Algebra and Logic, Vol. 35, No. 3, 1996

NORMAL AUTOMORPHISMS OF A FREE PRO-p-GROUP IN THE VARIETY $\mathcal{N}_2\mathcal{A}$

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UDC 512.5

An automorphism of a (profinite) group is called normal if each (closed) normal subgroup is left invariant by it. An automorphism of an abstract group is p-normal if each normal subgroup of p-power, where p is prime, is left invariant. Obviously, the inner automorphism of a group will be normal and p-normal. For some groups, the converse was stated to be likewise true. N. Romanovskii and V. Boluts, for instance, established that for free solvable pro-p-groups of derived length 2, there exist normal automorphisms that are not inner. Let $\mathcal{N}_2$ be the variety of nilpotent groups of class 2 and $\mathcal{A}$ the variety of Abelian groups. We prove the following results: (1) If p is a prime number distinct from 2, then the normal automorphism of a free pro-p-group of rank ≥ 2 in $\mathcal{N}_2\mathcal{A}$ is inner (Theorem 1); (2) If p is a prime number distinct from 2, then the p-normal automorphism of an abstract free $\mathcal{N}_2\mathcal{A}$-group of rank ≥ 2 is inner (Theorem 2).

An automorphism of a (profinite) group is called *normal* if each (closed) normal subgroup is left invariant by it. An automorphism of an abstract group is said to be *f-normal* (*p-normal*) if each normal subgroup of finite index (a normal subgroup of p-power, p is prime) is left invariant. It is obvious that the inner automorphism of a group will be normal. For some groups, the converse is also true. In this direction, it is worth noting the following results:

— normal automorphisms of absolute Galois groups of finite extensions of the field of p-adic numbers are inner (see [1]);

— each normal automorphism of a pro-K-group with n generators and m defining relations, where $n - m \geq 2$ and K is a class of finite groups closed under subgroups, homomorphic images, and extensions, is inner (see [1]);

— a p-normal automorphism of an abstract free group of rank ≥ 2 is inner (see [2]);

— an f-normal automorphism of an abstract free solvable group of derived length ≥ 2 is inner (see [3]);

— an f-normal automorphism of an abstract free group of the variety $\mathcal{A}\mathcal{N}_k$ is inner, with $\mathcal{A}$ the variety of Abelian groups and $\mathcal{N}_k$ the variety of nilpotent groups of class k;

In [4], normal automorphisms of a free solvable pro-p-group of derived length 2 were described. The description implied, in particular, that for the group there exist normal automorphisms that are not inner. In the present article we prove the following basic theorem.

THEOREM 1. If p is a prime number distinct from 2, then the normal automorphism of a free pro-p-group of rank ≥ 2 in the variety $\mathcal{N}_2\mathcal{A}$ is inner.

From this, we infer the following:

THEOREM 2. If p is a prime number distinct from 2, then the p-normal automorphism of an abstract free $\mathcal{N}_2\mathcal{A}$-group of rank ≥ 2 is inner.

*Supported by RFFR grant No. 93-01-01508.

Translated from *Algebra i Logika*, Vol. 35, No. 3, pp. 249-267, May-June, 1996. Original article submitted January 14, 1995.

In Section 1, we also prove the general result which says that if V is the variety of profinite groups, then a certain analog of the Shmelkin embedding will be valid for groups of type $F/V(R)$.

1. PRELIMINARY REMARKS CONCERNING VARIETIES OF PROFINITE GROUPS

1.1. In what follows, when speaking about profinite groups, we use the terms a "subgroup," a "homomorphism," etc., to refer to respective notions in the category of profinite groups, that is, a closed subgroup, a continuous homomorphism and so on. The necessary definitions related to profinite groups can be found in [5, 6, 7].

We recall that a *free profinite group* $F(X)$ with basis X is the completion of an abstract free group with basis X with respect to the profinite topology defined by subgroups of finite index containing almost all (that is, all but finitely many) elements from X. The basic property of that group is that any continuous map from the set $X \cup \{1\}$ to an arbitrary profinite group G such that $1 \to 1$ extends to the homomorphism $F(X) \to G$.

For a given profinite group A, consider the class of A-groups, that is, profinite groups on which A acts continuously. In this class, there are also free objects. A free profinite A-group with basis $\{y_i | \ i \in I\}$ can be constructed as follows. Represent A as the projective limit of finite groups A_λ $(\lambda \in \Lambda)$. For each λ, consider a free profinite group F_λ with basis $\{y_i^a | \ i \in I, \ a \in A_\lambda\}$. The canonical action of A_λ on the group F_λ can be treated as the continuous action of A. Consider the projective limit of groups F_λ $(\lambda \in \Lambda)$ on which A also acts. This limit is easily seen to be a free profinite A-group with basis $\{y_i | \ i \in I\}$.

1.2. A *variety* of profinite groups is the class of profinite groups closed under subgroups, homomorphic images, and direct (in the category of profinite groups) products. Varieties of profinite groups are in one-to-one correspondence with the classes K of finite groups closed under subgroups, homomorphic images, and direct (in the category of abstract groups) products. A corresponding variety of profinite groups consists of pro-K-groups only. As in the case of abstract groups, the variety can be defined via identities, in which case by an *identity* we mean an element of the free profinite group F_∞ with a countable basis. The identity $v \in F_\infty$ is satisfied on the profinite group G if, under any homomorphism $F_\infty \to G$, the image of an element v (the value of v) is equal to 1. Unlike the abstract case, we note, every variety of profinite groups can be defined by a single identity.

Let G be a profinite group and v some defining identity for V. The subgroup in G generated by all values of v is called *verbal* and is denoted by $v(G)$ or by $V(G)$. If $F(X)$ is a free profinite group with basis X, and V is a variety, then the factor-group $F(X)/V(F(X))$ will be a free group in V.

A *product* variety of V and W is the class of profinite groups that are extensions of groups from V by the groups in W. A *free* group in the variety VW is the factor-group $F(X)/V(W(F(X)))$.

1.3. For abstract groups, the Shmelkin embedding, which allows one to find a representation for the group $F/V(R)$ given F/R, is well known. Below, we give its analog for profinite groups.

Let V be some variety of profinite groups and let A be a profinite group represented as the factor-group $F(X)/R$, where $F(X)$ is a free profinite group with basis $X = \{x_i | \ i \in I\}$. Denote by a_i the canonical image of an element x_i in A. Consider the free profinite A-group F_0 with basis $\{y_i | \ i \in I\}$. The group $B = F_0/V(F_0)$ will be a free A-group with basis $\{b_i | \ i \in I\}$, where b_i is the canonical image of y_i in B, in the variety V. Let C be a subgroup in the semidirect product AB generated by elements $c_i = a_i b_i$ $(i \in I)$.

Proposition 1. If $\tau\colon F(X) \to C$ is a homomorphism determined by the mapping $x_i \to c_i$ $(i \in I)$, then $\ker \tau = V(R)$. In other words, $C \cong F(X)/V(R)$.

Proof. Obviously, $\ker r \geq V(R)$. To prove the inverse inclusion, it suffices to show that if $\psi: F(X) \to G$ is an epimorphism onto the finite group G such that $\ker\psi \geq V(R)$, then the map $c_i \to x_i\psi = g_i$ $(i \in I)$ yields an epimorphism $C \to G$.

Let $H = R\psi$, $A' = G/H$, a_i' be the canonical image of x_i in A', and $\sigma: A \to A'$ be an epimorphism determined by the map $a_i \to a_i'$. Consider the wreath product of finite groups, $H \wr A'$, represented as the semidirect product $A'\bar{H}$, where $\bar{H}$ is a basis subgroup in the wreath product. We can think of $\bar{H}$ as an A'-group and, hence, as an A-group, putting $h^a = h^{a\sigma}$, where $a \in A$ and $h \in \bar{H}$. There exists an embedding of G into $H \wr A'$ such that $g_i = a_i'h_i$, $h_i \in \bar{H}$ (see [8, Thm. 6.2.8]). The group $\bar{H}$ belongs to V. Therefore, the map $b_i \to h_i$ $(i \in I)$ gives a homomorphism of the profinite A-group B into the finite A-group $\bar{H}$; the map $c_i = a_ib_i \to g_i = a_i'h_i$ $(i \in I)$ yields an epimorphism $C \to G$, as desired.

1.4. Let W be the variety of profinite groups closed under extensions. Such a variety consists of pro-K-groups, where K is some class of finite groups closed under subgroups, homomorphic images, and extensions. Suppose that V is a subvariety of W. Consider the free group $F_W(X)$ with basis $X = \{x_i | i \in I\}$ in W and its factor-group $A = F_W(X)/R_W$. As before, a_i denotes the canonical image of x_i in A, and B stands for a free A-group with basis $\{b_i | i \in I\}$ in V. From Proposition 1, we easily infer

Proposition 2. The map $x_i \to a_ib_i$ $(i \in I)$ induces an embedding of the group $F_W(X)/V(R_W)$ into the semidirect product AB.

Indeed, $C \cong F(X)/V(R)$ by Proposition 1. Passing to the quotients on both sides of the equation with respect to the verbal subgroups corresponding to W, we obtain $C \cong F_W(X)/V(R_W)$.

1.5. In what follows, we deal with subvarieties of the variety of all pro-p-groups. This variety is closed under extensions and satisfies the hypothesis of Proposition 2.

Let A be a free Abelian pro-p-group with a finite basis $\{a_1, \dots, a_n\}$. The group algebra Z_pA of a group A over the ring Z_p of p-adic integers is identified with the ring $Z_p[[t_1, \dots, t_n]]$ of formal power series, where $t_1 = a_1 - 1, \dots, t_n = a_n - 1$ (see [7]). The additive group of this ring is a free Abelian pro-p-group with the basis consisting of monomials $M = t_1^{k_1} \dots t_n^{k_n}$, where $k_1, \dots, k_n$ are nonnegative integers. Consider a free A-pro-p-group F_A with basis $\{y_1 \dots y_n\}$. For the given monomial $M = t_1^{k_1} \dots t_s^{k_s}$, where $k_s > 0$, define an element y_i^M from F_A inductively by putting $y_i^M = (y_i^L)^{a_s}(y_i^L)^{-1}$, where $L = t_1^{k_1} \dots t_s^{k_s - 1}$. Obviously, y_i^M lies in the $(k_1 + \dots + k_s)$th term of the lower central series of the pro-p-group AF_A. Therefore, the set $\Omega = \{y_i^M | M \text{ are monomials}\}$ converges to 1. It is easy to see that the set generates F_A as a pro-p-group.

LEMMA 1. F_A is a free pro-p-group with basis S.

Proof. Let Ω_m be the set of all monomials of degree $< p^m$ in each of the variables. Then these monomials (or rather their canonical images) form a basis for the additive group of the group algebra Z_pA_m, where $A_m = A/A^{p^m}$. Consider the free A_m-pro-p-group F_m with basis $\{y_1, \dots, y_n\}$. It is a free pro-p-group with basis $\{y_i^a | i = 1, \dots, n; a \in A_m\}$, and we can also take the set $\{y_i^M | i = 1, \dots, n; M \in \Omega_m\}$ to be the basis of F_m, as a pro-p-group. The statement of the lemma now easily follows from $F_A = \varprojlim F_m$. The lemma is proved.

1.6. We are going to treat a free $\mathcal{N}_2\mathcal{A}$-pro-p-group F with basis $\{x_1, \dots, x_n\}$, where $\mathcal{N}_2$ is the variety of class 2 nilpotent pro-p-groups and $\mathcal{A}$ is the variety of Abelian pro-p-groups. Proposition 2 implies that F is embedded in the semidirect product AH, where A is a free Abelian pro-p-group with basis $\{a_1, \dots, a_n\}$ and H is a free nilpotent A-pro-p-group of class 2 with basis $\{y_1, \dots, y_n\}$. It follows from Lemma 1 that H, as a pro-p-group, is a free nilpotent group of class 2 with basis $\{y_i^M | M$ are monomials in $t_1 = a_1 - 1, \dots, t_n = a_n - 1\}$. The embedding of F is defined by the equalities $x_1 = a_1y_1, \dots, x_n = a_ny_n$.

2. PROOF OF THEOREM 1

We adopt the notation which will be used throughout. If a and b are elements of a group, then $a^b = b^{-1}ab$ and $[a,b] = a^{-1}b^{-1}ab$. The inner automorphism of a group G, which is a conjugation by an element x, is denoted by $\hat{x}$. A (topological) commutator subgroup of a (pro-p-) group G is denoted by G' or $[G,G]$; G'' is, respectively, a second commutator subgroup.

Let F be a free pro-p-group with basis X in the variety $\mathcal{N}_2\mathcal{A}$ and let φ be the normal automorphism of F. We argue that φ is an inner automorphism.

2.1. First note that the task we face reduces to the case where F has finite rank. For this to be the case, we need to represent F as the projective limit of groups F_λ ($\lambda \in \Lambda$), where F_λ is a free pro-p-group with basis X_λ in $\mathcal{N}_2\mathcal{A}$, and X_λ runs over all finite subsets of X. Suppose that on each group F_λ, φ induces an inner automorphism $\hat{f}_\lambda$, $f_\lambda \in F_\lambda$. Therefore, if f is the limit of the set $\{f_\lambda \mid \lambda \in \Lambda\}$, then $\varphi = \hat{f}$.

2.2. Let $X = \{x_1,\dots,x_n\}$ be a finite set, $n \geq 2$. In view of Sec. 1.6, we assume that the group F is embedded in the semidirect product $C = AH$, where A is a free Abelian pro-p-group with basis $\{a_1,\dots,a_n\}$ and H is a free class 2 nilpotent A-pro-p-group with basis $\{y_1,\dots,y_n\}$; $x_1 = a_1y_1, \dots, x_n = a_ny_n$. Then the following equalities hold:

$$F \cap H = F', \quad F \cap H' = F''. \tag{1}$$

Recall that H, as a pro-p-group, is a free class 2 nilpotent group with basis $\{y_i^M \mid i = 1,\dots,n;\ M$ are monomials in $t_1 = a_1 - 1, \dots, t_n = a_n - 1\}$. Order the monomials by putting $M = t_1^{k_1}\dots t_n^{k_n} < L = t_1^{l_1}\dots t_n^{l_n}$ if $k_1+\dots+k_n < l_1+\dots+l_n$, or $k_1+\dots+k_n = l_1+\dots+l_n$ and $k_1 = l_1, \dots, k_{i-1} = l_{i-1}$, $k_i < l_i$. The group H' is a free Abelian pro-p-group with basis $\Sigma = \{[y_i^M, y_j^L] \mid 1 \leq i \leq j \leq n;\ M$ and L are monomials; if $i = j$, then $M < L\}$. Order elements $[y_i^M, y_j^L]$ of that basis lexicographically, comparing the following parameters: the sum of degrees of the monomials M and L, i, j, M, and L. We say that an element $h \in H'$ *depends on* $[y_i^M, y_j^L]$ if the latter has a nonzero coefficient in the expansion of h (into a series) with respect to the basis Σ.

LEMMA 2. The automorphism φ induces an inner automorphism on F/F''.

Proof. The pro-p-group F/F'' is free, solvable of derived length 2, and is embedded in the semidirect product $\bar{C} = A\bar{H}$, where $\bar{H} = H/H'$. Denote by $\bar{c}$ the canonical image of an element $c \in C$ in $\bar{C}$. We can treat $\bar{H}$ as a free Z_pA-module with basis $\{\bar{y}_1,\dots,\bar{y}_n\}$. In [4], we described normal automorphisms of free solvable pro-p-groups of derived length 2. That description implies that there exists an element $u \in Z_pA$ such that $\varepsilon(u) = 1$ (ε is the unit augmentation map $Z_pA \to Z_p$), and that for any element $f \in F'$, the equality $\overline{f\varphi} = \bar{f}^u$ holds. In addition, from [4] it also follows that φ induces an inner automorphism on F/F'' if and only if $u \in A$. Thus, we need only prove that u lies in A.

Let $u \equiv 1 + m_1t_1 + \dots + m_nt_n \bmod \Delta^2$, where $m_1,\dots,m_n \in Z_p$, and $\Delta = \ker\varepsilon$ is the augmentation ideal of Z_pA. We have the following congruence: $u \equiv a = a_1^{m_1}\dots a_n^{m_n} \bmod \Delta^2$. Replacing φ by $\varphi\cdot\hat{f}^{-1}$, where $f = x_1^{m_1}\dots x_n^{m_n}$, reduces our problem to the case where $u \equiv 1 \bmod \Delta^2$. We prove that $u = 1$ in this case.

Assume the contrary, letting $u = 1 + v$, where $0 \neq v \in \Delta^2$. Choose a minimal monomial P on which v depends. The degree of the monomial is ≥ 2. Consider the element $z = [[x_1,x_2],[x_1,x_2]^{x_1}] \in F''$. It is immediately verified that $z = [y_1^{t_2}, y_2^{t_1^2}]^{-1}[y_1^{t_1t_2}, y_2^{t_1}][y_1^{t_2}, y_1^{t_1t_2}][y_2^{t_1}, y_2^{t_1^2}]$. We have

$$z\varphi = [[x_1,x_2]^u, [x_1,x_2]^{x_1u}] = [y_1^{t_2u}, y_2^{t_1^2u}]^{-1}[y_1^{t_1t_2u}, y_2^{t_1u}][y_1^{t_2u}, y_1^{t_1t_2u}][y_2^{t_1u}, y_2^{t_1^2u}]. \tag{2}$$

Further note that the action of the group F by conjugation on H' determines the structure of a Z_pA-module on the latter group. The normal subgroup in F generated by an element z is a submodule generated by z. Therefore, there exists an element $u' \in Z_pA$ such that $z\varphi = z^{u'}$. Let $u' = \alpha + w$, where $\alpha \in Z_p$ and $w \in \Delta$.

In what follows, we shall often use the following formula. Let $a \in A$, $t = a - 1$, and $h_1, h_2 \in H$. Then

$$[h_1, h_2]^t = [h_1^t, h_2][h_1, h_2^t][h_1^t, h_2^t]. \tag{3}$$

The formula implies that $z\varphi = z^\alpha z^w = [y_1^{t_2}, y_2^{t_1^2}]^{-\alpha} z'$, where z' does not depend on $[y_1^{t_2}, y_2^{t_1^2}]$. Comparing this with expression (2) produces $\alpha = 1$. Let S be the minimal monomial on which w depends. From formula (3), then, we infer that a minimal element of the basis Σ of the form $[y_1^M, y_2^L]$, on which z^w depends, is $[y_1^{t_2}, y_2^{t_1^2 S}]$. We have $z^w = z\varphi \cdot z^{-1}$. By (2), the minimal element of Σ of the form $[y_1^M, y_2^L]$, on which $z\varphi \cdot z^{-1}$ depends, is $[y_1^{t_2}, y_2^{t_1^2 P}]$. Consequently, $P = S$. The argument implies also that the monomial P has the same coefficient in the expansions of v and w. Let it be equal to β. In the expansion of $z\varphi \cdot z^{-1}$ with respect to the basis Σ, we isolate that part z_0 which is expressed in terms of basis elements of the form $[y_1^M, y_2^L]$, where $ML = t_1^2 t_2 P$. From (2), we infer that

$$z_0 = ([y_1^{t_2}, y_2^{t_1^2 P}]^{-1}[y_1^{t_2 P}, y_2^{t_1^2}]^{-1}[y_1^{t_1 t_2}, y_2^{t_1 P}][y_1^{t_1 t_2 P}, y_2^{t_1}])^\beta.$$

On the other hand, if we rely on the fact that $z\varphi \cdot z^{-1} = z^w$, then (3) will imply that to the above-mentioned representation of z_0, we must add another factor equal to the product

$$\prod_{P_1, P_2} ([y_1^{t_2 P_1}, y_2^{t_1^2 P_2}]^{-1}[y_1^{t_1 t_2 P_1}, y_2^{t_1 P_2}])^\beta, \tag{4}$$

taken over all monomials P_1 and P_2 such that $P_1 \neq 1$, $P_2 \neq 1$, and $P_1 P_2 = P$. If $P = t_i^{k_i} t_{i+1}^{k_{i+1}} \dots t_n^{k_n}$, where $k_i > 0$, then the element $[y_1^{t_2 t_i}, y_2^{t_1^2 t_i^{k_i - 1} t_{i+1}^{k_{i+1}} \dots t_n^{k_n}}]$ with exponent $-\beta$ occurs in (4) only once. Therefore, the product (4) is not equal to 1, a contradiction. The lemma is proved.

2.3. We need a more detailed information on the group H' and its subgroup F''. Obviously, H', as an Abelian pro-p-group or as a Z_p-module, is the (topological) exterior square of the Z_p-module H/H'. In turn, H/H' is identified with an additive group of the free Z_pA-module, denoted $\bar{H}$, with basis $\{\bar{y}_1 \dots \bar{y}_n\}$. In this section, we use the additive notation. Write $\bar{H} \wedge \bar{H} = \bigoplus_{1 \le i \le j \le n} H_i \wedge H_j$, where $H_i = \bar{y}_i \cdot Z_pA$ ($i = 1, \dots, n$). In addition, the action of the algebra Z_pA on $\bar{H} \wedge \bar{H}$ is defined by (3). Rewrite this formula additively as follows:

$$(u \wedge v)t = ut \wedge v + u \wedge vt + ut \wedge vt, \tag{5}$$

where $u, v \in \bar{H}$, $t = a - 1$, and $a \in A$. Under this action, $\bar{H} \wedge \bar{H}$ turns into a Z_pA-module.

LEMMA 3. (a) The module $H_i \wedge H_j$ with $i < j$ is a free Z_pA-module with basis $\{\bar{y}_i \wedge \bar{y}_j M \mid M$ are monomials in $t_1, \dots, t_n\}$.

(b) The module $H_i \wedge H_i$ is embedded in a free Z_pA-module.

Proof. (a) Define the weight of an element $\bar{y}_i M \wedge \bar{y}_j L$ ($i < j$) in the basis of the Z_p-module $H_i \wedge H_j$ as the sum of degrees of the monomials M and L. The weight of an arbitrary nonzero element in $H_i \wedge H_j$ is specified to be equal to the minimum of weights of the basis elements on which it depends. The weight of the zero element is assumed infinite. It is easy to see that if P is a monomial, then the weight of $(\bar{y}_i M \wedge \bar{y}_j L)P$ is equal to the sum of degrees of the monomials M, L, and P. By (5), the element $\bar{y}_i M \wedge \bar{y}_j L$ of weight k can be rewritten, modulo an element of larger weight, as a linear combination of elements $(\bar{y}_i \wedge \bar{y}_j P)S$ of weight k with coefficients from Z_p. Hence, the set $\{\bar{y}_i \wedge \bar{y}_j M \mid M$ are monomials in $t_1, \dots, t_n\}$ generates $H_i \wedge H_j$ as a Z_pA-module. We argue that this is a free system of generators. Indeed, suppose that there exists a nontrivial series $v = \sum_\alpha (\bar{y}_i \wedge \bar{y}_j M_\alpha) u_\alpha$, where $0 \neq u_\alpha \in Z_pA$. Let $\bar{y}_i \wedge \bar{y}_j M_0$ be minimal among the elements $\bar{y}_i \wedge \bar{y}_j M_\alpha$ and let L be the minimal monomial in the expansion of u_0. It then follows from (5) that v depends on $\bar{y}_i L \wedge \bar{y}_j M_0$; in particular, $v \neq 0$.

(b) In essence, here we must prove that $Z_pA \wedge Z_pA$ is embedded in a free Z_pA-module. Consider the algebra of formal power series, $Z_p[[s_1, \dots, s_n, r_1, \dots, r_n]]$, and the embedding of $Z_pA \wedge Z_pA$, as a Z_p-module, into this algebra, defined by the formula $\sigma\colon t_1^{k_1} \dots t_n^{k_n} \wedge t_1^{l_1} \dots t_n^{l_n} \to s_1^{k_1} \dots s_n^{k_n} r_1^{l_1} \dots r_n^{l_n} - s_1^{l_1} \dots s_n^{l_n} r_1^{k_1} \dots r_n^{k_n}$. If $u, v \in Z_pA$, then $((u \wedge v)t_i)\sigma = (u \wedge v)\sigma \cdot (s_i + r_i + s_i r_i)$, that is, the action of the ring Z_pA on $Z_pA \wedge Z_pA$ corresponds to the action of $Z_p[[s_1 + r_1 + s_1 r_1, \dots, s_n + r_n + s_n r_n]]$ by multiplication on $(Z_pA \wedge Z_pA)\sigma$. Obviously, $Z_p[[s_1, \dots, s_n, r_1, \dots, r_n]] = Z_p[[s_1, \dots, s_n, s_1 + r_1 + s_1 r_1, \dots, s_n + r_n + s_n r_n]]$. This implies that the algebra $Z_p[[s_1, \dots, s_n, r_1, \dots, r_n]]$, in which $(Z_pA \wedge Z_pA)\sigma$ is contained, is a free module over $Z_p[[s_1 + r_1 + s_1 r_1, \dots, s_n + r_n + s_n r_n]]$. The lemma is proved.

Thus we identify H' with the exterior square of the Z_p-module $\bar{H}$. The group F'' is embedded in H', and its image coincides with the exterior square of the module $\Gamma_0 = F'/F'' \leq \bar{H}$. It is known (see [5]) that the element $\bar{y}_1 u_1 + \dots + \bar{y}_n u_n$ of $\bar{H}$ lies in Γ_0 if and only if the relation $t_1 u_1 + \dots + t_n u_n = 0$ holds. In other words, Γ_0 is the kernel of the homomorphism $\tau : \bar{H} \to \Delta$, induced by the map $\bar{y}_1 u_1 + \dots + \bar{y}_n u_n \to t_1 u_1 + \dots + t_n u_n$. Denote the algebra $Z_p[[t_1, t_2, \dots, t_i]]$ $(i = 1, \dots, n)$ by R_i. Then the Z_p-module Δ decomposes into a direct sum $\Delta = t_1 R_1 \oplus t_2 R_2 \oplus \dots \oplus t_n R_n$. The Z_p-basis of the module is composed of elements of the form $t_1 t_1^{k_1}, t_2 t_1^{k_1} t_2^{k_2}, \dots, t_n t_1^{k_1} t_2^{k_2} \dots t_n^{k_n}$, where $k_1, \dots, k_n$ are nonnegative integers. This implies that the Z_p-module Γ decomposes into the direct sum of the module $\Gamma_0 = \ker \Delta$ and the submodule Γ_1, generated by elements of the form $\bar{y}_1 t_1^{k_1}, \bar{y}_2 t_1^{k_1} t_2^{k_2}, \dots, \bar{y}_n t_1^{k_1} t_2^{k_2} \dots t_n^{k_n}$ ($k_1, \dots, k_n$ are nonnegative integers). By construction, Γ_0 is a Z_pA-submodule and $\Gamma_1 \cdot t_1 \subseteq \Gamma_1$.

LEMMA 4. If $f \in \bar{H} \wedge \bar{H}$ and $f t_1 \in \Gamma_0 \wedge \Gamma_0$, then $f \in \Gamma_0 \wedge \Gamma_0$.

Proof. The decomposition $\bar{H} = \Gamma_0 \oplus \Gamma_1$ implies that $\bar{H} \wedge \bar{H} = (\Gamma_0 \wedge \Gamma_0) \oplus (\Gamma_0 \wedge \Gamma_1) \oplus (\Gamma_1 \wedge \Gamma_1)$. From (5), it follows that each summand in the latter decomposition is t_1-invariant. Therefore, if $f = f_1 + f_2 + f_3$, where $f_1 \in \Gamma_0 \wedge \Gamma_0$, $f_2 \in \Gamma_0 \wedge \Gamma_1$, and $f_3 \in \Gamma_1 \wedge \Gamma_1$, then $f t_1 \in \Gamma_0 \wedge \Gamma_0$ implies that $f_2 t_1 = f_3 t_1 = 0$. By the previous lemma, the module $\bar{H} \wedge \bar{H}$ is t_1-torsion-free. Therefore, $f_2 = f_3 = 0$ and $f \in \Gamma_0 \wedge \Gamma_0$. The lemma is proved.

For a given natural number k, consider the element $b_k = 1 + a_1 + \dots + a_1^{p^k - 1} \in Z_pA$. The following equalities are satisfied:

$$b_k = p^k + \binom{p^k}{2} t_1 + \dots + \binom{p^k}{p^k - 1} t_1^{p^k - 2} + t_1^{p^k - 1}, \quad b_k t_1 = a_1^{p^k} - 1. \tag{6}$$

The factor-algebra $Z_p[[t_1]]/(a_1^{p^k} - 1)$ is isomorphic to a group algebra of the cyclic group of order p^k over Z_p. As a Z_p-basis of this algebra we can take the set $\{1, t_1, \dots, t_1^{p^k - 1}\}$. The factor-algebra $Z_p[[t_1]]/(b_k)$ is also a free Z_p-module, whose basis is composed of elements $1, t_1, \dots, t_1^{p^k - 2}$. This implies that the Z_p-module $\bar{H}$ decomposes into a direct sum $\bar{H} b_k \oplus D_k$, where D_k is a free Z_p-submodule in $\bar{H}$ with the basis

$$\{\bar{y}_i t_1^l M \mid 1 \leq i \leq n, \ 0 \leq l \leq p^k - 2, \ M \text{ are monomials in } t_2, \dots, t_n\}. \tag{7}$$

We have $\bar{H} \wedge \bar{H} / \bar{H} \wedge \bar{H} b_k \cong D_k \wedge D_k$.

LEMMA 5. Let $p \geq 2$. If an element f in $\bar{H} \wedge \bar{H}$ is not divisible by t_1, then there exists a natural number k such that $f b_k \notin \bar{H} \wedge \bar{H} b_k$.

Proof. We recall that $\bar{H} = \bigoplus_{i=1,\dots,n} H_i$, and so $\bar{H} \wedge \bar{H} = \bigoplus_{1 \leq i \leq j \leq n} H_i \wedge H_j$ and $\bar{H} \wedge \bar{H} b_k = \bigoplus_{1 \leq i \leq j \leq n} (H_i \wedge H_j b_k + H_i b_k \wedge H_j)$. Therefore, we need only consider the case where $f \in H_i \wedge H_j$ and prove that $f b_k \notin H_i \wedge H_j b_k + H_i b_k \wedge H_j$ for a suitable k.

(a) First assume that $i < j$. By Lemma 3, $H_i \wedge H_j$ is generated as a $Z_p[[t_1]]$-module by elements of the form $\bar{y}_i M \wedge \bar{y}_j L$, where the monomial M does not depend on t_1. There then exists a representation $f = f_0 + f_1 t_1$, where f_0 belongs to the Z_p-module generated by the above-specified elements $\bar{y}_i M \wedge \bar{y}_j L$. Note that $f_1 t_1 b_k \in \bar{H} \wedge \bar{H} b_k$, which follows from $(u \wedge v) t_1 b_k = (u \wedge v)(a_1^{p^k} - 1) = u a_1^{p^k} \wedge v a_1^{p^k} - u \wedge v \in \bar{H} \wedge \bar{H} b_k$. Therefore, we must prove that $f_0 b_k \notin H_i \wedge H_j b_k + H_i b_k \wedge H_j$. Assume $f = f_0$. Moreover, we can also think that f is not divisible by p, since the quotient module $\bar{H} \wedge \bar{H} / \bar{H} \wedge \bar{H} b_k$ is p-torsion-free. Reduce the objects considered modulo p and mark the result by $\tilde{}$. We have $\tilde{H}/\tilde{H} b_k \cong \tilde{D}_k$, where $\tilde{D}_k$ is a free Z/pZ-module with basis (7). From (6), it follows that the element $t_1^{p^k - 1}$ annihilates the module $\tilde{H}/\tilde{H} b_k$. By assumption, $\tilde{f} \neq 0$. Take the minimal element $\bar{y}_i M_0 \wedge \bar{y}_j L_0$ on which $\tilde{f}$ depends. The monomial M_0 does not depend on t_1, and L_0 can be represented as $L_0 = t_1^l S$, where S is independent of t_1. Finally, choose the number k such that $p^k - 3 \geq l$. Since $t_1^{p^k - 1} \equiv b_k \bmod p \cdot Z_p A$, it follows that $\tilde{f} b_k = \tilde{f} t_1^{p^k - 1}$. We have the following formula:

$$(u \wedge v) t_1^s \equiv u t_1^s \wedge v + \binom{s}{1} u t_1^{s-1} \wedge v t_1 + \binom{s}{2} u t_1^{s-2} \wedge v t_1^2 + \ldots + u \wedge v t_1^s$$

modulo summands of the form $u t_1^{s_1} \wedge v t_1^{s_2}$, where $s_1 + s_2 > s$. Note that the binomial coefficients $\binom{p^k - 1}{m}$, where $1 \leq m \leq p^k - 1$, are not divisible by p. Therefore, the element $\tilde{f} t_1^{p^k - 1}$ depends on $\bar{y}_i t_1^{p^k - 2} M_0 \wedge \bar{y}_j t_1^{l+1} S$, and since $l + 1 \leq p^k - 2$, we have $\tilde{f} t_1^{p^k - 1} \notin \tilde{H} \wedge \tilde{H} b_k$.

(b) Let $i = j$. We will show that $Z_p[[t_1]] \wedge Z_p[[t_1]]$ is a free $Z_p[[t_1]]$-module with basis $\{t_1^l \wedge t_1^{l+1} \mid l \geq 0\}$. In the first place, the module in question is generated by this set. By induction (on r), we can assume that elements of the form $t_1^{l_1} \wedge t_1^{l_2}$, where $0 \leq l_2 - l_1 \leq r$, are expressed via elements $t_1^l \wedge t_1^{l+1}$. We then have $(t_1^l \wedge t_1^{l+r}) t_1 = t_1^l \wedge t_1^{l+r+1} + t_1^{l+1} \wedge t_1^{l+r} + t_1^{l+1} \wedge t_1^{l+r+1}$, whence the desired expression for $t_1^l \wedge t_1^{l+r+1}$. Second, the set $\{t_1^l \wedge t_1^{l+1} \mid l \geq 0\}$ is independent over $Z_p[[t_1]]$. In fact, consider the nontrivial sum $\sum_l (t_1^l \wedge t_1^{l+1}) u_l$, where $u_l \in Z_p[[t_1]]$. Choose the minimal index $l = l_0$ for which $u_l \neq 0$. If t_1^m is a minimal monomial on which u_{l_0} depends, from (5) it follows that the sum $\sum_l (t_1^l \wedge t_1^{l+1}) u_l$ depends on $t_1^{l_0} \wedge t_1^{l_0 + m + 1}$; in particular, it cannot be zero.

The above argument implies that $H_i \wedge H_i$, if treated as an $Z_p[[t_1]]$-module, is free, and its basis is the set $\Sigma_1 \cup \Sigma_2$,

$$\Sigma_1 = \{\bar{y}_i M \wedge \bar{y}_i L t_1^l \mid l \geq 0; \ \ M \text{ and } L \ \text{ are monomials in } t_2, \ldots, t_n; \ M < L\}$$

and

$$\Sigma_2 = \{\bar{y}_i M t_1^l \wedge \bar{y}_i M t_1^{l+1} \mid l \geq 0; \ \ M \ \text{ are monomials in } t_2, \ldots, t_n\}.$$

As in item (a), our problem reduces to the case where f is not divisible by p and expands, with respect to the basis $\Sigma_1 \cup \Sigma_2$, into a series with coefficients from Z_p. Again we reduce all the objects modulo p. If the minimal element of the set $\Sigma_1 \cup \Sigma_2$ on which $\tilde{f}$ depends lies in Σ_1, we need only repeat the argument of (a). Let that minimal element be equal to $\bar{y}_i M t_1^l \wedge \bar{y}_i M t_1^{l+1}$. Choose the number k satisfying $p^k \geq 2l + 5$. Under this condition, if $m = \frac{p^k - 1}{2}$, then $l + 1 + m \leq p^k - 2$. We have

$$(\bar{y}_i M t_1^l \wedge \bar{y}_i M t_1^{l+1}) t_1^{2m} = \ldots + \binom{2m}{m-1} \bar{y}_i M t_1^{l+m+1} \wedge \bar{y}_i M t_1^{l+m} +$$

$$\binom{2m}{m} \bar{y}_i M t_1^{l+m} \wedge \bar{y}_i M t_1^{l+m+1} + \ldots =$$

$$\ldots + \left(\binom{2m}{m} - \binom{2m}{m-1} \right) \bar{y}_i M t_1^{l+m} \wedge \bar{y}_i M t_1^{l+m+1} + \ldots .$$

It is easy to see that in the expression given, the coefficient at $\bar{y}_i M t_1^{l+m} \wedge \bar{y}_i M t_1^{l+m+1}$ is equal to $\binom{2m}{m} - \binom{2m}{m-1} = \binom{2m}{m-1} : m$ and is not divisible by p. Therefore, the element $\tilde{f} t_1^{p^k - 1} = \tilde{f} t_1^{2m}$ depends on $\bar{y}_i M t_1^{l+m} \wedge \bar{y}_i M t_1^{l+m+1}$, whence $\tilde{f} t_1^{p^k-1} \notin \tilde{H} \wedge \tilde{H} b_k$. The lemma is proved.

2.4. We turn directly to the proof of Theorem 1. Let φ be a normal automorphism of the group F. Relying on Lemma 2, we can assume that φ induces an identity automorphism on F/F''. Then $x_1\varphi = x_1 f_1, \ldots, x_n\varphi = x_n f_n$, where $f_1, \ldots, f_n \in F''$. For a given natural number k, F_k denotes the normal subgroup in F generated by $x_1^{p^k}$. Since $x_1^{p^k}\varphi = (x_1 f_1)^{p^k} = x_1^{p^k} f_1^{1+a_1+\ldots+a_1^{p^k-1}}$, we have

$$f_1^{1+a_1+\ldots+a_1^{p^k-1}} \in F_k \cap F''. \tag{8}$$

In Sec. 2.3, the group F'' was identified with the additive group of the submodule $\Gamma_0 \wedge \Gamma_0$ of the Z_pA-module $\bar{H} \wedge \bar{H}$. It is not hard to see that $F_k \cap F'' \subseteq \bar{H} \wedge \bar{H} b_k$. From (8), we obtain the following inclusion (written additively): $f_1 b_k \in \bar{H} \wedge \bar{H} b_k$ for any k. By Lemma 5, then, f_1 must be divisible by t_1 in $\bar{H} \wedge \bar{H}$. By Lemma 4, it is divisible by t_1 also in $\Gamma_0 \wedge \Gamma_0$. Coming back to the multiplicative notation, we can assert that there exists an element $g \in F''$ such that $f_1 = g^{a_1} g^{-1} = g^{x_1} g^{-1}$. Consider the inner automorphism $\hat{g}$ and the automorphism $\psi = \varphi\hat{g}$. We have $x_1\psi = x_1$. Let $x_i\psi = x_i g_i$, where $g_i \in F''$, $2 \leq i \leq n$. For any natural number m, we also have $(x_1^m x_i)\psi = (x_1^m x_i) g_i$. The preceding argument implies that g_i, as an element of the Z_pA-module $\bar{H} \wedge \bar{H}$, is divisible by $a_1^m a_i - 1 \equiv m t_1 + t_i \bmod \Delta^2$. By Lemma 3, $\bar{H} \wedge \bar{H}$ is embedded in a free Z_pA-module. It is not hard to see that a nonzero element from Z_pA cannot be divisible by elements $a_1^m a_i - 1$ for all natural m. This means that $g_2, \ldots, g_n$ are trivial elements, ψ is an identity automorphism, and φ is inner. The theorem is proved.

3. PROOF OF THEOREM 2

Let G be an abstract free $\mathcal{N}_2\mathcal{A}$-group with basis $X = \{x_i | \, i \in I\}$. Based on the matrix representation [9], we can assert that G is a residually finite p-group for any prime number p. Therefore, it is embedded in the completion with respect to the pro-p-topology (the latter is defined by all normal subgroups of finite p-index containing almost all elements from X), which will be the free $\mathcal{N}_2\mathcal{A}$-pro-p-group F with basis X. Let φ be a p-normal automorphism of G. We need to prove that φ is inner.

3.1. First we show that our problem reduces to the case where G has finite rank. Let $\{x_1, \ldots, x_n\} \subseteq X$, $n \geq 2$, and $G_1 = \langle x_1, \ldots, x_n \rangle$. Then there exists a subset $\{x_1, \ldots, x_n, \ldots, x_m\}$ in X such that $G_1\varphi \leq G_2 = \langle x_1, \ldots, x_n, \ldots, x_m \rangle$. Represent G_2 as the factor-group F/N, where N is a normal subgroup of F generated by all elements from $X \backslash \{x_1, \ldots, x_n, \ldots, x_m\}$. Let φ_2 be a normal automorphism of the group $F/N = G_2$ induced by φ. We can say that φ and φ_2 coincide on G_1. Let $\varphi_2 = \hat{u}$, where $u \in G_2$. We prove that $\varphi = \hat{u}$. Indeed, if g is an arbitrary element in G and $g \in G_3 = \langle x_1, \ldots, x_n, \ldots, x_m, \ldots, x_k \rangle$, the above argument implies that there exists an element $v \in F$ such that $\varphi|_{G_3} = \hat{v}$. It suffices to prove that $u = v$. If not, a nontrivial element uv^{-1} centralizes the subgroup G_1. Consider the matrix representation of G from [9] or the representation of F given in Sec. 2. We may conclude that the centralizer of G_1 in G is equal to 1, whence $u = v$.

3.2. Assume that $X = \{x_1, \ldots, x_n\}$ is a finite set. The automorphism φ of an abstract group G uniquely extends to the normal automorphism of a pro-p-group F, denoted by the same symbol φ. By Theorem 1, there exists an element $f \in F$ such that $\varphi = \hat{f}$. It remains to prove that $f \in G$.

Consider a nontrivial element $\bar{y} = yG'' \in G'/G''$. The group G'/G'' is embedded in F'/F''. The normal subgroup of G/G'' generated by an element $\bar{y}$ is identified with a free one-generated module over the group ring $Z[G/G']$. The normal subgroup generated by $\bar{y}$ in F/F'' is the corresponding module over Z_pA, where $A = F/F'$. In a module, the action of φ on $\bar{y}$ corresponds to multiplication by the canonical image of f in F/F'. Therefore, there exists $g \in G$ such that $f \equiv g \bmod F'$, which reduces the problem to the case $f \in F'$.

3.3. Let $f \in F'$. We have $x_1\varphi = x_1 \cdot [x_1, f]$ and $[x_1, f] \in G'$. Recall that the group F'/F'' is embedded in an additive group of the free topological Z_pA-module $\bar{H}$ with basis $\{\bar{y}_1, \ldots, \bar{y}_n\}$. Denote by R the ring $Z[a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}]$, and by R_0 its subring $Z[t_1, \ldots, t_n]$. Also let $\bar{R} = Z_pA = Z_p[[t_1, \ldots, t_n]]$. The abstract group G'/G'' is embedded in an additive group of an abstract free R-module E with basis $\{\bar{y}_1, \ldots, \bar{y}_n\}$. We have $\overline{[x_1, f]} = \bar{f} \cdot t_1 \in E$. There exists an element a in the abstract group $\langle a_1, \ldots, a_n\rangle$ such that $\bar{f} \cdot t_1 a \in \bar{y}_1 R_0 + \ldots + \bar{y}_n R_0 = E_0$. Let $\bar{f} \cdot t_1 a = \bar{y}_1 u_1 + \ldots + \bar{y}_n u_n$, where $u_1, \ldots, u_n \in R_0$. Since the elements $u_1, \ldots, u_n$ are divisible by t_1 in $\bar{R}$, they are also divisible by t_1 in R_0. Let $u_i = v_i \cdot t_1$, where $v_i \in R_0$ $(i = 1, \ldots, n)$. We have $\bar{f} \cdot a = \bar{y}_1 v_1 + \ldots + \bar{y}_n v_n$. Recall that the element $\bar{y}_1 w_1 + \ldots + \bar{y}_n w_n$ of E lies in G'/G'' if and only if $w_1 t_1 + \ldots + w_n t_n = 0$. Since $u_1 t_1 + \ldots + u_n t_n = 0$, we have $v_1 t_1 + \ldots + v_n t_n = 0$. Therefore, $\bar{f} \cdot a \in G'/G''$ and $\bar{f} \in G'/G''$, which reduces the proof to the case $f \in F''$.

3.4. Here we prove an auxiliary statement. Let $R = Z[a, a^{-1}]$ be the integral group ring of the infinite cyclic group; $t = a - 1$; $R_0 = Z[t]$; $\bar{R} = Z_p[[t]]$. Let $e_i\bar{R}$ be a free (topological) one-generated R-module with generator e_i $(i = 1, 2)$. Consider the exterior product of $e_1\bar{R}$ and $e_2\bar{R}$, treated as Z_p-modules, on which the action of the ring $\bar{R}$ is defined by formula (5). The exterior product of Z-modules e_1R and e_2R is contained in $e_1\bar{R} \wedge e_2\bar{R}$.

LEMMA 6. Let $f \in e_1\bar{R} \wedge e_2\bar{R}$ and $ft \in e_1R_0 \wedge e_2R_0$. Then $f \in e_1R_0 \wedge e_2R_0$.

Proof. Consider a homogeneous element g of $e_1R_0 \wedge e_2R_0$ of weight n, that is, an element of the form $g = \sum_{i=0}^{n} \alpha_i \cdot e_1t^i \wedge e_2t^{n-i}$, $\alpha_i \in Z$. Define $\operatorname{tr}(g)$, the trace of that element, as $\sum_{i=0}^{n}(-1)^i\alpha_i$ and the derivative $d(g)$ as $\sum_{i=0}^{n} \alpha_i d(e_1t^i \wedge e_2t^{n-i})$, where $d(e_1 \wedge e_2t^n) = 0$, $d(e_1t^i \wedge e_2t^{n-i}) = e_1t^i \wedge e_2t^{n-i+1} - e_1t^{i-1} \wedge e_2t^{n-i+2} + \ldots + (-1)^i e_1t \wedge e_2t^n$ for $1 \le i \le n$. This definition implies that $d(g)$ is a homogeneous element of degree $n+1$. From (5), we infer that $g \equiv \operatorname{tr}(g) \cdot e_1 \wedge e_2t^n + d(g) \bmod (e_1R_0 \wedge e_2R_0)t$.

Note that an element of $e_1\bar{R} \wedge e_2\bar{R}$ of the form $\alpha \cdot e_1 \wedge e_2t^n + u$, where $\alpha \neq 0$ and the weight $u > n$, cannot be divisible by t. Assume the contrary and represent that element as ht. Then the weight of h is equal to $n-1$. Let $e_1t^k \wedge e_2t^{n-k-1}$ be a maximal element of the form $e_1t^i \wedge e_2t^{n-i-1}$ in the Z_p-basis occurring in the expansion of h. By formula (5), then, ht should depend on $e_1t^{k+1} \wedge e_2t^{n-k-1}$, which conflicts with the initial representation of this element.

We come back to the element ft. Now we represent it as $f_k + f_{k+1} + \ldots + f_n$, where f_i is a homogeneous element of weight i in $e_1R_0 \wedge e_2R_0$. The preceding remark implies that $\operatorname{tr}(f_k) = 0$. Then $ft \equiv (d(f_k) + f_{k+1}) + f_{k+2} + \ldots + f_n \bmod (e_1R_0 \wedge e_2R_0)t$. This allows us to reduce the problem to the case where ft is a homogeneous element of degree n.

Let $ft = \alpha_k \cdot e_1t^k \wedge e_2t^{n-k} + \alpha_{k-1} \cdot e_1t^{k-1} \wedge e_2t^{n-k+1} + \ldots + \alpha_l \cdot e_1t^l \wedge e_2t^{n-l}$. We have $\operatorname{tr}(ft) = (-1)^k\alpha_k + \ldots + (-1)^l\alpha_l = 0$ and $ft \equiv d(ft) \bmod (e_1R_0 \wedge e_2R_0)t$. From the definition of a derivative, we obtain $d(ft) = -\alpha_k \cdot e_1t^k \wedge e_2t^{n-k+1} + (\alpha_k - \alpha_{k-1}) \cdot e_1t^{k-1} \wedge e_2t^{n-k+2} + \ldots - (-1)^{k-l}((-1)^k\alpha_k + \ldots + (-1)^l\alpha_l) \cdot e_1t^l \wedge e_2t^{n-l} + \ldots + ((-1)^k\alpha_k + \ldots + (-1)^l\alpha_l) \cdot e_1t \wedge e_2t^n$. Since $\operatorname{tr}(ft) = (-1)^k\alpha_k + \ldots + (-1)^l\alpha_l = 0$, we have the expression for $d(ft)$ with a lesser number of summands $\alpha \cdot e_1t^i \wedge e_2t^j$ than for ft. From this, by repeatedly applying the function d to ft, we obtain the zero element. This means that $ft \in (e_1R_0 \wedge e_2R_0)t$.

Let $ft = gt$, where $g \in e_1R_0 \wedge e_2R_0$. From Lemma 3, it follows that $e_1\bar{R} \wedge e_2\bar{R}$ is a free $\bar{R}$-module. Therefore, $f = g \in e_1R_0 \wedge e_2R_0$. The lemma is proved.

3.5. We come back to the proof of Theorem 2. Let $f \in F''$. We have $x_1\varphi = x_1[x_1, f]$ and $x_2\varphi = x_2[x_2, f] \in G$, whence $[x_1, f]$, $[x_2, f] \in G''$. In Sec. 3.4, the group F'' was identified with an additive subgroup of the exterior square $\bar{H} \wedge \bar{H}$. We will use the same notation as in Secs. 2.3 and 3.3. In the additive notation, we have ft_1, $ft_2 \in G''$. In particular, the elements ft_1 and ft_2 lie in the exterior square of the Z-module $E = \bar{y}_1R + \ldots + \bar{y}_nR$. Moreover, we can assume that ft_1, $ft_2 \in (E_0 \cap G'/G'') \wedge (E_0 \cap G'/G'')$, where $E_0 = \bar{y}_1R_0 + \ldots + \bar{y}_nR_0$.

LEMMA 7. Let w be an arbitrary element in $\bar{H} \wedge \bar{H}$. Then $w \in E_0 \wedge E_0$ if wt_1, $wt_2 \in E_0 \wedge E_0$.

Proof. For arbitrary monomials M and L in $t_2, \ldots, t_n$, where $M \leq L$, write $S_{M,L}$ for a Z_p-module generated in $\bar{H} \wedge \bar{H}$ by all elements of the form $Mt_1^m \wedge Lt_1^l$ $(m, l \geq 0)$. Obviously, $S_{M,L}$ is also a $Z_p[[t_1]]$-module and $\bar{H} \wedge \bar{H} = \bigoplus_{M \leq L} S_{M,L}$. From $wt_1 \in E_0 \wedge E_0$, it follows that w lies in the sum of finitely many modules $S_{M,L}$. Let $w = w_1 + w_2$, where $w_1 \in \bigoplus_{M<L} S_{M,L}$, $w_2 \in \bigoplus_M S_{M,M}$. Then $w_1t_1 \in E_0 \wedge E_0$, $w_2t_1 \in E_0 \wedge E_0$. Lemma 6 implies that $w_1 \in E_0 \wedge E_0$. Obviously, the element w_2 is expressed over Z_p via elements of the form $Mt_2^m \wedge Lt_1^l$ $(m, l \geq 0)$, where M and L are the monomials in $t_1, t_3, \ldots, t_n$, and $M < L$. Since $w_2t_2 \in E_0 \wedge E_0$, the preceding argument (with t_1 replaced by t_2) yields $w_2 \in E_0 \wedge E_0$. The lemma is proved.

It follows from the lemma that $f \in E_0 \wedge E_0$. In Lemma 4, we proved that $F'' = \Gamma_0 \wedge \Gamma_0$, as a $Z_p[[t_1]]$-module, is a direct summand in $\bar{H} \wedge \bar{H}$. Carrying that proof over to the abstract case, we see that $(E_0 \cap G'/G'') \wedge (E_0 \cap G'/G'')$, as a $Z[t_1]$-module, is a direct summand in $E_0 \wedge E_0$. The inclusion $ft_1 \in (E_0 \cap G'/G'') \wedge (E_0 \cap G'/G'')$ then implies that $f \in G'' = G'/G'' \wedge G'/G''$. Theorem 2 is proved.

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