$ are $\hat{r}$-free. Then assume notationally that $\mathbf{B}_{(n)}$ is $\mathbf{NB}$ so that $\mathbf{B}_{(n+1)}$ is $\mathbf{N}'r_{\pi}\mathbf{B}$. Thus $\tilde{\mathbf{B}}_{(n)}$ is $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{B}}_{(n+1)}$ is $r_{\pi}\tilde{\mathbf{B}}$, i.e., $\mathfrak{S}\text{ig}O_{(n+1)}\tilde{\mathbf{B}}_{(n)}$. If $O_{(n+1)}$ applies the converse and $\mathbf{B}_{(n)}$ is $\mathbf{N}'r_{\pi}\mathbf{B}$, then $\tilde{\mathbf{B}}_{(n+1)}$ is $\tilde{\mathbf{B}}$, $\tilde{\mathbf{B}}_{(n)}$ is $r_{\pi}\tilde{\mathbf{B}}$, i.e., $\overline{\mathfrak{S}\text{ig}O_{(n+1)}}\tilde{\mathbf{B}}_{(n+1)}$. Thirdly, as to **17**.3, assume $O_{(n+1)}$ not a $q^{i\text{R}}$-shift. If $O_{(n+1)}$ is not performed in $\mathbf{B}_{(n)}$ then clearly $\mathbf{B}_{(n+1)}$ is $\mathbf{B}_{(n)}$, hence $\tilde{\mathbf{B}}_{(n+1)}$ is $\tilde{\mathbf{B}}_{(n)}$. If $O_{(n+1)}$ is performed in $\mathbf{B}_{(n)}$ let $\mathbf{B}_{(n)}$ be $\mathbf{B}'\mathbf{I}\mathbf{B}''$, $\mathbf{B}_{(n+1)}$ be $\mathbf{B}'\mathbf{J}\mathbf{B}''$ where $O_{(n+1)}$ applies $\mathbf{I} \to \mathbf{J}$. If $\mathbf{I} \to \mathbf{J}$ is $\hat{r}$-free, then both $\tilde{\mathbf{B}}_{(n)}$ and $\tilde{\mathbf{B}}_{(n+1)}$ are $\tilde{\mathbf{B}}'\tilde{\mathbf{B}}''$; if $\mathbf{I} \to \mathbf{J}$ is the rule of $\mathfrak{U}_{2.6}$ $r_{\pi}s_{\theta} \to s_{\theta}xr_{\pi}x$ or the converse then both $\mathbf{B}_{(n)}$ and $\mathbf{B}_{(n+1)}$ are $\tilde{\mathbf{B}}'r_{\pi}\tilde{\mathbf{B}}''$. If $\mathbf{I} \to \mathbf{J}$ is an $\bar{r}$-insertion or $\bar{r}$-deletion then an application of $\mathbf{I} \to \mathbf{J}$ effects a proof of $\mathbf{B}_{(n)}/\mathbf{B}_{(n+1)}$ in $(\mathfrak{Z}_r, \varnothing)$. Lemma 23 verifies **17**.4. As to **17**.5, if $T'_{(0,N)}$ were of form $\mathbf{M}\mathfrak{S}\text{ig}O_{(s)}\mathfrak{S}\text{ig}O_{(s+n)}\mathbf{N}$ where $\mathfrak{S}\text{ig}O_{(s+n)}$ is $\overline{\mathfrak{S}\text{ig}O_{(s)}}$ then H would be $q^{i\text{R}}$-redundant in $O_{(s)}$ and $O_{(s+n)}$ contrary to hypothesis. Clearly $\tilde{\mathbf{B}}_{(N)}$ is 1 since $O_{(N)}]$ is $[del(q^{i\text{R}}_{\alpha}\bar{q}^{i\text{R}}_{\alpha})$.

Thus (VII††) follows by Lemma 21.2. Cases $3q$ and $4q$ are dual⁺ to $1q$ and $2q$ respectively. This shows Theorem VII.

25. *Demonstration of Theorem* VIII. For proofs in $\mathfrak{T}_5$ let $Q_{qs} = Q_q + Q_s$. All proofs in $\mathfrak{T}_5$ having no $\bar{q}$-deletions are ordered in the following way: $H^u(\mathbf{A}/\mathbf{B})$ precedes $H^v(\mathbf{C}/\mathbf{D})$ if either (i) $Q^u_{qs} < Q^v_{qs}$, or (ii) $Q^u_{qs} = Q^v_{qs}$ and $N^u_s < N^v_s$.

As is sufficient to show Theorem VIII, we assume given a $\bar{q}$-free $H(\Sigma/\Xi q\Omega)$ in $\mathfrak{T}_5$ and show that (VIII†) *if* $N_s > 0$, *then there is a $\bar{q}$-free $H'(\Sigma/\Xi q\Omega)$ which precedes* $H(\Sigma/\Xi q\Omega)$. By Reduction A—noting particularly A2—and since Σ and $\Xi q\Omega$ are $\bar{s}$-less we may assume H is s-malcev to show (VIII†) for H in general. We let q^2 be the single q-marker occurring in each step of the proof H. (Lemma 11.)

While the following lemma is not essential it is an expositional convenience.[37]

LEMMA 25. *In demonstrating* (VIII†) *we may assume that the given proof* $H(\Sigma/\Xi q\Omega)$ *contains no occurrences of* $\hat{l}$- *or* $\hat{y}$-*markers right of* q^2 *in any step and no occurrences of* $\hat{r}$- *or* $\hat{x}$-*markers left of* q^2 *in any step.*

Simply erasing all violating marker occurrences in the given proof results in a valid $\bar{q}$-free s-malcev proof in $\mathfrak{T}_5$ (except for repetitious steps) without increasing either the number of $\bar{s}$-deletions or the number of s-shifts and q-shifts and without adding $\bar{q}$-deletions. Let C_n be the n^{th} step of the given proof H and C_n^0 be the result of erasing all occurrences of $\hat{l}$- and $\hat{y}$-markers right of q^2 and $\hat{r}$- and $\hat{x}$-markers left of q^2 in step C_n. Suppose O_n, the n^{th} operation of H, (1) is not a q^2-shift and (2) either (2A) is not both $\hat{l}$- and $\hat{y}$-free and applied right of q^2 or (2B) is not both $\hat{r}$- and $\hat{x}$-free and applied left of q^2. Then C_{n+1}^0 is C_n^0; otherwise C_{n+1}^0 follows from C_n^0 by O_n and O_n does not violate **9**.2 or **9**.3 in the proof C_n^0/C_{n+1}^0 since the proof H itself is s-malcev. But since C_1^0 is C_1, i.e., Σ, and C_N^0 is C_N, i.e., $\Xi q\Omega$. This shows the lemma.

We show (VIII†) by means of the following two auxiliary theorems.

THEOREM VIII′. *If* $N_s > 0$ *and* H *is either* q^2-*redundant or* s^v-*redundant for some* s^v, *then there is a* $\bar{q}$-*free* $H'(\Sigma/\Xi q\Omega)$ *in* $\mathfrak{T}_5$ *such that* $Q'_{qs} < Q_{qs}$.

THEOREM VIII″. *If* $N_s > 0$ *and* H *is neither* q^2-*redundant nor* s^v-*redundant for any* s^v, *then there is a* $\bar{q}$-*free* $H'(\Sigma/\Xi q\Omega)$ *such that* $Q'_{qs} \leqq Q_{qs}$ *and* $N'_s = N_s - 1$.

Theorem VIII′ is immediate by Reduction F—noting particularly F3 but ignoring F4—taking g to be q and i to be 2 if H is q^2-redundant and g to be s and i to be v if H is s^v-redundant.

As to Theorem VIII″, since H is s-malcev there are by Lemmas 14.1 and 14.3 initially six possibilities for $\{O_s$, viz., 1s through 6s of Lemma 14.1. If Case 1s holds then Theorem VIII″ follows by Reduction D since $Cond_4(\mathfrak{T}_5;\, s,\, q)$ noting that D2 implies that $Q'_{qs} \leqq Q_{qs}$.

It is convenient to show Theorem VIII″ for Cases 2s and 5s together by the following unified argument.

26. *Demonstration of Theorem* VIII″, *residual case.* Suppose $O_s = del(s_\alpha^u \bar{s}_\alpha^{iR})$ where either u is iR or s^u has entered the proof via a q^2-shift. Now $O_{(0)}$ is to be $ins(s_\alpha^{iR}\bar{s}_\alpha^{iR})$; we assume O_s is the $N+1^{st}$ operation following $O_{(0)}$ in H and let $O_{(n)}$, $n = 1, 2, \cdots, N$ be the n^{th} operation following

[37] Its inclusion actually lengthens the argument, but it seems a natural step in the development.

$O_{(0)}$. By Lemma 10 either $q^2 < \bar{s}_\alpha^{iR}$ or $\bar{s}_\alpha^{iR} < q^2$ in all steps containing $\bar{s}_\alpha^{iR}$; we call these two situations subcases a and b respectively. We now suppose subcase a. We write $O_{(n)}]$ in the form

$$\mathbf{A}_{(n)} q^2_{\beta_{(n)}} \mathbf{B}_{(n)} \Omega_{(n)} \bar{s}_\alpha^{iR} \mathbf{C}$$

where either the right-most marker of $\mathbf{B}_{(n)}$ is an $\hat{s}$-marker or $\mathbf{B}_{(n)}$ is 1.[38]

LEMMA 26. *No $O_{(n)}$ is an $\bar{s}$-insertion performed in or immediately left of $\Omega_{(n-1)}$.*

Suppose $O_{(n)}$ is the first counter-example to Lemma 26. If $O_{(n)}$ is $ins(s_\varepsilon^{tR}\bar{s}_\varepsilon^{tR})$ then $\bar{s}_\varepsilon^{tR} < \bar{s}_\alpha^{iR}$ in $O_{(n)}]$, hence $\bar{s}_\varepsilon^{tR}$ must leave the proof prior to $\bar{s}_\varepsilon^{iR}$ since H is s-malcev thus contradicting the definition of $\bar{s}_\alpha^{iR}$. If $O_{(n)}$ is $ins(\bar{s}_\varepsilon^{tL}s_\varepsilon^{tL})$ then $q^2 < \bar{s}_\varepsilon^{tL} < s_\varepsilon^{tL} < \bar{s}_\alpha^{iR}$ in $O_{(n)}]$. Neither s_α^{iR} nor s-markers entering via q^2-shifts occur between $\bar{s}_\varepsilon^{tL} < \bar{s}_\alpha^{iR}$ in $O_{(n)}]$,—but this situation also obtains in succeeding steps containing $\bar{s}_\varepsilon^{tL}$ and $\bar{s}_\alpha^{iR}$ by the fact that H is s-malcev (or, alternatively, by Lemma 10). This is again a contradiction the form assumed for the first $\bar{s}$-deletion, i.e., O_s.

It is immediate by Lemma 26 that for non-void $\mathbf{B}_{(n)}$ the right-most $\hat{s}$-marker occurring in $\mathbf{B}_{(n)}$ is unbarred; we let $s^{u(n)}_{\gamma(n)}$ be this marker occurrence.

If $O_{(n)}$ is a non-trivial operation and $\Omega_{(n-1)}$ and $\Omega_{(n)}$ are distinct words then $O_{(n)}$ is called *extreme*. Obviously there are two kinds of extreme operations $s^{u(n-1)}$-shifts and q^2-shifts removing markers of $\Omega_{(n-1)}$ from the proof or entering markers of $\Omega_{(n)}$ into the proof. We use $T'_{(m,n)}$ for the sub-sequence of extreme operations of $T_{(m,n)}$, $T''_{(m,n)}$ for the sub-sequence of q^2-shifts of $T_{(m,n)}$.

LEMMA 27. *$\mathfrak{sig}_{rx}$ and $\{\Omega_{(n)}\}_{n=0}^{N}$ form a right signature complex for $T'_{(0,N)}$ in $T_{(0,N)}$.*

Clearly $\Omega_{(0)}$ is 1 since no $\hat{r}$- or $\hat{x}$-markers occur between s_α^{iR} and $\bar{s}_\alpha^{iR}$ in $ins(s_\alpha^{iR}\bar{s}_\alpha^{iR})]$. This verifies **17**.1. Suppose $O_{(n)}$ is an extreme operation. By Lemma 25, $O_{(n)}$ is $\hat{l}$-free and $\hat{y}$-free. Then **17**.2 is checked by the following table, interchanging $\Omega_{(n)}$ with $\Omega_{(n+1)}$ and $\mathbf{B}_{(n)}$ with $\mathbf{B}_{(n+1)}$ for $O_{(n)}$ the converse of the cases listed. (The superscripts for markers are dropped in the table, $\beta = \beta_{(n-1)}$, $\delta = \beta_{(n)}$, $\gamma_{(n-1)} = \gamma_{(n)} = \gamma$.

Rule applied by $O_{(n)}$	$\mathbf{B}_{(n)}$	$\mathbf{B}_{(n+1)}$	$\Omega_{(n+1)}$
$\Delta q_\beta \Pi \to l_\theta \Delta' q_\delta \Pi' r_\vartheta$	Π	Π'	$r_\vartheta \Omega_{(n)}$
$r_\eta s_\gamma \to s_\gamma x r_\eta x$	$\mathbf{B} r_\eta s_\gamma$	$\mathbf{B} s_\gamma$	$x r_\eta x \Omega_{(n)}$
$x s_\gamma \to s_\gamma x x$	$\mathbf{B} x s_\gamma$	$\mathbf{B} s_\gamma$	$x x \Omega_{(n)}$

[38] The **C** need not carry a subscript because of the s-malcev property of the proof H.

Now suppose $O_{(n)}$ is not extreme. If $O_{(n)}$ is not performed in $\Omega_{(n)}$, then $\Omega_{(n)}$ is $\Omega_{(n+1)}$ and **17**.3 is clear. If $O_{(n)}$ is performed in $\Omega_{(n)}$ then it must be an $\bar{x}$- or $\bar{r}$-insertion or deletion. For $O_{(n)}$ is $\hat{l}$-free and $\hat{y}$-free by Lemma 25, is not an $\bar{s}$-insertion by Lemma 26, is not an $\bar{s}$-deletion by the definition of O_s, and is $\hat{q}$-free since q^2 is the sole $\hat{q}$-marker occurring in the proof. Thus, as an $\bar{r}$- or $\bar{x}$-insertion or deletion, $O_{(n)}$ itself effects a proof of $\Omega_{(n)}/\Omega_{(n+1)}$ in $(\mathfrak{Z}_{rx}, \varnothing)$ again verifying **17**.3.

27. *A stronger version of Lemma* 27. The idea behind the next four lemmas is to show the complex of Lemma 27 faithful. Let, $\tilde{\mathbf{W}}$, $\mathbf{W}$ any word on $\mathfrak{Z}_5$, be the word obtained from $\mathbf{W}$ by erasing all symbols except $\hat{r}$-symbols everywhere.

LEMMA 28. *Suppose* $O_{(e)}$ *and* $O_{(f)}$ *are extreme* q^2*-shifts and* $T'_{(e,f-1)}$ *is empty. Then* $\mathfrak{Sig}_r$ *and* $\{\tilde{\mathbf{B}}_{(t)}\}_{t=e}^{f}$ *form a faithful right signature complex for* $T''_{(e,f-1)}$ *in* $T_{(e,f-1)}$). *Further,* $\tilde{\mathbf{B}}_{(f)}$ *is* 1.

No $\hat{r}$-marker lies between q^2 and $\Omega_{(t)}$ if either $O_{(t)}$ or $O_{(t-1)}$ is an extreme q^2-shift. Thus both $\tilde{\mathbf{B}}_{(e)}$ and $\tilde{\mathbf{B}}_{(f)}$ are 1. If $O_{(t)}$, $e < t < f$, is an application of the rule $\mathbf{I} \to \mathbf{J}$ of $\mathfrak{U}_{2.6}$ then since $T'_{(e,f-1)}$ is empty, $\mathbf{B}_{(t-1)}$ is $\mathbf{B}'\mathbf{I}\mathbf{B}''$ for some $\mathbf{B}'$ and *non-void* $\mathbf{B}''$ so that $\mathbf{B}_{(t)}$ is $\mathbf{B}'\mathbf{J}\mathbf{B}''$. The remainder of the argument is exactly that of Theorem VII†† with the following changes.

For	substitute
$q^{i\mathrm{R}}$	q^2
$T'_{(0,N)}$	$T''_{(e,f-1)}$
n	t

LEMMA 29. *Suppose* $O_{(e)}$ *and* $O_{(f)}$ *are extreme* q^2*-shifts such that* $T'_{(e,f-1)}$ *is empty. Then* $O_{(e)}$ *and* $O_{(f)}$ *are not converses of each other.*

Using Lemmas 20.2 and 28, $T_{(e,f-1)}$ is empty. Thus Lemma 29 is immediate from the fact that H is not q^2-redundant.

LEMMA 30. $\mathfrak{Sig}_{rx}T'_{(0,N)}$ *is* M_{rx}*-reduced.*

Consider any two extreme operations $O_{(e)}$ and $O_{(f)}$, $e < f$, such that $T'_{(e,f-1)}$ is empty. Thus $\mathfrak{Sig}_{rx}T'_{(0,N)}$ is of the form $\mathbf{E}\mathfrak{Sig}_{rx}O_{(e)}\mathfrak{Sig}_{rx}O_{(f)}\mathbf{F}$. Suppose $\mathfrak{Sig}_{rx}O_{(e)}$ is $\overline{\mathfrak{Sig}_{rx}O_{(f)}}$. Then by the definition of $\mathfrak{Sig}_{rx}$ either (1) both operations are s-shifts, or (2) both operations are q^2-shifts. Suppose 1. Then $O_{(e)}$ is an $s^{u(e)}$-shift and $O_{(f)}$ an $s^{u(f)}$-shift. But $s^{u(e)}_{\gamma(e)}$ occurs in $O_{(c)}$ and in fact is $s^{u(c)}_{\gamma(c)}$, $c = e + 1, \cdots, f$, since no $O_{(c)}$, $c < f$, is extreme. Thus 1 implies that H is $s^{u(e)}$-redundant, contrary to hypothesis. Clearly 2 contradicts Lemma 29.

LEMMA 31. *$\mathfrak{Sig}_{rx}$ and $\{\Omega_{(n)}\}_{n=0}^{N}$ form a faithful right signature complex for $T'_{(0,N)}$ in $T_{(0,N)}$. Further, Ω_N is* 1.

Clearly Ω_N is 1 since no $\hat{r}$- or $\hat{x}$-markers occur between s_α^u and $\bar{s}_\alpha^{iR}$ in $[del(s_\alpha^u \bar{s}_\alpha^{iR})$. The first sentence is clear by Lemmas 23, 27, and 30.

LEMMA 32. *$T'_{(0,N)}$ is empty; hence s_α^u is s_α^{iR}.*

Immediate, by Lemmas 21.2 and 31.

By Lemma 32 and Reduction B, Theorem VII″ is immediate, but it must be recalled that we have been assuming subcase a of the first paragraph of Section **26**.

28. *Argument if s^{iR} occurs left of q^2.* Subcase b (of Section **26**, first paragraph) is shown by a degenerate form of the argument for subcase a. We proceed as follows, writing $\mathrm{O}_{(n)}]$ in the form

$$\mathbf{B}_{(n)}\Xi_{(n)}\bar{s}_\alpha^{iR}\mathbf{C}$$

where either the right-most marker of $\mathbf{B}_{(n)}$ is an $\hat{s}$-marker or $\mathbf{B}_{(n)}$ is 1. By *the correspondent* of a lemma or argument used under subcase a, we now mean the result of substituting l, y, $\mathfrak{Z}'_{ly}$, $\mathfrak{U}'_{ly}$, $\mathfrak{M}'_{ly}$, for r, x, $\mathfrak{Z}'_{rx}$, $\mathfrak{U}'_{rx}$, $\mathfrak{M}'_{rx}$, respectively throughout that lemma or argument (but *not* interchanging left and right). The correspondent of Lemma 26 is true and the demonstration of Lemma 26 in Section **26** becomes a valid demonstration here if $q^2 < \bar{s}_\varepsilon^{tL} < s_\varepsilon^{tL} < \bar{s}_\alpha^{iR}$ is replaced by $\bar{s}_\varepsilon^{tL} < s_\varepsilon^{tL} < \bar{s}_\alpha^{iR} < q^2$. By Lemma 10 and the correspondent of Lemma 26 each $\mathbf{B}_{(n)}$ is non-void and has s_α^{iR} as the right-most marker occurrence; hence $\mathrm{O}_s = del(s_\alpha^{iR}\bar{s}_\alpha^{iR})$. Extreme operations are defined as before (with Ξ for Ω) but note that now these operations are identical with the s^{iR}-shifts. The correspondent of Lemma 27 holds. For, verifying **17**.1, we note that $\Xi_{(0)}$ is 1 since no $\hat{l}$- or $\hat{y}$-markers occur between s_α^{iR} and $\bar{s}_\alpha^{iR}$ in $ins(s_\alpha^{iR}\bar{s}_\alpha^{iR})]$. To verify conditions **17**.2 and **17**.3, take the correspondent of the argument about conditions **17**.2 and **17**.3 in the proof of Lemma 27 in Section **26** substituting the following table for the earlier one:

Rule applied by $\mathrm{O}_{(n)}$	$\mathbf{B}_{(n)}$	$\mathbf{B}_{(n+1)}$	$\Xi_{(n+1)}$
$yl_\eta ys_\alpha \rightarrow s_\alpha l_\eta$	$\mathbf{B}yl_\eta ys_\alpha$	$\mathbf{B}s_\alpha$	$l_\eta\Xi_{(n)}$
$yys_\alpha \rightarrow s_\alpha y$	$\mathbf{B}yys_\alpha$	$\mathbf{B}s_\alpha$	$y\Xi_{(n)}$

No reference to q-shifts appears in this table since no extreme operation is a q-shift in this subcase b.

No analogues of Lemmas 28 and 29 are needed. The correspondent of Lemma 30 is valid—since H is not s^{iR}-redundant. The correspondent of

Lemma 31 holds—noting Lemmas 23, the correspondents of Lemmas 27 and 30, and the fact that $\Xi_{(N)}$ is 1 since no $\hat{l}$- or $\hat{y}$-markers occur between s_α^{iR} and $\bar{s}_\alpha^{iR}$ in $[O_s$.

Thus, the correspondent of Lemma 32 holds by Lemmas 21.2 and the correspondent of Lemma 31. By the correspondent of Lemma 32 and Reduction B, Theorem VIII″ again follows for subcase b, so that Theorem VIII″ has now been shown for Cases 2*s* and 5*s* of Lemma 14.1.

The demonstration of Theorem VIII″ for Cases 3*s*, 4*s*, and 6*s* are the exact duals⁺ of those for 1*s*, 2*s*, and 5*s*.

This completes the demonstration of Theorem VIII.

29. *Demonstration of Theorem* IX. Let $\mathbf{W}^*$, $\mathbf{W}$ any word on $\mathfrak{Z}_5$, be the result of erasing $\hat{r}$-, $\hat{l}$-, $\hat{x}$-, and $\hat{y}$-markers at all occurrences. If $\mathbf{A} \to \mathbf{B}$ is any rule of $\mathfrak{U}_5$ then either $\mathbf{A}^* \to \mathbf{B}^*$ is a rule of $\mathfrak{U}_1$ (this being the case, for example, when $\mathbf{A} \to \mathbf{B}$ is a q-shift) or $\mathbf{A}^*$ is $\mathbf{B}^*$ (this being the case, for example, when $\mathbf{A} \to \mathbf{B}$ is an s-shift). Thus if Σ, $\mathbf{C}_2, \cdots, \mathbf{C}_N$, $\Xi q \Omega$ is a valid proof in $\mathfrak{T}_5$, then Σ, $\mathbf{C}_2^*, \cdots, \mathbf{C}_N^*$, q with repetitious step omitted is a valid proof in $\mathfrak{T}_1$, which shows Theorem IX.

30. *The unsolvability of the word problem for a certain choice of* $\mathfrak{T}_2$. Result a follows from the Main Theorem, whose demonstration we have just completed, and Lemma 1. (The full argument is given on page 216 immediately following the statement of the Main Theorem.)

PART IV

In Part IV we discuss certain matters, described in the Introduction which are related to the word problem. Section **31** gives a result of a general nature like those of Part II—especially like the key lemma of Reduction C, Lemma 15 (page 231). But Section **31** is not used in the arguments for Results b and c which immediately follow.

31. *Rightward g-shifts.*

DEFINITION OF RIGHTWARD g-SHIFT RULE. For any system under any marker convention a g-shift rule of the form

$$\mathbf{A} g_\eta \mathbf{B} \to \mathbf{C} g_\theta$$

is called *rightward.*

Reduction G. *For any* $\mathfrak{T}$ *under any marker convention suppose there is a* $K(\mathbf{M} g_\gamma^i \mathbf{E}/\mathbf{F} g_\alpha^i \mathbf{N})$ *in* $\mathfrak{T}$ *such that all* g^i*-shifts are rightward. Then there is a* $TU(\mathbf{M} g_\gamma^i \mathbf{E}/\mathbf{F} g_\alpha^i \mathbf{N})$ *such that*

(G1) *T contains no* g^i*-shifts;*

(G2) *U contains no operations right of* g^i;
(G3) $\mathrm{M}(TU, \mathbf{C} \to \mathbf{D}) = \mathrm{M}(K, \mathbf{C} \to \mathbf{D})$ *for any* $\mathbf{C} \to \mathbf{D}$ *of* $\mathfrak{U}$.

Let O_F be the first operation of K applied right of g^i and preceded by a (rightward) g^i-shift. If O_F does not exist, K is TU. If O_F does exist let K be $V_1\mathrm{O}_E V_3 \mathrm{O}_F V_5$ where O_E is the last g^i-shift preceding O_F. Clearly $V_1\mathrm{O}_F\mathrm{O}_E V_3 V_5(\mathbf{M}g^i_\gamma\mathbf{E}/\mathbf{F}g^i_\alpha\mathbf{N})$ is a valid proof in $\mathfrak{T}$ by Lemma 6. Thus Reduction G follows by induction using this argument.

32. *The demonstration of* Result b. The reasons why the argument of Parts I, II, and III does not yield Result b directly are the following. (1) An assumption was made about the form of the operation rules of $\mathfrak{U}_1$, i.e., each member of each rule is a special word. (2) The Main Theorem does not relate the equality of *any* two words of $\mathfrak{T}_1$ to the equality of certain words of $\mathfrak{T}_2$, but only so relates the equality of a special word and the word q to equality in $\mathfrak{T}_2$. Theorems X and XII nullify the effect of 1, Theorem XI, the effect of 2. Cf. footnote 5, page 208.

We consider the Thue system, $\mathfrak{T}_*$, defined in terms of the arbitrary Thue system $\mathfrak{T}$.

$\mathfrak{T}_*$

$\mathfrak{Z}_*$: The symbols of $\mathfrak{Z}$; q_1
$\mathfrak{U}_*$: $*.1$ $q_1\mathbf{A} \leftrightarrow q_1\mathbf{B}$ where $\mathbf{A} \leftrightarrow \mathbf{B}$ is a rule couple of $\mathfrak{U}$;
$*.2$ $q_1\mathbf{a} \leftrightarrow \mathbf{a}q_1$ where $\mathbf{a}$ is a symbol of $\mathfrak{Z}$.

THEOREM X. *Where* $\mathfrak{T}$ *is any Thue system and* $\mathbf{W}$ *and* $\mathbf{V}$ *are any words on* $\mathfrak{Z}$, $q_1\mathbf{W} \vdash_{\mathfrak{T}_*} q_1\mathbf{V}$ *is a necessary and sufficient condition that* $\mathbf{W} \vdash_{\mathfrak{T}} \mathbf{V}$.

Erasing q_1 at all occurrences in a given proof of $q_1\mathbf{W}/q_1\mathbf{V}$ in $\mathfrak{T}_*$ results in a valid proof of $\mathbf{W}/\mathbf{V}$ in $\mathfrak{T}$ except for repetitious steps. This shows the sufficiency of the theorem. As to the necessity, we first note the following lemma which is obvious by induction on $|\mathbf{P}|$ using the rules $\mathfrak{U}_{*\cdot 2}$.

LEMMA 33. *For any Thue system* $\mathfrak{T}$, $q_1\mathbf{P} \vdash_{\mathfrak{T}_*} \mathbf{P}q_1$ *where* $\mathbf{P}$ *is any word on* $\mathfrak{Z}$.

Clearly $q_1\mathbf{A} \vdash_{\mathfrak{T}_*} q_1\mathbf{B}$ where $\mathbf{A} \to \mathbf{B}$ is a rule of $\mathfrak{U}$, by the rules $\mathfrak{U}_{*\cdot 1}$. Thus by Lemma 3 we have (X†) $q_1\mathbf{PAQ} \vdash_{\mathfrak{T}_*} q_1\mathbf{PBQ}$ *where* $\mathbf{P}$, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{Q}$ *are words on* $\mathfrak{Z}$ *and* $\mathbf{A} \to \mathbf{B}$ *is a rule of* $\mathfrak{U}$. From (X†) the necessity of Theorem X clearly follows by induction on the number of steps of a given proof of $\mathbf{W}/\mathbf{V}$ in $\mathfrak{T}$.

We now consider the Thue system $\mathfrak{T}_0$ defined in terms of the arbitrary Thue system $\mathfrak{T}$.

$\mathfrak{T}_0$

$\mathfrak{Z}_0$: $\mathbf{a}$, $\mathring{\mathbf{a}}$ where $\mathbf{a}$ is a symbol of $\mathfrak{Z}$;

p, v

$\mathfrak{U}_0$: 0.1 The rules of $\mathfrak{U}$;

0.2 $\mathring{\mathbf{a}}\mathbf{b} \leftrightarrow \mathbf{b}\mathring{\mathbf{a}}$

0.3 $p\mathring{\mathbf{a}}\mathbf{a} \leftrightarrow \mathring{\mathbf{a}}p$ } where $\mathbf{a}$, $\mathbf{b}$ are symbols of $\mathfrak{Z}$.

0.4 $\mathring{\mathbf{a}}pv \leftrightarrow pv$

We relate the Thue system $\mathfrak{T}$—arbitrary, but fixed throughout the discussion—to the dependent system $\mathfrak{T}_0$ by means of Theorem XI. We use Greek capitals for words on $\mathfrak{Z}$. The word Θ_0 is to be the word obtained from Θ by everywhere replacing $\mathbf{a}$ by $\mathring{\mathbf{a}}$; Θ_d, by everywhere replacing $\mathbf{a}$ by $\mathring{\mathbf{a}}\mathbf{a}$.

THEOREM XI. *For any Thue system* $\mathfrak{T}$—$p\Lambda_0\Theta v \vdash_{\mathfrak{T}_0} pv$ *is a necessary and sufficient condition that* $\Theta \vdash_{\mathfrak{T}} \Lambda$.

We first note the following lemma.

LEMMA 34. *For any Thue system* $\mathfrak{T}$:

(34.1) $\Theta_0\Theta \vdash_{\mathfrak{T}_0} \Theta_d$.

(34.2) $p\Theta_d \vdash_{\mathfrak{T}_0} \Theta_0 p$.

(34.3) $\Theta_0 pv \vdash_{\mathfrak{T}_0} pv$.

Using induction on word lengths, Lemma 34.1 is immediate from the rules $\mathfrak{U}_{0.2}$, Lemma 34.2 from the rules $\mathfrak{U}_{0.3}$, and Lemma 34.3 from the rules $\mathfrak{U}_{0.4}$.

Lemma 3 is used throughout the following argument. Now suppose $\Theta \vdash_{\mathfrak{T}} \Lambda$ so that $p\Lambda_0\Theta v \vdash_{\mathfrak{T}_0} p\Lambda_0\Lambda v$ in view of the rules $\mathfrak{U}_{0.1}$. Then $p\Lambda_0\Lambda v \vdash_{\mathfrak{T}_0} p\Lambda_d v$ by Lemma 34.1, $p\Lambda_d v \vdash_{\mathfrak{T}_0} \Lambda_0 pv$ by Lemma 34.2, and $\Lambda_0 pv \vdash_{\mathfrak{T}_0} pv$ by Lemma 34.3. Thus the necessity of Theorem XI is clear.

We now show the sufficiency of Theorem XI. Where $\mathbf{A}$ is any p-free v-free word on $\mathfrak{Z}_0$, $\mathbf{A}_e$ is to be the word obtained from $\mathbf{A}$ by everywhere erasing all symbols on $\mathfrak{Z}$; $\mathbf{A}_f$, by everywhere erasing all symbols not on $\mathfrak{Z}$; the word $\mathbf{A}_g$ is to be the word obtained from $\mathbf{A}_e$ by everywhere replacing $\mathring{\mathbf{a}}$ by $\mathbf{a}$. We adopt the marker convention for $\mathfrak{T}_0$ wherein the rules of $\mathfrak{U}_{0.3}$ and $\mathfrak{U}_{0.4}$ are designated p-shifts, the rules of $\mathfrak{U}_{0.4}$ v-shifts. The system $\mathfrak{T}_0$ is both p-stable and v-stable. Clearly any proof in $\mathfrak{T}_0$ contains the same p-markers and the same v-markers in every step. (Cf. Lemma 11 which states the corresponding result for groups.)

LEMMA 35. *For any Thue system* $\mathfrak{T}$, *suppose there is an* $H(\mathbf{C}p\mathbf{D}/\mathbf{E}p\mathbf{F})$ *in* $\mathfrak{T}_0$ *where* $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, *and* $\mathbf{F}$ *are p-free v-free words on* $\mathfrak{Z}_0$. *Then*

(35.1) $\mathbf{C}_e\mathbf{D}_e$ *is* $\mathbf{E}_e\mathbf{F}_e$;

(35.2) $\mathbf{C}_g\mathbf{D}_f \vdash_{\mathfrak{T}} \mathbf{E}_g\mathbf{F}_f$.

Throughout the demonstration of this lemma, we restrict the range of the variables $\mathbf{A}, \mathbf{B}, \cdots$ to the *p-free v-free* words on $\mathfrak{Z}_0$. We let p^1 be the p-marker occurring in the first step of the proof H of the lemma; thus any step of this proof is $\mathbf{G}p^1\mathbf{H}$ for some $\mathbf{G}$ and $\mathbf{H}$ which, of course, depend on the step.

If the proof H consists of a single step the lemma holds since in that case $\mathbf{C}$ is $\mathbf{E}$ and $\mathbf{D}$ is $\mathbf{F}$. We shall show that if the operation sequence H consists of a single operation, O, the lemma holds. Using Lemma 3.3 and the last sentence of the preceding paragraph this suffices to show the lemma in general by induction on the number of steps of the proof H of the lemma. Suppose O applies the rule $\Phi \rightarrow \Psi$ of $\mathfrak{U}_{0.1}$ right of p^1 and let $\mathbf{C}p^1\mathbf{D}$ be $\mathbf{C}p^1\mathbf{P}\Phi\mathbf{Q}$ so that $\mathbf{E}p^1\mathbf{F}$ is $\mathbf{C}p^1\mathbf{P}\Psi\mathbf{Q}$. Using parentheses with the obvious meaning, $\mathbf{C}_e(\mathbf{P}\Phi\mathbf{Q})_e$ is $\mathbf{C}_e\mathbf{P}_e\mathbf{Q}_e$, which is also $\mathbf{C}_e(\mathbf{P}\Psi\mathbf{Q})_e$. Thus Lemma 35.1 for this case. The word $\mathbf{C}_g(\mathbf{P}\Phi\mathbf{Q})_f$ is $\mathbf{C}_g\mathbf{P}_f\Phi\mathbf{Q}_f$. Since $\Phi \rightarrow \Psi$ is a rule of $\mathfrak{U}$ as well as of $\mathfrak{U}_{0.1}$, by Lemmas 2, 3.4, $\mathbf{C}_g\mathbf{P}_f\Phi\mathbf{Q}_f \vdash_{\mathfrak{T}} \mathbf{C}_g\mathbf{P}_f\Psi\mathbf{Q}_f$. But $\mathbf{C}_g\mathbf{P}_f\Psi\mathbf{Q}_f$ is $\mathbf{C}_g(\mathbf{P}\Psi\mathbf{Q})_f$. Thus Lemma 35.2 for this case.

The following table verifies Lemmas 35.1 and 35.2 under the remaining possibilities for O. Each of the last three rows is understood to be listed a second time with the entries of the first and second columns interchanged.

$\mathbf{C}p^1\mathbf{D}$	$\mathbf{E}p^1\mathbf{F}$	*Both* $\mathbf{C}_e\mathbf{D}_e$ *and* $\mathbf{E}_e\mathbf{F}_e$	*Both* $\mathbf{C}_g\mathbf{D}_f$ *and* $\mathbf{E}_g\mathbf{F}_f$
$\mathbf{P}\Phi\mathbf{Q}p^1\mathbf{D}$	$\mathbf{P}\Psi\mathbf{Q}p^1\mathbf{D}$	$\mathbf{P}_e\mathbf{Q}_e\mathbf{D}_e$	$\mathbf{P}_g\mathbf{Q}_g\mathbf{D}_f$
$\mathbf{C}p^1\mathring{\mathbf{a}}\mathbf{a}\mathbf{Q}$	$\mathbf{C}\mathring{\mathbf{a}}p^1\mathbf{Q}$	$\mathbf{C}_e\mathring{\mathbf{a}}\mathbf{Q}_e$	$\mathbf{C}_g\mathbf{a}\mathbf{Q}_f$
$\mathbf{A}\mathring{\mathbf{a}}\mathbf{b}\mathbf{B}p^1\mathbf{D}$	$\mathbf{A}\mathbf{b}\mathring{\mathbf{a}}\mathbf{B}p^1\mathbf{D}$	$\mathbf{A}_e\mathring{\mathbf{a}}\mathbf{B}_e\mathbf{D}_e$	$\mathbf{A}_g\mathbf{a}\mathbf{B}_g\mathbf{D}_f$
$\mathbf{C}p^1\mathbf{A}\mathring{\mathbf{a}}\mathbf{b}\mathbf{B}$	$\mathbf{C}p^1\mathbf{A}\mathbf{b}\mathring{\mathbf{a}}\mathbf{B}$	$\mathbf{C}_e\mathbf{A}_e\mathring{\mathbf{a}}\mathbf{B}_e$	$\mathbf{C}_g\mathbf{A}_f\mathbf{b}\mathbf{B}_f$

LEMMA 36. *For any Thue system* $\mathfrak{T}$, *suppose there is an* $H(p\mathbf{D}v/pv)$ *in* $\mathfrak{T}_0$, $\mathbf{D}$ *any p-free v-free word on* $\mathfrak{Z}_0$. *Then for some p-free v-free word* $\mathbf{E}$ *on* $\mathfrak{Z}_0$, $p\mathbf{D} \vdash_{\mathfrak{T}_0} \mathbf{E}p$.

Let v^2 be the single v-marker appearing in each step of the proof H, and let the proof $K(p\mathbf{D}v^2/\mathbf{E}pv^2\mathbf{M})$ be those steps of the proof H preceding the first application of a rule of $\mathfrak{U}_{0.4}$. Lemma 36 now follows by Lemma 6—identifying v^2 with the $\hat{g}^k$ of Lemma 6.

To show the sufficiency of Theorem XI assume, for arbitrary Thue system $\mathfrak{T}$, that $p\Lambda_0\Theta \vdash_{\mathfrak{T}_0} pv$. By Lemma 36, $p\Lambda_0\Theta \vdash_{\mathfrak{T}_0} \mathbf{E}p$ for some p-free v-free word $\mathbf{E}$ on $\mathfrak{Z}_0$. The word $(\Lambda_0\Theta)_e$ is Λ_0. Hence Λ_0 is $\mathbf{E}_e$ by Lemma 35.1. Thus $\mathbf{E}_g$ is Λ. But the word $(\Lambda_0\Theta)_f$ is Θ, so that $\Theta \vdash_{\mathfrak{T}_0} \Lambda$ by Lemma 35.2.

Let $\mathfrak{T}_{(\mathbf{W})}$ be the Thue system defined as follows in terms of the arbitrary Thue system $\mathfrak{T}$ and fixed word $\mathbf{W}$ on $\mathfrak{Z}$.

$\mathfrak{T}_{(\mathbf{W})}$

$\mathfrak{Z}_{(\mathbf{W})}$: The symbols of $\mathfrak{Z}$; q
$\mathfrak{U}_{(\mathbf{W})}$: **W**.1 The rules of $\mathfrak{U}$;
W.2 $\mathbf{W} \leftrightarrow q$

THEOREM XII. *Where* **U** *and* **W** *are any words on* $\mathfrak{Z}$, $\mathbf{U} \vdash_{\mathfrak{T}_{(\mathbf{W})}} q$ *if and only if* $\mathbf{U} \vdash_{\mathfrak{T}} \mathbf{W}$.

Adding the step q to a proof of $\mathbf{U}/\mathbf{W}$ in $\mathfrak{T}$ yields a proof of $\mathbf{U}/q$ in $\mathfrak{T}_{(\mathbf{W})}$ by the rules $\mathfrak{U}_{\mathbf{W}.2}$. Replacing q by **W** throughout a proof $\mathbf{U}/q$ in $\mathfrak{T}_{(\mathbf{W})}$ produces a proof of $\mathbf{U}/\mathbf{W}$ in $\mathfrak{T}$ except for repetitious steps.

THEOREM XIII. *For any Thue system* $\mathfrak{T}$ *and words* **A** *and* **B** *on* $\mathfrak{Z}$, *the following statements are equivalent*:

$$\mathbf{A} \vdash_{\mathfrak{T}} \mathbf{B}$$
$$p\mathbf{B}_0\mathbf{A}v \vdash_{\mathfrak{T}_0} pv$$
$$q_1 p\mathbf{B}_0\mathbf{A}v \vdash_{\mathfrak{T}_{0*}} q_1 pv$$
$$q_1 p\mathbf{B}_0\mathbf{A}v \vdash_{\mathfrak{T}_{0*(q_1 pv)}} q$$

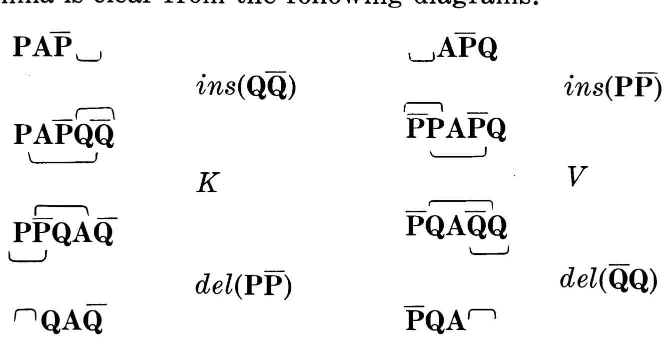
This theorem follows at once from Theorems XI, X, and XII.

We now show Result b. Where $\mathfrak{T}$ is an arbitrary Thue system each rule couple of $\mathfrak{T}_{0*(q_1 pv)}$ is of the form $\Delta q_\alpha \Pi \leftrightarrow \Delta' q_\beta \Pi'$, Δ, Π, Δ', Π' q-free, i.e., having the form of $\mathfrak{T}_1$ when the symbols q_1, q here are identified with the q-symbols of $\mathfrak{Z}_1$, the remaining symbols here with the s-symbols of $\mathfrak{Z}_1$. Moreover, the word $q_1 p\mathbf{B}_0\mathbf{A}v$ of the last statement of Theorem XIII is a special word in the terminology of $\mathfrak{T}_1$ when such an identification is made. Thus the first and last statements of Theorem XIII together with the Main Theorem give Result b.

33. *Application of a technique of Tietze.* The following lemma is a generalization of Lemma 4.5.

LEMMA 37. *For any group presentation* $\mathfrak{T}$, *suppose the words* **P** *and* **Q** *are* $\hat{g}$*-free words on* $\mathfrak{Z}$. *Then* $\mathbf{A}\overline{\mathbf{P}}\mathbf{Q} \vdash_{\mathfrak{T}_g} \overline{\mathbf{P}}\mathbf{Q}\mathbf{A}$ *if and only if* $\mathbf{P}\mathbf{A}\overline{\mathbf{P}} \vdash_{\mathfrak{T}_g} \mathbf{Q}\mathbf{A}\overline{\mathbf{Q}}$.

The lemma is clear from the following diagrams.



THEOREM XIV. *Let $\mathfrak{T}$ and $\mathfrak{T}'$ be any two group presentations such that $\mathfrak{Z}$ is $\mathfrak{Z}'$, and $\mathfrak{U}$ is $\mathfrak{U}'$ with the non-trivial rule couple* $\mathbf{A} \leftrightarrow \mathbf{B}$ *replaced by* $\mathbf{A}' \leftrightarrow \mathbf{B}'$, $\mathbf{A}, \mathbf{B}, \mathbf{A}', \mathbf{B}'$ *$\bar{g}$-free. Suppose* $\mathbf{A}' \vdash_{\mathfrak{T}_g} \mathbf{B}'$ *and* $\mathbf{A} \vdash_{\mathfrak{T}'_g} \mathbf{B}$. *Then for any words* $\mathbf{C}$ *and* $\mathbf{D}$ *on* $\mathfrak{Z}$, $\mathbf{C} \vdash_{\mathfrak{T}_g} \mathbf{D}$ *if and only if* $\mathbf{C} \vdash_{\mathfrak{T}'_g} \mathbf{D}$.

Given a $\bar{g}$-free proof of $\mathbf{C}/\mathbf{D}$ in $\mathfrak{T}$, any consecutive steps of form $\mathbf{PAQ}$, $\mathbf{PBQ}$ can be replaced by a $\bar{g}$-free proof of $\mathbf{PAQ}/\mathbf{PBQ}$ in $\mathfrak{T}'$.

34. *The demonstration of Result* c. This, then, is a revision of the constructions of Parts I, II, and III so as to obtain certain group presentations of comparatively simple form and with unsolvable word problem. As before, $\mathfrak{T}_1$ is to be an arbitrary Thue system having the form stipulated on page 214 of Section **2**, Part I but we specify a new version of $\mathfrak{T}_2$. We omit a listing of the barred symbols in the new $\mathfrak{Z}_2$ and of the trivial rules in the new $\mathfrak{U}_2$. We now specify words by means of integral *exponents* (positive, negative, or zero) in the familiar way. Lower case Greek letters are variables for exponents.

$\mathfrak{T}_2$–Second Version.

$\mathfrak{Z}_2$: All symbols of $\mathfrak{Z}_1$;
$t_1, t_2, k; a, b, c, d, e$

$\mathfrak{U}_2$: Where $\iota = 1, 2, \cdots, P,$ $\alpha = 1, 2,$ and $\beta = 1, 2, \cdots, M,$ the rule couples 2.1 through 2.7 are rules of $\mathfrak{U}_2$:

2.1 $\Sigma_\iota \leftrightarrow d^\iota e^\iota a c^\iota d^\iota \Gamma_\iota b^\iota c^\iota a e^\iota b^\iota$

2.2 $s_\beta \mathbf{z} \leftrightarrow \mathbf{z} s_\beta$, where $\mathbf{z}$ is e, c, or a

2.3 $s_\beta d \leftrightarrow d^{P+1} a d^{P+1} s_\beta$

2.4 $t_\alpha \mathbf{u} \leftrightarrow \mathbf{u} t_\alpha$, where u is e, c, a, or d

2.5 $b s_\beta \leftrightarrow s_\beta b^{P+1} a b^{P+1}$

2.6 $\mathbf{v} k \leftrightarrow k \mathbf{v}$, where $\mathbf{v}$ is e, c, a, or b

2.7 $k \bar{q} \bar{t}_1 t_2 q \leftrightarrow \bar{q} t_1 \bar{t}_2 q k$

We shall first show that the Main Theorem holds under this reinterpretation. The general idea is to let the item in the second column given below play the role of the corresponding item in the first. As are the old (Lemma 23), the new sets of words referred to are independent. That fact is the central idea of this revision.

$\left.\begin{matrix}\mathfrak{Z}_r\\ \mathfrak{Z}_l\end{matrix}\right\}$ The symbols[39] $\hat{c}$, $\hat{e}$

$\mathfrak{M}_r$ The words $c^\iota e^\iota$
$\mathfrak{M}_l$ The words $e^\iota c^\iota$ $\Big\}\ \iota = \pm 1, \pm 2, \cdots, \pm P$

[39] We now write $\hat{c}$ for c, $\bar{c}$ in listing symbols.

$$\left.\begin{array}{l}\mathfrak{Z}_{rx}\\ \mathfrak{Z}'_{ly}\\ \mathfrak{Z}_{ly}\\ \mathfrak{Z}_{rx}\end{array}\right\}\text{The symbols } \hat{c},\ \hat{e},\ \hat{a},\ \hat{b},\ \hat{d}$$

$$\left.\begin{array}{l}\left.\begin{array}{l}\mathfrak{M}_{rx}\\ \mathfrak{M}'_{ly}\end{array}\right\}\text{The words } c^{\pm1},\ e^{\pm1},\ d^{\pm1},\ b^{P+1}ab^{P+1},\ b^{\iota}c^{\iota}ae^{\iota}b^{\iota}\\ \left.\begin{array}{l}\mathfrak{M}_{ly}\\ \mathfrak{M}'_{rx}\end{array}\right\}\text{The words } c^{\pm1},\ e^{\pm1},\ b^{\pm1},\ d^{P+1}ad^{P+1},\ d^{\iota}e^{\iota}ac^{\iota}d\end{array}\right\}\ \iota = 0,\ \pm 1, \cdots, \pm P$$

We shall explain in detail the revisions which should be made in the original argument for the Main Theorem so as to obtain a valid argument for the Main Theorem under this new interpretation of $\mathfrak{T}_2$. There is, of course, no need to check through Section **4** of Part I or Sections **7** through **17** of Part II as the results shown there hold for systems in general. Beginning with Section **5** of Part I let Ξ be a variable for word on $\hat{e}$, $\hat{c}$, $\hat{d}$, $\hat{a}$; Ω, for words on $\hat{e}$, $\hat{c}$, $\hat{b}$, $\hat{a}$. Then Lemma 5 remains valid using the new rules $\mathfrak{U}_{2.2}$ and $\mathfrak{U}_{2.3}$ for Lemma 5.1, the new rules $\mathfrak{U}_{2.2}$ and $\mathfrak{U}_{2.5}$ for Lemma 5.2, the new rules $\mathfrak{U}_{2.4}$ for Lemma 5.3, and the new rules $\mathfrak{U}_{2.6}$ for Lemma 5.4. The form of the argument is otherwise unchanged. Thus Theorem I still holds, — the original argument being completely valid under the new interpretation. Consequently the new version of the Main Theorem is clear in the trivial direction.

As to the program for the demonstration in the non-trivial direction, in the definition of $\mathfrak{T}_3$ and $\mathfrak{T}_4$ replace the phrase in the first column following by that opposite it in the second:

of $\mathfrak{U}_{2.8}$ and $\mathfrak{U}_{2.9}$	of $\mathfrak{U}_{2.6}$
and $\mathfrak{U}_{2.10}$	and $\mathfrak{U}_{2.7}$
of $\mathfrak{U}_{2.4}$ and $\mathfrak{U}_{2.5}$	of $\mathfrak{U}_{2.4}$

With these changes the chain of theorems to be shown remains as before.

The marker convention adopted for the new systems is the same as for the old ones (see bottom of page 221) except that the new rules $\mathfrak{U}_{2.2}$, $\mathfrak{U}_{2.3}$, and $\mathfrak{U}_{2.5}$ are now designated as s-shifts.

The discussion of Part II, Section **18**, is almost entirely revamped. Consider the table just given on the bottom of the preceding and top of this page to be made a part of the new Section **18** taking the item in the second column as the definition of the corresponding item in the first and adding the following additional definitions:

$$\left.\begin{array}{l}\mathfrak{U}_r\\ \mathfrak{U}_l\end{array}\right\}\mathfrak{U}_{2.1}$$

$$\left.\begin{array}{l}\mathfrak{U}_{rx}\\ \mathfrak{U}'_{ly}\\ \mathfrak{U}_{ly}\\ \mathfrak{U}'_{rx}\end{array}\right\}\text{The rules of } \mathfrak{U}_{2.1},\ \mathfrak{U}_{2.2},\ \mathfrak{U}_{2.5}, \text{ and } \mathfrak{U}_{2.3}$$

With the understanding that $\mathfrak{Sig}_{rx}(\mathbf{B}\rightarrow\mathbf{A})$ is $\overline{\mathfrak{Sig}_{rx}(\mathbf{A}\rightarrow\mathbf{B})}$ and $\mathfrak{Sig}_r(\mathbf{B}\rightarrow\mathbf{A})$ is $\overline{\mathfrak{Sig}_r(\mathbf{A}\rightarrow\mathbf{B})}$, the right signature of $\mathfrak{U}_r$ on $\mathfrak{M}_r$, $\mathfrak{Sig}_r$, is given by the requirement that $\mathfrak{Sig}_r(\Sigma\rightarrow d^\iota e^\iota ac^\iota d^\iota \Gamma b^\iota c^\iota ae^\iota b^\iota)$ is $c^\iota e^\iota$, and the right signature of $\mathfrak{U}_{rx}$ on $\mathfrak{M}_{rx}$ is given by the following table.

$\mathbf{A}\rightarrow\mathbf{B}$	$\mathfrak{Sig}_{rx}(\mathbf{A}\rightarrow\mathbf{B})$
$\Sigma\rightarrow d^\iota e^\iota ac^\iota d^\iota \Gamma b^\iota c^\iota ae^\iota b^\iota$	$b^\iota c^\iota ae^\iota b^\iota$
$\mathbf{z}s_\beta\rightarrow s_\beta\mathbf{z}$, where $\mathbf{z}$ is e, c, or a	$\mathbf{z}$
$bs_\beta\rightarrow s_\beta b^{P+1}ab^{P+1}$	$b^{P+1}ab^{P+1}$
$d^{P+1}ad^{P+1}s_\beta\rightarrow s_\beta d$	d

We give $\mathfrak{Sig}_l$ and $\mathfrak{Sig}_{ly}$ the definitions which are dual⁺ to those for $\mathfrak{Sig}_r$ and $\mathfrak{Sig}_{rx}$. We take $\mathfrak{Sig}'_{ly}$ to be $\mathfrak{Sig}_{rx}$, and $\mathfrak{Sig}'_{rx}$ to be $\mathfrak{Sig}_{ly}$. Of course it is intended that b and d are dual, and that a, c, and e are each self dual. We retain the abbreviations $\vdash^{rx}$, etc., under the new interpretation.

Lemma 23 is valid under the new interpretation. This⁺ fact, i.e., the set of words $c^\iota e^\iota$, $\iota=\pm 1, \pm 2, \cdots, \pm P$, and the set $c^{\pm 1}$, $e^{\pm 1}$. $d^{\pm 1}$, $b^{P+1}ab^{P+1}$, $b^\iota c^\iota ae^\iota b^\iota$, $\iota = 0, \pm 1, \cdots, \pm P$, are independent, is obvious. By Lemma 19 a completely combinatorial demonstration is effected.

We now consider the revisions necessary to Part III. As before, Theorem III remains an instance of Reduction E. In the demonstration of Theorem IV simply replace the words "$\mathfrak{U}_{2.8}$ or $\mathfrak{U}_{2.9}$" by "$\mathfrak{U}_{2.6}$". Theorem V remains an instance of Reduction E. In the demonstration of Theorem VI replace "(here **a** is y or an l-symbol)" in the fifth last line of Section **21** by "(here **a** is c, e, a, or d)". As to the demonstration of Theorem VII, up to the definition of $\tilde{\mathbf{W}}$ the text is to stand as given. We now define $\tilde{\mathbf{W}}$ to be $\mathbf{W}$ with all symbols except $\hat{c}$, $\hat{e}$, everywhere erased. Then Lemma 24 remains valid, it being only necessary to replace "$\hat{r}$-" by "$\hat{c}$- or $\hat{e}$-" — or "$\hat{c}$- and $\hat{e}$-" in the context "$\hat{r}$-free"—and "the rule of $\mathfrak{U}_{2.6}$ $r_\pi s_\theta\rightarrow s_\theta x r_\pi x$" by "either the rule $cs_\beta\rightarrow s_\beta c$ or the rule $es_\beta\rightarrow s_\beta e$ of $\mathfrak{U}_{2.2}$" in the original demonstration to obtain the new one needed.

To demonstrate the new Theorem VIII we modify the argument for the original Theorem VIII in the following way. Lemma 25 is dropped completely but all other lemmas and auxiliary theorems are retained in the present development. We now let Ω and Ξ be variables for words on $\hat{c}$-, $\hat{e}$-, $\hat{a}$-, $\hat{b}$-, and $\hat{d}$-symbols. Up to Lemma 26 these are the only changes to be made. (Thus we are continuing to distinguish between subcases a and b of the residual case of Theorem VIII″.) The demonstration of Lemma 26 is valid as it stands. The demonstration of Lemma 27 is to be changed. Statements said to follow by Lemma 25 must be dropped. A reference to r and x, e.g., "no $\hat{r}$- or $\hat{x}$-markers" must be replaced by a corresponding reference to c, e, a, b, and d. The table given is to be replaced by the following table:

Rule applied by $O_{(n)}$	$\mathbf{B}_{(n)}$	$\mathbf{B}_{(n+1)}$	$\Omega_{(n+1)}$
$\Delta q_\beta \Pi \to d^\iota c^\iota a e^\iota d^\iota \Delta' q_\delta \Pi' b^\iota e^\iota a c^\iota b^\iota$	Π	Π'	$b^\iota e^\iota a c^\iota b^\iota \Omega_{(n)}$
$\mathbf{z} s_\gamma \to s_\gamma \mathbf{z}$ where $\mathbf{z}$ is c, e, or a	$\mathbf{B}\mathbf{z}s_\gamma$	$\mathbf{B}s_\gamma$	$\mathbf{z}\Omega_{(n)}$
$b s_\gamma \to s_\gamma b^{p+1} a b^{p+1}$	$\mathbf{B}bs_\gamma$	$\mathbf{B}s_\gamma$	$b^{p+1}ab^{p+1}\Omega_{(n)}$
$d^{p+1} a d^{p+1} s_\gamma \to s_\gamma d$	$\mathbf{B}d^{p+1}ad^{p+1}s_\gamma$	$\mathbf{B}s_\gamma$	$d\Omega_{(n)}$

For Lemma 28 we define $\tilde{\mathbf{W}}$ to be the word obtained from $\mathbf{W}$ by erasing everywhere all symbols except $\hat{c}$- and $\hat{e}$-symbols. In the demonstration the reference to $\mathfrak{U}_{2.6}$ should be replaced by a reference to the rules $s_\beta \mathbf{z} \leftrightarrow \mathbf{z} s_\beta$ where $\mathbf{z}$ is c or e of the new $\mathfrak{U}_{2.2}$. The reference to Theorem VII†† should be interpreted as a reference to the new Theorem VII††.

The demonstrations of Lemmas 29 and 30 are valid as given. The argument for Lemma 31 must be altered only in the first sentence by replacing "$\hat{r}$- or $\hat{x}$-markers" by "$\hat{c}$-, $\hat{e}$-, $\hat{b}$-, $\hat{a}$-, or $\hat{d}$-markers". Lemma 32 and Theorem VII″, subcase a of the residual case, follow as before.

To obtain the revised version of Section **28** it is only necessary to replace "$\hat{l}$- or $\hat{y}$-markers" by "$\hat{c}$-, $\hat{e}$-, $\hat{b}$-, $\hat{a}$-, or $\hat{d}$-markers" and to replace the table given there by the table we have just given on this page with the first line omitted. Of course, the references to Lemma 28, Lemma 29, etc., should be interpreted as references to the new Lemmas 28 and 29. The demonstration of Theorem VIII is thus completed.

In the demonstration of Theorem IX it is only necessary to replace "$\hat{r}$-, $\hat{l}$-, $\hat{x}$-, or $\hat{y}$-" by "$\hat{a}$-, $\hat{b}$-, $\hat{c}$-, $\hat{d}$-, or $\hat{e}$-".

Thus the Main Theorem is valid under the new interpretation of $\mathfrak{T}_2$.

We now make a final but trivial change in this new version of the Main Theorem by interpreting the q of $\mathfrak{U}_{2.7}$ of the second version of $\mathfrak{T}_2$ and of the statement of the Main Theorem itself to be a fixed special word on $\mathfrak{Z}_1$. All our earlier arguments are valid under this notational reinterpretation as may be directly verified.[40]

[40] This corresponds, of course, to dropping Theorem XII in the demonstration of Result b. (See footnote 21 on page 215.)

We now show Result c. Let $\mathfrak{T}$ be an arbitrary Thue system. Then the system $\mathfrak{T}_*$ (as defined in Section **32**) has the form of the system $\mathfrak{T}_1$, identifying the symbol q_1 of $\mathfrak{T}_*$ with the q-symbols of $\mathfrak{T}_1$. Where $\mathfrak{T}_1$ is identified with $\mathfrak{T}_*$, and $\mathbf{P}$ is a fixed word on $\mathfrak{Z}$, let $\mathfrak{T}_{\langle \mathbf{P} \rangle}$ be $\mathfrak{T}_2$ with the q of $\mathfrak{U}_{2.7}$ replaced by $q_1\mathbf{P}$ (and of course $\bar{q}$ by $\bar{\mathbf{P}}\bar{q}_1$). Let $\mathfrak{T}_{\mathbf{P}}$ be $\mathfrak{T}_{\langle \mathbf{P} \rangle}$ with $k\bar{\mathbf{P}}\bar{q}_1\bar{t}_1t_2q_1\mathbf{P} \leftrightarrow \bar{\mathbf{P}}\bar{q}_1\bar{t}_1t_2q_1k$ replaced by $t_1q_1\mathbf{P}k\bar{\mathbf{P}}\bar{q}_1\bar{t}_1 \leftrightarrow t_2q_1\mathbf{P}k\bar{\mathbf{P}}\bar{q}_1\bar{t}_2$. Then, where $\mathbf{W}$ is any word on $\mathfrak{Z}$, the following statements are all equivalent:

$$\mathbf{W} \vdash_{\mathfrak{T}} \mathbf{P}$$

By Theorem X.

$$q_1\mathbf{W} \vdash_{\mathfrak{T}_*} q_1\mathbf{P}$$

By the Main Theorem with $q_1\mathbf{P}$ substituted for q.

$$t_1\mathbf{W}k\bar{\mathbf{W}}\bar{t}_1 \vdash_{\mathfrak{T}_{\langle \mathbf{P} \rangle}} t_2\mathbf{W}k\bar{\mathbf{W}}\bar{t}_2$$

By Lemma 37 and Theorem XIV (Section **33**).

$$t_1\mathbf{W}k\bar{\mathbf{W}}\bar{t}_1 \vdash_{\mathfrak{T}_{\mathbf{P}}} t_2\mathbf{W}k\bar{\mathbf{W}}\bar{t}_2$$

The equivalence of the first and fourth statements is Result c when the subscript 1 is dropped from q_1.

Certain applications of Result c, some in connection with Theorem XI, are discussed in the Introduction.

35.[41] *The equivalence of the word problem and Magnus' extended word problem.* Let $\mathfrak{T}$ be any finite presentation of a group and $\mathfrak{Z}'$ a subset of $\mathfrak{Z}$ such that if $\mathbf{a}$ is a symbol of $\mathfrak{Z}'$ so also is $\bar{\mathbf{a}}$. The *extended word problem for $\mathfrak{T}$ relative to $\mathfrak{Z}'$* is the problem of determining for an arbitrary word $\mathbf{W}$ on $\mathfrak{Z}$ whether or not there is a word $\mathbf{W}'$ on $\mathfrak{Z}'$ such that $\mathbf{W} \vdash_{\mathfrak{T}} \mathbf{W}'$. We shall show directly that for arbitrary $\mathfrak{T}$ and $\mathfrak{Z}'$ there is another finite presentation of a group, $\mathfrak{T}_-$, such that the solution of the word problem for $\mathfrak{T}_-$ implies the solution of the extended word problem for $\mathfrak{T}$ relative to $\mathfrak{Z}'$. The converse reduction is trivial: for to solve the word problem for any finite presentation of a group, $\mathfrak{T}$, is no more general than the solution of the problem of words being equal to 1 in $\mathfrak{T}$,—and this last problem is the extended word problem for $\mathfrak{T}$ relative to the empty set of symbols.

Where the group presentation $\mathfrak{T}$ has as symbols $g_1, \bar{g}_1, \cdots, g_N, \bar{g}_N$ and non-trivial operation rules $\Sigma_1 \leftrightarrow \Sigma_1', \cdots, \Sigma_M \leftrightarrow \Sigma_M'$, let $\mathfrak{T}_-$ be the following group presentation where m is a fixed integer, $0 \leqq m \leqq N$. We omit the barred symbols and trivial rules.

[41] Presented to the American Mathematical Society, Houston, December 1955, (abstract, Bulletin AMS, 62 (1956), 148). Michael Rabin has noted (independently) a very simple argument for the general result given here using free products of groups with amalgamations. Our interest in this problem of course arises from the unsolvability result of [2], Parts I-IV.

$\mathfrak{T}_-$

$\mathfrak{Z}_-$: $g_1, g_2, \cdots, g_N; k$
$\mathfrak{U}_-$: $-.1 \ \Sigma_\nu \leftrightarrow \Sigma'_\nu, \ \nu = 1, 2, \cdots, M;$
$-.2 \ kg_\iota \leftrightarrow g_\iota k, \ \iota = 1, 2, \cdots, m$

We now use Γ as a variable for words on $g_1, \bar{g}_1, \cdots, g_m, \bar{g}_m$ and, as exemplified above, Σ for words on $\mathfrak{Z}$.

THEOREM XV. *$k\Sigma \vdash_{\mathfrak{T}_-} \Sigma k$ if and only if there is a Γ such that $\Sigma \vdash_{\mathfrak{T}} \Gamma$.*

LEMMA 38. *For any group presentation $\mathfrak{T}$:* (38.1) $k\Gamma \vdash_{\mathfrak{T}_-} \Gamma k$; (38.2) $k\bar{g}_\iota \vdash_{\mathfrak{T}_-} \bar{g}_\iota k, \ \iota = 1, 2, \cdots, m$.

Using the rules $\mathfrak{U}_{-.1}$, Lemma 38.1 follows by induction on $|\Gamma|$; using the rules $\mathfrak{U}_{-.2}$, Lemma 38.2 follows by Lemmas 2 and 4.3.

Let $\mathfrak{T}_\div$ be the group presentation obtained from $\mathfrak{T}_-$ by adding the rules $k\bar{g}_\iota \leftrightarrow \bar{g}_\iota k, \ \iota = 1, 2, \cdots, m$. For $\mathfrak{T}_\div$ we take the marker convention stipulating the universal k-qualification with all other qualifications null. In a $\bar{k}$-free proof of $k\Sigma/\Sigma k$ in $\mathfrak{T}_\div$ each step contains a single k-marker (Lemma 11) which we call k^0.

LEMMA 39. *If $k\Sigma \vdash_{\mathfrak{T}_\div k} \Sigma k$, then $k\Sigma \vdash_{\mathfrak{T}_\div k} \Sigma k$ in which every k-shift is rightward.*

If $H(k\Sigma/\Sigma k)$ is a $\bar{k}$-free proof in $\mathfrak{T}_\div$ containing D, D > 0, k-shifts which are not rightward, then $H \| \mathcal{K} | \mathcal{L}$ is a $\bar{k}$-free proof in $\mathfrak{T}_\div$ with D-1 k-shifts which are not rightward. Thus Lemma 39 follows by induction on D. In Diagrams $\mathcal{K}$ and $\mathcal{L}$, $\hat{g}_\alpha$ is g_α where $\check{g}_\alpha$ is $\bar{g}_\alpha$ and vice versa.

$\mathcal{K}$

$k\Sigma$

K_1

$\Sigma'\hat{g}^u_\alpha k^0 \Sigma''$

O

$\Sigma' k^0 \hat{g}^v_\alpha \Sigma''$

K_3

Σk

$\mathcal{L}$

$ins(\check{g}^\iota_\alpha \hat{g}^\iota_\alpha)$

$\Sigma'\hat{g}^u_\alpha k^0 \check{g}^\iota_\alpha \hat{g}^\iota_\alpha \Sigma''$

O′

$\Sigma'\hat{g}^u_\alpha \check{g}^j_\alpha k^0 \hat{g}^\iota_\alpha \Sigma''$

$del(\hat{g}^u_\alpha \check{g}^j_\alpha)$

$\Sigma' \ k^0 g^\iota_\alpha \Sigma''$

We now show Theorem XV. Suppose $\Sigma \vdash_{\mathfrak{T}} \Gamma$ for some Γ. The $\Sigma \vdash_{\mathfrak{T}_-} \Gamma$ by the rules $\mathfrak{U}_{-.1}$, $k\Sigma \vdash_{\mathfrak{T}_-} k\Gamma$ by Lemma 3.4, $k\Gamma \vdash_{\mathfrak{T}_-} \Gamma k$ by Lemma 38.1, hence $k\Sigma \vdash_{\mathfrak{T}_-} \Sigma k$ by Lemma 3. Suppose we are given an $H(k^0\Sigma/\Sigma k^0)$ in $\mathfrak{T}_-$. Noting that $\mathfrak{T}_-$ is a k-translation group presentation, by Reduction E (Section **15** of Part II) there is a $\bar{k}$-free $H'(k^0\Sigma/\Sigma k^0)$ in $\mathfrak{T}_-$ but which we regard as a proof in $\mathfrak{T}_\div$. By Lemma 39, there is a $\bar{k}$-free $H''(k^0\Sigma/\Sigma k^0)$ in

$\mathfrak{T}_{\div}$ in which all k-shifts are rightward. By Reduction G (of Part IV, Section **31**) and Lemma 11, there are proofs $T(k^0\Sigma/\Sigma' k^0\Sigma'')$ and $U(\Sigma' k^0\Sigma''/\Sigma k^0)$ in $\mathfrak{T}_{\div}$ for some Σ' and Σ'', where T contains no k-shifts, U no operations right of k^0. (The $\bar{k}$-freeness is clear from G3.) By Lemma 6, $T(\Sigma/\Sigma'')$ is a valid $\hat{k}$-free proof in $\mathfrak{T}_{\div}$ since no operations of T are applied to k^0—hence also a valid proof in $\mathfrak{T}$. It thus remains only to show (XV*) Σ'' *is a word on* $g_1, \bar{g}_1, \cdots, g_m, \bar{g}_m$. But consider the proof $U(\Sigma' k^0\Sigma''/\Sigma k^0)$. Since no operations of U are applied right of k^0 and the last step of this proof has no markers right of k^0 each marker of Σ'' must leave the proof via a k-shift. This shows (XV*), hence Theorem XV.

36.[42] *A generalization of the word problem.* We now relax the condition that the operation rules of a system must be finite. We show, using the techniques of Part II, that the word problem for a certain presentation of a group, consisting of a finite set of symbols and an infinite—but recursively given — set of operation rules, is unsolvable.

Where S is any set of ordered pairs of positive integers, let $\mathfrak{T}_S$ be the following group presentation, omitting barred symbols and trivial rules.

$$\mathfrak{T}_S$$

$\mathfrak{Z}_S$: z, x_1, x_2, q
$\mathfrak{U}_S$: $z^\mu x_1^\nu q x_1^{-\nu} \leftrightarrow x_2^\nu q x_2^{-\nu}$ for each (μ, ν) of S.
$z \leftrightarrow 1$

THEOREM XVI. $x_1^\nu q x_1^{-\nu} \vdash_{\mathfrak{T}_S} x_2^\nu q x_2^{-\nu}$ *if and only if there is a* μ *such that* (μ, ν) *is a member of* S.

It is immediate that Theorem XVI implies the unsolvability result just mentioned; for we may take S to be a set S_0 with the following properties: (1) There is a recursive procedure to determine for an arbitrary pair of positive integers, (m, n), whether or not (m, n) is a member of S. (2) There is no recursive procedure to determine for arbitrary n whether or not there is an m such that (m, n) is a member of S_0. Such an S_0 is known to exist.[43]

In the demonstration of Theorem XVI the group presentations $\mathfrak{T}'_S$, $\mathfrak{T}''_S$, and $\mathfrak{T}'''_S$ are all to have as unbarred symbols x_1, x_2, q; $\mathfrak{U}'_S$, $\mathfrak{U}''_S$, and $\mathfrak{U}'''_S$ are $\mathfrak{U}$ with $z^\mu x_1^\nu q x_1^{-\nu} \leftrightarrow x_2^\nu q x_2^{-\nu}$ replaced by $x_1^\nu q x_1^{-\nu} \leftrightarrow x_2^\nu q x_2^{-\nu}$, $q x_1^{-\nu} x_2^\nu \leftrightarrow x_1^{-\nu} x_2^\nu q$, and $x_1^\nu q \leftrightarrow x_2^\nu q$ respectively and with $z \leftrightarrow 1$, $\bar{z}$-insertions, and -deletions

[42] Included in an N.S.F. interim report on contract G-1974, May 28, 1956, but in a form more akin to [2], Part V than Result a as presently shown. The idea was evolved from the corresponding result for the infinitely generated, infinitely related abelian case, communicated to us by Hillary Putnam and Dana Scott in September 1955. The infinitely-generated infinitely-related (general) case is shown unsolvable by J. L. Britton in [5].

[43] See, e.g., [9]

omitted. We write $\vdash$, $\vdash'$, $\cdots$, for $\vdash_{\mathfrak{T}_S}$, $\vdash_{\mathfrak{T}'_S}$, $\cdots$, $\mathbf{P}_{\alpha\iota}$ for $x_\alpha^\iota q x_\alpha^{-\iota}$, $\alpha = 1, 2$, and assume the universal q-qualification with all other qualifications null in showing this theorem. Each of the following statements implies the succeeding one:

$\mathbf{P}_{1\iota} \vdash \mathbf{P}_{2\iota}$	
	Erase $\hat{z}$-symbols from the $\mathfrak{T}_S$ proof (Cf. Section **29**).
$\mathbf{P}_{1\iota} \vdash' \mathbf{P}_{2\iota}$	
	By Lemma 37 and Theorem XIV (Section **33**).
$\mathbf{P}_{1\iota} \vdash'' \mathbf{P}_{2\iota}$	
	As $\mathfrak{T}''_S$ is a q-translation group presentation, by Reduction E.
$\mathbf{P}_{1\iota} \vdash''_q \mathbf{P}_{2\iota}$	
	By Lemma 37 and Theorem XIV (Section **33**).
$\mathbf{P}_{1\iota} \vdash'_q \mathbf{P}_{2\iota}$	
	Erase all symbols right of q in each step of the $\mathfrak{T}_S$ proof.
$x_1^\iota q \vdash'''_q x_2^\iota q$	
	See i below.
$x_1^\iota q \vdash'''_{qx} x_2^\iota q$	
	See ii below.
$x_1^\iota q \leftrightarrow x_2^\iota q$ is a rule couple of $\mathfrak{U}'''_S$	
	By the definition of $\mathfrak{T}'''_S$.
$x_1^\iota q x_1^{-\iota} \leftrightarrow x_2^\iota q x_2^{-\iota}$ is a rule couple of $\mathfrak{U}_S$.	

All proofs referred to now are to be $\bar{q}$-free and in $\mathfrak{T}'''_S$.

i. Order all proofs first by the number of q-shifts; for two proofs with the same number of q-shifts, the one with the fewer $\bar{x}$-deletions precedes. (This is the ordering of the first paragraph of Section **26.**) We assume given an $H(x_1^\iota q / x_2^\iota q)$ with $\mathrm{N}_x > 0$ and show (XVI†) *there is an* $H^0(x_1^\iota q / x_2^\iota q)$ *preceding the proof* H *in this ordering.* Let q^0 be the single q-marker of the proof H. (Lemma 11.) By Reduction F the proof H may be assumed not q^0-redundant (Cf. Theorem VIII′, Section **25**) and x-malcev by Reduction A. According as O_x, the first $\bar{x}$-deletion of H, is $del(x_\alpha^{j\mathrm{L}} \bar{x}_\alpha^{i\mathrm{R}})$, $i \neq j$, or $del(x_\alpha^{i\mathrm{R}} \bar{x}_\alpha^{i\mathrm{R}})$, (XVI†) follows by Reduction D or B. If $\mathrm{O}_x = del(\bar{x}_\alpha^{i\mathrm{L}} x_\alpha^u)$, x^u entering the proof via a q^0-shift, then, inductively, the conclusion of a q^0-shift following $ins(\bar{x}_\alpha^{i\mathrm{L}} x_\alpha^{i\mathrm{L}})$ and applying $x_\beta^\theta q \rightarrow x_\gamma^\theta q$ is of form $\mathbf{A} \bar{x}_\alpha^{i\mathrm{L}} \mathbf{B} x_\beta^s x_\gamma^\theta q^0 \mathbf{C}$, $\beta \neq \gamma$, $\mathbf{B}$ being $\bar{x}$-free and not of form $x_\alpha^u \mathbf{B}'$. Since H is not q^0-redundant the left-most x-marker of x_γ^θ does not leave the proof via the next q^0-shift which therefore applies $x_\gamma^\eta q \rightarrow x_\beta^\eta q$, for some η, $\eta < \theta$. Thus this case cannot occur.

ii. If $V_1 O_E O_F V_4$ effects an $\bar{x}$-free proof of $x_1^\iota q / x_2^\iota q$ in $\mathfrak{T}_S'''$ and O_E and O_F are converses, so does $V_1 V_4$ so that we may assume a given $H(x_1^\iota q/x_2^\iota q)$ in $\mathfrak{T}_S'''$ not q^0-redundant. By exactly the same argument as the last case of i, H consists of a single operation.

A kind of reciprocal relationship holds between the unsolvability result just shown and B. H. Neumann's exhibition [17] of a group given by a finite number of symbols and infinite number of operation rules but having no finite presentation. Both his proof and the present argument depend upon the non-trivial rules of the presentation discussed being independent, i.e., for each non-trivial $\mathbf{A} \leftrightarrow \mathbf{B}$, not $\mathbf{A} \vdash \mathbf{B}$ in the system obtained by discarding $\mathbf{A} \leftrightarrow \mathbf{B}$.[44] Theorem XVI shows Neumann's Theorem. Most simply, use $\mathfrak{T}_S'$ with ν having the range of the positive integers.[45] But, conversely, as pointed out to us by John Milnor, and Dana Scott in May 1956, Neumann's example can be adapted to fit the present argument for the unsolvability result using Craig's device [6] in the same way it has been used here. Add z, $\bar{z}$, to Neumann's presentation and take $z^\mu c_\nu = 1$, for each (μ, ν) of S_0 as the non-trivial rules, c_ν being as defined by Neumann.[46]

Lastly we note how simply the results of this section may be obtained using the theory of free products of groups with amalgamated subgroups, the following argument being due entirely to Graham Higman. We show directly for $\mathfrak{T}_S'$, where ν has the range S', a set of non-negative. integers including zero, that (*) *not* $\mathbf{P}_{1\iota} \vdash' \mathbf{P}_{2\iota}$ *unless* ι *is in* S'. Let $F(x_1, q)$ be the free group with the free generators x_1, q. Then the elements $\mathbf{P}_{1\nu} = x_1^\nu q x_1^{-\nu}$, ν in S', are free generators of the subgroup they generate. (See [17], page 514, 4.4.) Hence this subgroup does not contain $\mathbf{P}_{1\iota}$ if ι is not in S'. Form the free product of $F(x_1, q)$ with $F(x_2, q')$, identifying the subgroup generated by the $\mathbf{P}_{1\nu}$ with that generated by $\mathbf{P}_{2\nu}' = x_2^\nu q' x_2^{-\nu}$, ν in S', according to the isomorphism $\mathbf{P}_{1\nu}$ *corresponds to* $\mathbf{P}_{2\nu}$. Thus $\mathbf{P}_{1\iota}$, ι not in S', is not in the identified subgroup, hence $\mathbf{P}_{1\iota} \neq \mathbf{P}_{2\iota}$. Since $q = \mathbf{P}_{10} = \mathbf{P}_{20}' = q'$ we may eliminate q'. This shows (*).

44 It has been shown by B. H. Neumann in [17] that if a group has a finite presentation then for any finitely-generated infinitely-related presentation of that group all but a finite number of the defining relations are dependent.

45 But Graham Higman has shown that the group *just described* can be embedded in a finitely presented group. Cardinality considerations alone show that there exist finitely-generated infinitely-related groups that cannot be so embedded, but an example has yet to be exhibited.

46 As still another alternative one can use the two-generator embedding result of Higman, Neumann, and Neumann [8] together with the infinite-generator results mentioned in footnote 42. The resulting defining relations would of course be rather complicated.

37. *Connections with a theorem of Higman, Neumann, and Neumann.* There is a well-known result [8] (or see [18]) by these mathematicians that for any group G with isomorphic subgroups A and B there is an extension of G in which A and B are transforms of each other. In September 1955, Ralph Fox pointed out to us that Theorem XV of Section **33** was easily obtained by the methods of the theorem of [8] just described. In November 1957, Higman explained to us how Theorem III (stated on page 219) could also be obtained by the argument of [8].

To what extent the other combinatorial arguments about the word problem used in this article can be replaced by more familiar group theoretic arguments is an interesting question.[47]

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[47] (*Added in proof.*) A brief account of the present article appears under the same title in Proc. Natl. Acad. of Sc., vol. 44 (1958), pp. 1061–1065. In line 15 from bottom of page 1062, for Γ read Γ_ι.

K. A. Hirsch's English translation of [19] now appears as vol. 9, pp. 1–122, of the Amer. Math. Soc. Translations, Series 2. The present author regrets that at this writing he has not had the opportunity for a detailed study of that translation and cannot, therefore, compare Novikov's argument with this paper. (But see J. L. Britton's remarks noted in footnote 3 of this article.) Britton's review of [19] now appears in The Jour. Sym. Logic, vol. 23 (1958), pp. 50–53. In this review Britton says of [19]: ". . . . the reviewer now firmly believes that this proof is correct and contains no serious gaps." Cf. Markov [14].

Britton's proof of the unsolvability of the word problem (see our footnote 3) now appears under the title *The word problem for groups* in Proc. London Math. Soc., Series 3, vol. 8 (1958), pp. 493–506. The argument uses his earlier paper, *Solution of the word problem for certain types of groups* II, Proc. of the Glasgow Math. Association, Vol. III (1957), pp. 68–90. (His interesting exploitation of the theorem of Higman, Neumann, and Neumann referred to in Section **37** of this paper was effected independently of the remarks of Fox and Higman mentioned in that section. The author should like also to note here that at the time of his original announcement (see footnote 3) Britton was unaware of the existence of [2].)

Michael Rabin's [21] now appears under the title *Recursive unsolvability of group theoretic problems* in the Ann. of Math., vol. 67 (1958), pp. 172–194. Reference should also be made here to S. I. Adán, *The algorithmic unsolvability of the problem of checking certain properties of groups* (in Russian), Dok. Akad. Nauk SSSR, vol. 103 (1955), pp. 533-535; and, also by Adán, *Finitely generated groups and algorithms* (in Russian), Uspéhi mat. nauk, vol. 12 no. 3 (1957), pp. 248-249. Reviews of this work of Adán by Andrzej Ehrenfeucht appear in The Jour. Sym. Logic, vol. 23 (1958). (Rabin's work [21] was carried out without his knowing of Adán's 1955 paper.)

The abstract by G. S. Céjtin, Dok. Akad. Nauk SSSR, vol. 103 (1955), pp. 370-371, gives, independently, the result of Dana Scott in [22]. See the review by Mostowski, the Jour. of Sym. Logic, vol. 22, (1957), page 219.

A review of [2] by Michael Rabin now appears in The Jour. Sym. Logic, vol. 22 (1957), pp. 372-374. As the author wishes [2] itself to be a readable and self-contained proof of the unsolvability of the word problem, it is suggested on the basis of certain comments by Rabin in his review that footnote 38, page 228 of [2] Part VI be amplified by the addition of the following remark to this footnote as presently given: "These processes introduce no new $\bar{q}$-deletions. For let Diagram $\mathcal{A}$ with q everywhere replaced by k be *rep* the proof H of Theorem IX such that $N_q=0$. Then, since $\mathfrak{T}_9$ contains no rules of form $\mathbf{U}k_\eta\mathbf{V}q_\theta\mathbf{W}\longleftrightarrow\mathbf{P}q_\pi\mathbf{Q}k_\omega\mathbf{R}$, q^0 does not occur in the subproof $k_\gamma^j\mathbf{E}/_9\mathbf{F}k_\alpha^j$. Note now that it is only this subproof which is modified."

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