AWARD OF THE 1977 CHAUVENET PRIZE TO PROFESSOR GILBERT STRANG

The Board of Governors of the Mathematical Association of America voted to award the 1977 Chauvenet Prize to Professor Gilbert Strang for his paper, "Piecewise Polynomials and the Finite Element Method," Bulletin of the American Mathematical Society, 79 (1973) 1128–37.

A certificate and monetary award in the amount of five hundred dollars was presented to Professor Strang at the Business Meeting of the Association on January 30, 1977 in St. Louis, Missouri.

The Chauvenet Prize is awarded for a noteworthy paper of an expository or survey nature published in English, which comes within the range of profitable reading for members of the Association. The purpose of the Prize is to stimulate the writing of expository and survey articles. The 1977 Prize, awarded for a paper published in the three year period 1973–75, is the twenty-fifth award of the Chauvenet Prize since its institution by the Association in 1925. For the list of names of the previous winners, see this MONTHLY, 71 (1964), p. 589; 72 (1965), pp. 2–3; 74 (1967), p. 3; 75 (1968), pp. 3–4; 77 (1970), pp. 117–118; 78 (1971), pp. 112–113; 79 (1972), pp. 112–113; 80 (1973), pp. 120; 81 (1974), pp. 113–114; 82 (1975), pp. 108–109; and 83 (1976), pp. 84–85.

Professor Strang was born on November 27, 1934, in Chicago. He received the S.B. degree from M.I.T. in 1955, and was elected to a Rhodes Scholarship for postgraduate study in England. He was awarded First Class Honours at Balliol College, Oxford in 1957. After completing the Ph.D. degree at U.C.L.A. in 1959, under the supervision of Professor Peter Henrici, he returned to M.I.T. as a Moore Instructor and has remained a member of the M.I.T. faculty ever since. He was promoted to Professor in 1970, and became chairman of the Committee on Pure Mathematics in 1975.

Professor Strang was recently elected to the Council of the Society for Industrial and Applied Mathematics, after service on the AMS-SIAM Committee on Applied Mathematics. He was chosen for a NATO Fellowship in 1961 and for a Sloan Fellowship in 1966.

His mathematical interests have centered on partial differential equations and their discrete approximations. His earliest papers studied the stability of different methods for the Cauchy problem, and with Hermann Flaschka he investigated correctness in the presence of multiple characteristics. Professor Strang's more recent work has been concerned especially with the finite element method, which was the subject of a lecture to the American Mathematical Society and of the paper which has been awarded the Chauvenet Prize. With George Fix, he is the author of *An Analysis of the Finite Element Method* (Prentice-Hall, 1973). His teaching and his work with students have concentrated very strongly on applied linear algebra, and they led in 1976 to a new undergraduate textbook on *Linear Algebra and Its Applications* (Academic Press).

In his acceptance, Professor Strang expressed his gratitude to the Association for the honor of the 1977 Chauvenet Prize, and his happiest thanks to teachers and friends (and in particular to Peter Lax and Erwin Canham) for their encouragement over many years.

D.P. ROSELLE

ALL GAMES BRIGHT AND BEAUTIFUL

J. H. CONWAY

Our topic is the addition theory of partizan games. This means that although this paper is written after [3], it should naturally precede [3] on grounds of logic as well as euphony, since the number system of [3] was in fact suggested by the more general system described here. I thank Donald Knuth for suggesting that I write a survey paper with this title.

No proofs will be found in this paper. We hope that most readers will be interested enough to prove the more basic results for themselves, and rich enough to buy at least one of [1] and [4] if they find themselves stuck with the more difficult ones. The games described here are all treated more fully in [1] or [4], and in many cases their descriptions are taken almost verbatim from one of those two references.

To see how addition of games comes about, we consider two particular cases.

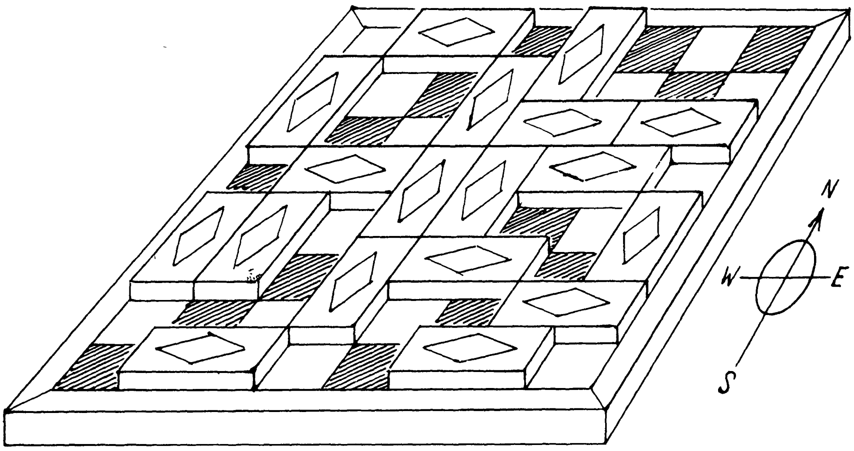


FIG. 1. A game of Domineering.

Domineering (proposed by Göran Andersson). This is played on a checkerboard, with a number of dominoes each of which can cover exactly two squares of the board. The player called Left, when it is his turn to move, must place a domino in North-South orientation so as to cover two currently empty squares, while Right, at each of his turns, places a domino oriented East-West, again so as to

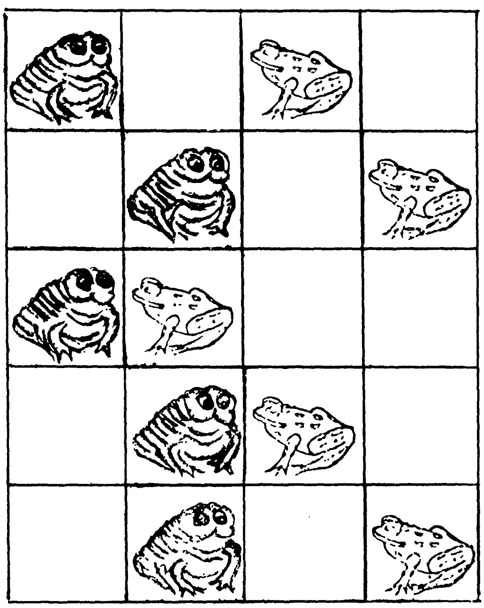


FIG. 2. A game of Toads and Frogs.

exactly cover two previously empty squares. If either player, when it is his turn to move, finds himself unable to place a domino in the required orientation, that player loses.

Toads and Frogs (proposed by Richard Guy). In Figure 2, Left has trained a number of Toads (*Bufo bufo*), and Right a number of Frogs (*Rana rana*) to play the following game. When it is Left's turn to move, he must *either* make one of his toads move just one place Eastward onto an empty square of the board, *or* persuade some toad to jump over a frog just to the East of it and land on the square just beyond that frog, which must be empty. Right moves his frogs in a similar way, but in the Westward direction. If a player whose turn it is to move cannot move any of his creatures in the prescribed way, then that player loses.

Partizan games in general. We now use these examples to illustrate our terminology. Both games are played according to the *normal play convention*, according to which a player loses *if and only if* he is unable to move when it is his turn to do so. And both satisfy the *finishing condition*, that there can be no infinite sequence of legal moves, whether made alternately by the two players or not. From now on, we shall understand that these two conditions apply to every game we consider. Note that the normal play convention enables us to avoid defining the winner in each individual case—he is simply that player who makes the last move of the game, and of course the finishing condition ensures that there will always be a last move. It should soon become obvious why we do not restrict the finishing condition to alternating sequences only. Because we do *not* impose the frequently added restriction that exactly the same moves are available to each player, we shall refer to our games as *partizan games* (and deliberately use the less common spelling of this word).

Sums of partizan games. The most interesting feature of our two examples is the way their typical positions break up into sums of rather smaller positions. Thus the available space in the Domineering position of Figure 1 is composed of separate regions of the shapes shown in Figure 3. When it is some

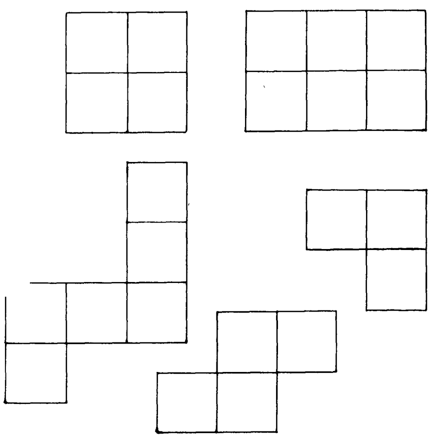
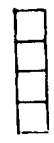
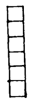


FIG. 3. Available regions in the Domineering position of Figure 1.

player's turn to move, he is forced to choose just one of these regions, and make a move legal for him in that region, and moves made in one region do not interfere with those made in another. Again, in a Toads and Frogs game, a player must choose just one of the East-West lanes into which the board is divided, and make a move legal for him in that lane, and moves made in one lane will not interfere with those made in another.

More generally, we can play the (*disjunctive*) *sum* of any finite collection of partizan games, the individual games being called the *components* of the sum. Each player, when it is his turn to move, must choose just one of the components and make a move legal for him in that component. If he finds that there is no component in which he has a legal move, then of course he loses, by the normal play convention. The moves of the sum that affect any particular component need not of course be made alternately by the two players, but the strong finishing condition that we have adopted still ensures that the sum will necessarily end. In fact the finishing condition on the components is enough to ensure the finishing condition on their sum, which is therefore another partizan game.

Evaluating positions. It turns out that it is possible to assign *values* to the positions of partizan games in such a way that the value assigned to a sum of games is just the sum of the values of the components. In many cases the values are ordinary numbers, which are added according to the rules we learned at school, but in many more cases they are not. The theory of partizan games is concerned with the rules for adding and comparing the many other weird and wonderful values that arise.

Integer values. The positions  ,  ,  , in Domineering are easy to evaluate, since Right, who must place his Dominoes East-West, can never move, and Left has at most 1, 2, 3 moves in the respective cases. Since we shall always reckon values from Left's point of view, we call the values of these three positions 1, 2, and 3 in the order given, and the corresponding positions  ,  ,  will have values -1, -2, -3 since in them the free moves are reserved for Right. A position in which neither player has a legal move, for example the position in Domineering, has the value 0.

Suppose that for some $n = 0, 1, 2, 3, \ldots$, Left has a move to some position of value n, but no other move, and that Right has no legal move whatever. Then of course the value of that position is $n + 1$. In symbols, we write

$$\{n \mid \ \} = n + 1 \qquad (n = 0, 1, 2, 3, \ldots)$$

indicating (the values of) Left's options before the bar, and Right's after it. It will not affect matters if we give Left some additional moves to positions of integer values less than $n + 1$; so for example

$$\{0, 5, 3, 5 \mid \ \} = 6.$$

Nor will it matter if Right is allowed to move to some positions whose values are integers, provided these are greater than $n + 1$, so

$$\{1, 4, 7 \mid 13, 20\} = 8.$$

By reversing the roles of Left and Right in this equation, we find also

$$\{-13, -20 \mid -1, -4, -7\} = -8.$$

The simplest formula of this type is of course the equation

$$\{\ \mid \ \} = 0$$

which expresses the fact that positions in which neither player can move have value zero. But here too we can add certain options without affecting the value—namely moves for Left to positions of negative value, or for Right to positions of positive values only. So for example

$$\{-1 \mid 3, 5\} = 0.$$

We can summarize these statements as follows:

When the value of a position is a number, it is the *simplest* number that is neither less than or equal to any of Left's options nor greater than or equal to any of Right's.

We shall call this the *simplicity principle.* The number 0 is the simplest number of all, then come 1 and -1, then 2 and -2, and so on.

Many positions in partizan games have integer values, and a few games consist entirely of such positions, for example:

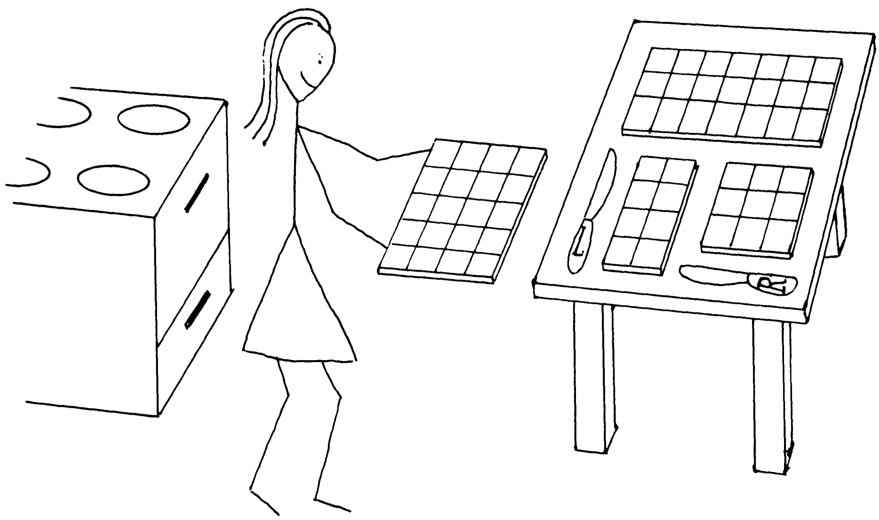


FIG. 4. Ready for a game of Cutcake.

Cutcake is played by two children with a number of rectangular pieces of cake which their mother has already scored along horizontal and vertical lines so as to be ready for breaking into little squares as shown in Figure 4. Lefty moves by breaking some piece along one of the vertical lines, and his sister Rita by breaking a piece along one of the horizontal lines. The values are shown in Table 1. We leave

	1	2	3	4	5	6	7	8	9	10	11	12
1	0	1	2	3	4	5	6	7	8	9	10	11
2	-1	0	0	1	1	2	2	3	3	4	4	5
3	-2	0	0	1	1	2	2	3	3	4	4	5
4	-3	-1	-1	0	0	0	0	1	1	1	1	2
5	-4	-1	-1	0	0	0	0	1	1	1	1	2
6	-5	-2	-2	0	0	0	0	1	1	1	1	2
7	-6	-2	-2	0	0	0	0	1	1	1	1	2
8	-7	-3	-3	-1	-1	-1	-1	0	0	0	0	0
9	-8	-3	-3	-1	-1	-1	-1	0	0	0	0	0
10	-9	-4	-4	-1	-1	-1	-1	0	0	0	0	0

TABLE 1. Values of positions in Cutcake.

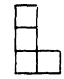
the reader to verify that the pattern indicated by the dividing lines continues indefinitely. The values are computed using only the rules we have given—for example the 5×10 rectangle has the options shown in

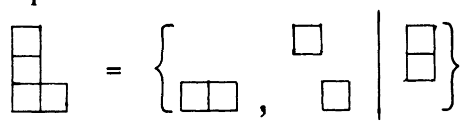
$$\{5\times 1+5\times 9, 5\times 2+5\times 8, 5\times 3+5\times 7, \ldots \mid 1\times 10+4\times 10, 2\times 10+3\times 10\}$$

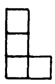
and so has the value $\{-4+1, -1+1, -1+0, 0+0, 0+0 \mid 4+1, 4+4\}$ or simply

$$\{-3, 0, -1, 0, 0 \mid 5, 8\} = 1,$$

by the simplicity rule.

Fractional Values. When we try to evaluate the position  in Domineering, we find, in the symbolic notation, the equation

 $=\{-1,0\mid 1\}$

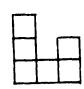
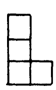
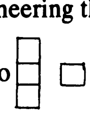
and unfortunately there is no *integer* strictly between -1 and 0 (on the left), and 1 (on the right). However, there is a *fraction*, and in fact the simplest such fraction is $\frac{1}{2}$. It turns out that in fact the position  is really worth exactly $\frac{1}{2}$ a move to Left, in a suitable sense, and that adding two such regions to a Domineering position confers exactly the same advantage on Left as adding  would.

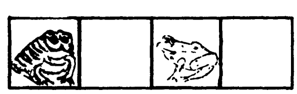
Other positions in finite games can have values involving quarters or eighths of moves etc., and the simplicity rule continues to hold for such values, provided we add the conditions that all integers are simpler than fractions with denominator 2, while these are simpler than those with denominator 4, in turn simpler than those with denominator 8, and so on. So for example

$$\{1 \mid 1\tfrac{3}{8}\} = 1\tfrac{1}{4}$$

since there is no integer or half-integer between 1 and $1\frac{3}{8}$, but $1\frac{1}{4}$ *is* between these two numbers. Other examples are

$$\{-1 \mid 21\tfrac{1}{2}\} = 0,\ \{0 \mid 3\tfrac{1}{4}\} = 1,\ \{\tfrac{1}{4} \mid 2\} = 1,\ \{\tfrac{1}{4} \mid 1\} = \tfrac{1}{2}.$$

Fractional positions arise in many games. In Domineering the position  has value $\frac{3}{4}$, since Left's best option is to  (value $\frac{1}{2}$), and Right's is to  (value 1), and we have $\{\frac{1}{2} \mid 1\} = \frac{3}{4}$.

In Toads and Frogs the position 

turns out to have value $\frac{1}{2}$. The first of these examples is rather tricky to evaluate, since one of Left's options does not have a numerical value, but the second example is easy, and the reader will probably benefit by verifying that all the values shown in Figure 5 are consequences of the simplicity rule.

In **Red-blue Hackenbush** every position has a numerical value, which may be integral or fractional. This game is played with *pictures* made of red and blue edges, and every edge must touch the ground or be connected to the ground by some chain of other edges. Left moves by chopping some bLue edge, and Right by chopping a Red one, and after each chop, any edges no longer connected to the ground are deleted.

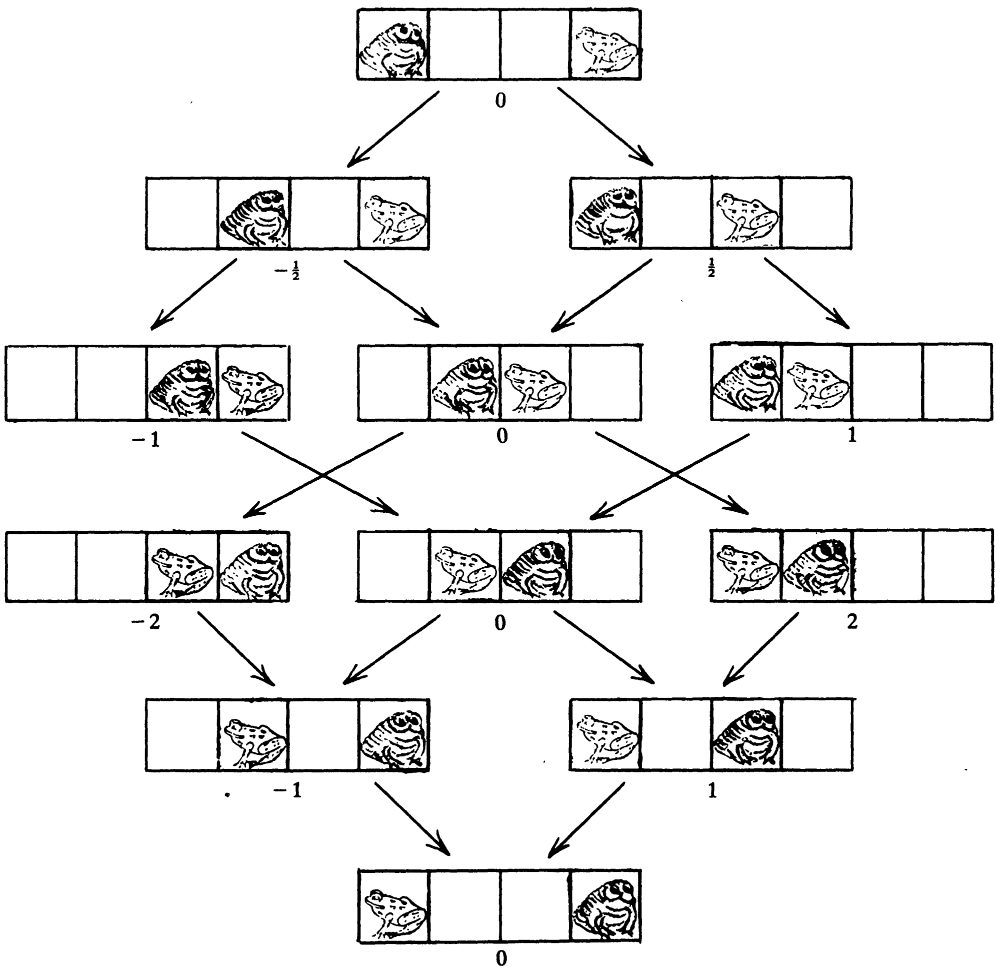
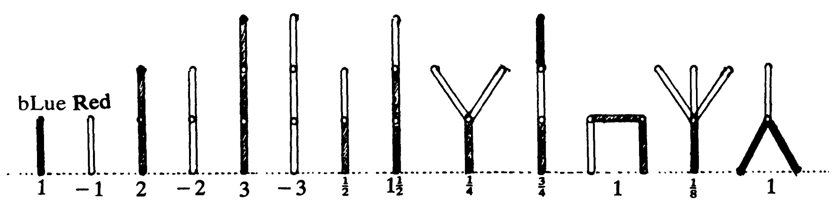
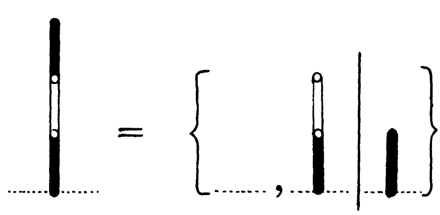


FIG. 5. Anatomy of 4-place Toads and Frogs.

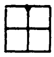
Exercises:



Example:

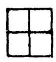
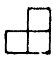


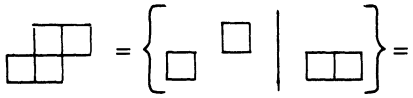
$= \{0, \tfrac{1}{2} \mid 1\} = \tfrac{3}{4}.$

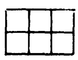
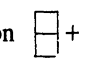
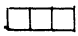
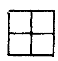
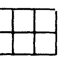
Fuzzy values. In Domineering, the position  has value $\{$  $|$  $\} = \{1|-1\}$, which is not a number, since Left's option is greater than Right's. We call this value ± 1, and in general use $\pm x$ for $\{x|-x\}$, which for $x \geqq 0$ is a non-numerical value. The position  $= \{$  $|$  $\}$ has value ± 0, which we also call $*$, and which is a very common value indeed. More generally, there is a non-numerical value $\{x \mid y\}$ for every pair of numbers x and y with $x \geqq y$. In a sense soon to be explained, the value $\{x \mid y\}$ is *strictly less* than every number greater than x, *strictly greater* than every number less than y, but *incomparable* with all numbers between x and y inclusive. There is a simple policy for finding the best move from any sum of such values, possibly together with numerical values:

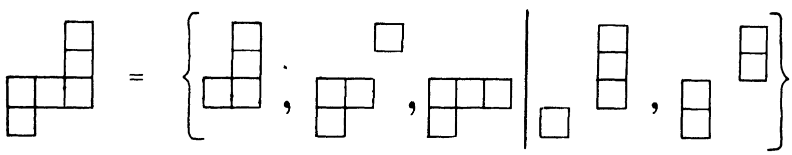
Never move in a component whose value is a number unless you have no other alternative. Of the various components $\{x \mid y\}$ with $x \geqq y$, move in one with the largest possible value of $x - y$.

Since $\frac{1}{2}(x-y)$ is called the *temperature* of $\{x \mid y\}$, the policy may be summarized more briefly: move in the *hottest* $\{x \mid y\}$. If several components are equally hot, it will not matter which of them we choose. A similar temperature policy applies in many other situations, but not in all.

We can now analyze the position of Figure 1. We have already met the values $\{1 \mid -1\}$ and $\{0 \mid 0\}$ of  and , and the evaluation

 $= \{0 \mid -1\}$

is easy. The region  is a little harder. Left has the option  $= 2$, and all Right's options are similar to  $= -\frac{1}{2}$. But Left has also the option , of value ± 1. But since $\pm 1 < 2$, this is worse than his previous option, and so we have  $= \{2 \mid -\frac{1}{2}\}$. Finally, we have the equation



showing that this region has value $\{\frac{1}{2}, *, -\frac{1}{2} \mid 1, 2\}$. Here since $\frac{1}{2}$ is greater than $*$ and $-\frac{1}{2}$, and 1 is less than 2, the value is $\{\frac{1}{2} \mid 1\} = \frac{3}{4}$.

Since we can neglect the isolated squares, which neither player can use, this calculation yields

$$\{2 \mid -\tfrac{1}{2}\} + \{1 \mid -1\} + \{0 \mid -1\} + \{0 \mid 0\} + \tfrac{3}{4}$$

for the total value of Figure 1, where we have kindly rearranged the terms $\{x \mid y\}$ in decreasing order of their temperatures $\frac{1}{2}(x-y)$. If both players play according to the temperature policy, the value after the first four moves will be

$$2 - 1 + 0 + 0 + \tfrac{3}{4} = 1\tfrac{3}{4}$$

if Left starts, and

$$-\tfrac{1}{2} + 1 - 1 + 0 + \tfrac{3}{4} = \tfrac{1}{4}$$

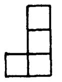
if Right starts. Since both values are positive, Left can win no matter who starts.

If however, Left moves first and makes his move rather stupidly in the top right hand corner, this has the effect of replacing $\{2 \mid -\frac{1}{2}\}$ by ± 1, and making the total value

$$\{1 \mid -1\} + \{1 \mid -1\} + \{0 \mid -1\} + \{0 \mid 0\} + \tfrac{3}{4}$$

and if the next four moves are played sensibly, the resulting value will be

$$-1+1-1+0+\tfrac{3}{4}=-\tfrac{1}{4},$$

and since this is negative, Right will win. If instead Left's opening move had been in the bottom left-hand corner, replacing $\frac{3}{4}$ by  $=\frac{1}{2}$, the value would have become

$$\{2\mid -\tfrac{1}{2}\}+\{1\mid -1\}+\{0\mid -1\}+\{0\mid 0\}+\tfrac{1}{2},$$

and after four sensible moves, which lead to the value

$$-\tfrac{1}{2}+1-1+0+\tfrac{1}{2}=0,$$

we can pretend that the game has finished. Since the most recent move here was Left's, he has won, but it was a close shave, and another false step would have been disastrous.

In general, the winner, assuming best play, can be determined from the value according to the following rule:

> If the value is *positive*, *Left* can win, no matter who starts.
> If the value is *negative*, *Right* can win, no matter who starts.
> If the value is *zero*, whoever plays *second* can win.
> If the value is *fuzzy*, the *first* player to move can win.

The *fuzzy* games are those that are neither ***positive***, ***negative***, nor ***zero***, but rather, ***confused*** with zero. Such a game is $\{x\mid y\}$, when $x \geqq 0 \geqq y$.

Comparing and adding values.in general. We write G^L for the typical option available to Left from G, and G^R for the typical option of Right, so that symbolically

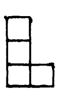
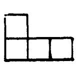
$$G=\{G^L \mid G^R\}.$$

This notation should not be taken to imply either the uniqueness or even the existence, of the options for either player. So for instance from the game $G=\{1, \pm 2\mid\ \}$, from which Left can move to options of value either 1 or ± 2, but Right cannot move, G^L means either 1 or ± 2, but G^R takes no value (which is different from saying that it takes the value 0). In this notation, the sum of two games is given by the formula:

$$G+H=\{G^L+H, G+H^L \mid G^R+H, G+H^R\}.$$

We can also define the ***negative*** of a game by the formula

$$-G=\{-G^R \mid -G^L\},$$

which merely expresses the fact that the roles of Left and Right are to be reversed throughout. Thus the negative of  is  , and the negative of the Hackenbush position  is 

The nomenclature is justified by the following theorems:

1. *Adding a zero game (that is, one won by the second player), never affects the outcome under best play.*
2. *The sum of any game and its negative is zero.*
3. *The sum of two games that are both positive or zero is itself positive or zero.*
4. *The sum of two negative or zero games is another of the same kind.*
5. *If the difference $G-H$, by which we mean $G+(-H)$, is zero, we can replace G by H in any sum without affecting the outcome of that sum.*

6. *If $G \geqq H$, by which we mean that $G - H \geqq 0$, then Left need not object when we replace H by G in any sum, and Right need not object to the replacement of G by H.*

We can use these results to justify the formal definition of value. If $G - H$ is a zero game, then we say that G and H have the same *value*, and write $G = H$. In general order relations between games and their values can be decided by the condition that ? has the same value in the relation G ? H as it does in $G - H$? 0. We always have one of the four cases

$G > H$, meaning that $G - H$ can always be won by *Left*
$G < H$, meaning that $G - H$ can always be won by *Right*
$G = H$, meaning that $G - H$ can be won by the *second* player
$G \| H$, meaning that $G - H$ can be won by the *first* player.

We also abbreviate various compounds of these relations in natural ways. Thus $G \geqq H$ means $G > H$ *or* $G = H$, $G \lhd\!| H$ means $G < H$ *or* $G \| H$.

We can use these conditions to check our assertions about values $\{x \mid y\}$ when x and y are numbers with $x \geqq y$. For if z is any number in the range $x \geqq z \geqq y$, then from the difference $\{x \mid y\} - z$, Left can win by moving to $x - z \geqq 0$, and Right by moving to $y - z \leqq 0$. On the other hand, if $z > x$, then $\{x \mid z\}$ is some number between x and z, so we have $\{x \mid y\} \leqq \{x \mid z\} < z$.

In checking inequalities between values we can use some other obvious remarks—the value of a game G will be unaltered or increased if we increase the value of any Left or Right option of G, add a new Left option, or remove some Right option.

Dominated and reversible options. In this section, we sketch a method by which one can reduce any value to a simplest form. Two games have the same value if and only if their simplest forms are identical. We describe some modifications to the form of a game which do not affect its value.

If $A \leqq B$, and A and B are both Left options from G, then since Left will in any case prefer his move to B over his move to A, the value of G will not be affected if we omit A, while retaining B. If A and B were Right options we could instead have omitted B and retained A. In each case, we say that the option we intend to omit is *dominated* by the retained one.

A more subtle concept is that of *reversible* options. Suppose that the Left option G^{L_0} of G has itself a Right option $G^{L_0R_0} \leqq G$. Then in turns out that the value of G is unaffected if we delete the

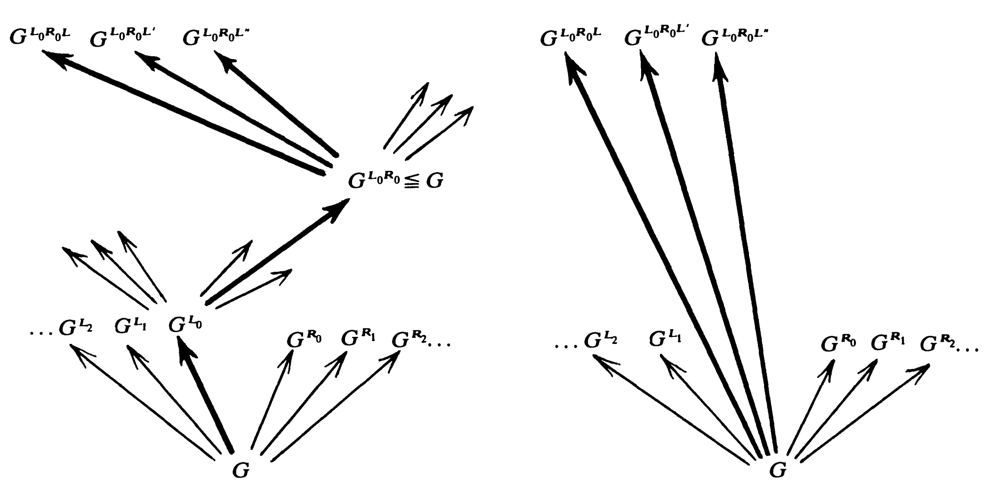


FIG. 6. Bypassing the reversible move G^{L_0}.

G^{L_0} as a Left option of G, and insert instead all the Left options $G^{L_0R_0L}$ of $G^{L_0R_0}$ as new Left options of G. In this case we say that the option G^{L_0} was reversible *through* $G^{L_0R_0}$ *to* the $G^{L_0R_0L}$, and we call this process the *bypassing* of G^{L_0}. See Figure 6. Of course we call the Right option G^{R_0} reversible if it has some Left option $G^{R_0L_0} \geqq G$, and we can then replace it by all the $G^{R_0L_0R}$ as new Right options of G. In general, an option is reversible if the opponent can move from it to a position which is better for him than the original game G.

Here is a simple example. The game $G = \{0, \pm 1 \mid 2\}$ plainly has no dominated options, since 0 and ± 1 are incomparable. But if Left moves to ± 1, Right's reply to -1 is obviously better for Right than the original game, which was clearly positive. So the option ± 1 is reversible through -1, and we can replace it as a Left option of G by the list of all Left options of -1 without affecting the value of G. Since in fact -1 has no Left options, we find $G = \{0 \mid 2\} = 1$.

If we eliminate all dominated and reversible options from all positions of some game G with finitely many positions, we finally obtain *the simplest form of G*. The following result justifies the name:

> Two games G and H have the same value if and only if their simplest forms are identical.

There is also a slightly weaker result valid for infinite games, which we do not discuss here.

Impartial games and the game of Nim. A game is called *impartial* if from every position the moves available to the two players are exactly the same. The theory for such games was described by R. P. Sprague **[10]**, and independently by P. M. Grundy **[6]** and seems to have been independently rediscovered several times since then. It fits very naturally inside our more general theory.

The game of Nim (Bouton **[2]**) is played with a number of heaps, of beans say, and the legal move, for either player, is to reduce the size of some heap by removing some of its beans. We write $*n$ for the value of a Nim-heap of size n, so that

$$*n = \{*0, *1, \ldots, *(n-1) \mid *0, *1, \ldots, *(n-1)\}.$$

The three simplest cases are

$$*0 = 0, *1 = *, \quad \text{and} \quad *2 = \{0, * \mid 0, *\}.$$

Then the Sprague–Grundy theory is contained in the following assertions:

(i) Every impartial game with only finitely many positions has one of the values $*0, *1, *2, \ldots$.

(ii) $\{*a, *b, *c, \ldots \mid *a, *b, *c, \ldots\} = *m$, where m is the least number from $0, 1, 2, \ldots$ that does *not* appear among $a, b, c, \ldots$ (m is called the *mex* of $a, b, c, \ldots$).

(iii) If $a, b, c, \ldots$ are distinct, we have

$$*2^a + *2^b + *2^c + \cdots = *(2^a + 2^b + 2^c + \cdots).$$

(iv) $*n + *n = 0 (= *0)$.

So for example, from property (ii) we have $\{*0, *1, *4, *7 \mid *0, *1, *4, *7\} = *2$ and, as an example of addition:

$$*3 + *5 = (*2 + *1) + (*4 + *1) = *2 + *4 = *6$$

using properties (iii) and (iv).

These results are in fact very easy to prove, for we can see that any $*n$ with $n > m$ will be a reversible option in

$$*m = \{*0, *1, \ldots, *(m-1) \mid *0, *1, \ldots, *(m-1)\}$$

which proves (ii), from which (i) follows inductively, and establishes that $*x + *y$ is equal in value to *some* Nim-heap $*z$, and then the rules of (iii) and (iv) for evaluating z are easily established. The whole theory generalizes trivially to infinite impartial games, there being a value $*\alpha$ for every ordinal number α.

The game of Kayles. Grundy's game. Figure 7 shows a position in the old English bowling game of Kayles, introduced by Dudeney [5]. The players alternately remove either *any one*, or *any two adjacent* skittles, and the first who is unable to do so is the loser.

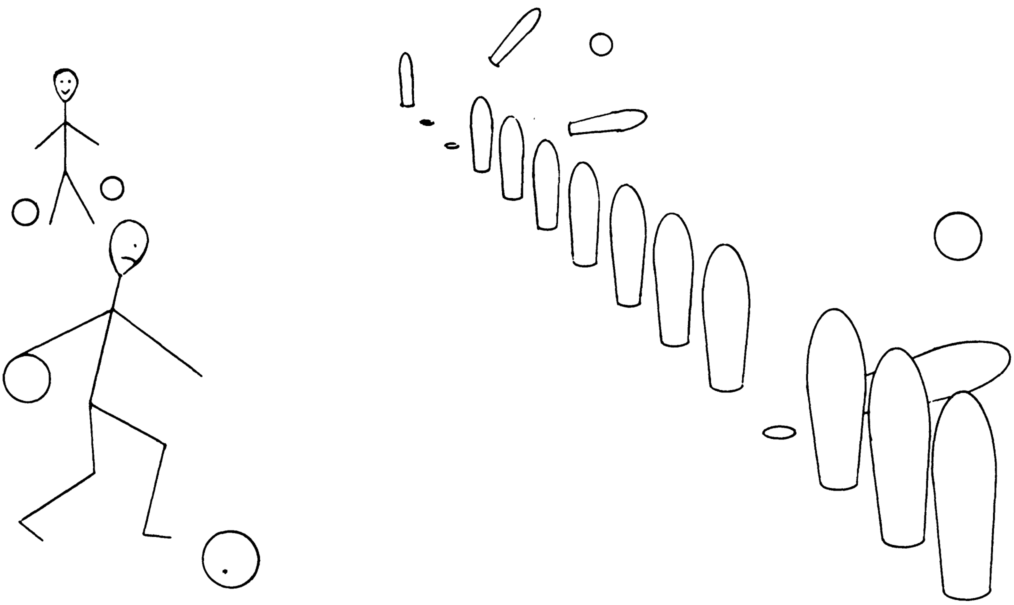


FIG. 7. A game of Kayles.

Kayles is obviously an impartial game, and if we write K_n for the value of a line of n skittles in Kayles, we have

$$K_n = \{K_a + K_b \mid K_a + K_b\} \quad a + b = n - 1 \text{ or } n - 2.$$

Thus for example, neglecting $K_0 = 0$, we find

$$K_5 = \{K_4, K_3 + K_1, K_2 + K_2, K_3, K_2 + K_1 \mid \text{ditto}\},$$

and using the values $*0, *1, *2, *3, *1$ for K_0, K_1, K_2, K_3, K_4 that we can suppose ourselves to have already found, this becomes

$$\{*1, *3 + *1, *2 + *2, *3, *2 + *1 \mid \text{ditto}\}$$

$$= \{*1, *2, *0, *3, *3 \text{ ditto}\} = *4, \text{ by the mex rule.}$$

R. K. Guy discovered in this way the remarkable fact that the sequence of values K_n in Kayles is periodic with period 12 for $n \geqq 71$, and many similar results have been found for other games with heaps (Guy and Smith [7]). *Grundy's game* (split any heap into two smaller heaps of distinct sizes) has been analyzed by Berlekamp to 240,000 values and has not yet become periodic. However, a number of structural features were observed in the values, and if these persist we can prove that the values *will* ultimately become periodic though perhaps only for much larger n.

Seating couples and larger families. Superstars. Figure 8 shows the dinner party which celebrates the end of one of the chapters of our forthcoming book. Left and Right are responsible for the seating

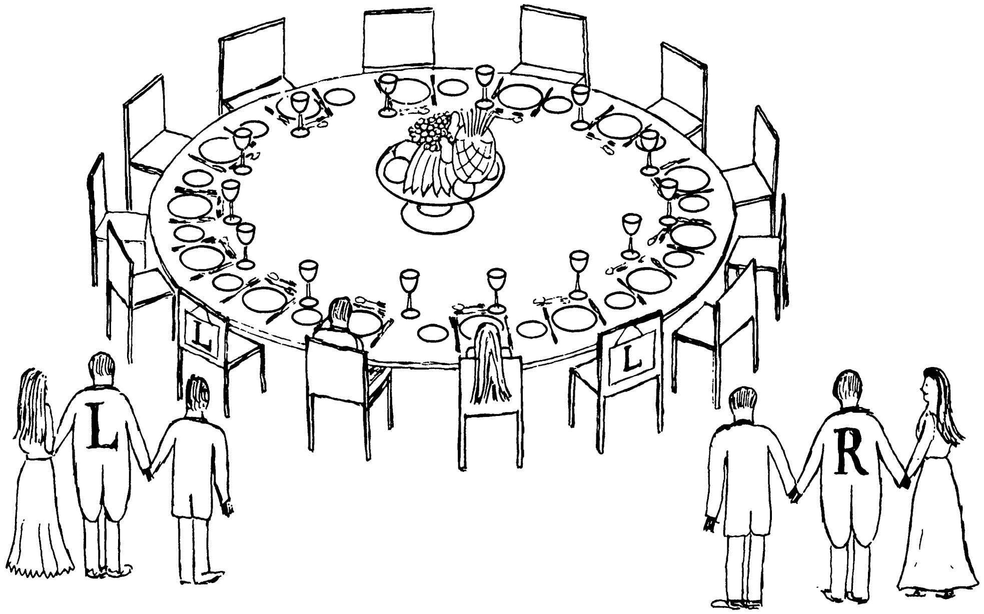


FIG. 8. Seating Couples.

arrangements, and will alternately seat each pair of guests as they arrive. Left thinks it proper to seat a lady only to the *left* of her partner, while Right prefers to seat her to the *right.* No lady may be seated next to a gentleman who is not her partner. Whoever is first unable to seat a couple according to these self-imposed rules has the embarrassing task of explaining the situation to the remaining guests, and may be said to *lose.*

In this game, whenever one of the two players seats a couple, he effectively reserves the two adjacent seats for use of his opponent, since neither player can seat two couples in four consecutive seats. So after the first move, each line of n seats is terminated by reserved seats at either end, and so has the form

LnL, if the two end seats are reserved for *Left*
RnR, if they are reserved for *Right*, and
LnR or RnL if they are reserved one for each player.

We have the equations:

$$LnL\,(=-RnR)=\{LaR+RbL \mid LbL+LaL\}$$

$$LnR\,(=RnL)=\{LaR+RbR \mid LbL+LaR\},$$

where a and b range over all non-negative integers with $a+b=n-2$, except that the number x in any position LxR or RxL must be strictly positive.

There is a generalization of the impartial theory which is very useful in analyzing such games, namely:

> If a game of the form
>
> $$\{A,B,C,\ldots \mid -A,-B,-C,\ldots\}$$
>
> has the properties $A \leqq -A, B \leqq -B, C \leqq -C,\ldots$, then it has value $*m$, where m is the least number, from $0,1,2,3,\ldots$, for which $*m$ does not appear among the values $A,B,C,\ldots$.

Since we can see easily that $RbR \leqq LbL$, this theorem shows in particular that each LnR has a value of form $*m$, where m is the least number for which $*m$ is not an option of LnR. The values that arise in Seating Couples are very simple, and the reader might care to work them out himself as an exercise.

We may modify the game by considering families of size $n > 2$ instead of couples. For example in the game of Seating Families of Five, each family consists of a Mother, a Father, and three children who must always be seated between their parents. Once again, Left will seat each lady at the left end of her family, while Right prefers to seat her to the right, and no lady may be seated next to the husband of another. The values are then given by the same equations as for our game of seating couples, except that we have $a + b = n - 5$, rather than $n - 2$.

We mention this particular example because of the rather special behavior of its values. It can be shown that the sequence

$$L0R, L1R, L2R, \ldots$$

for Seating Families of Five, consists of the values for the (impartial) game of *Dawson's Kayles* (defined like ordinary Kayles, except that the *only* move is to remove two adjacent skittles), each repeated three times. Moreover, the values LnL and RnR usually have the form

$$G = \{*a, *b, *c, \ldots \mid *A, *B, *C, \ldots\}.$$

How do we analyze these?

It turns out that the value of the above game depends rather critically on the two numbers

$$m = \operatorname{mex}(a, b, c, \ldots) \text{ and } M = \operatorname{mex}(A, B, C, \ldots).$$

If $m = M = n$, say, we have $G = *n$. If, however, $m > M$, it turns out that the value of G is otherwise independent of the particular numbers $a, b, c, \ldots$, and we write it as

$$\uparrow_{ABC\ldots} = \{*0, *1, *2, \ldots \mid *A, *B, *C, \ldots\}$$

where the lefthand side of the bracket may be thought of as containing $*n$ for *all* integers n. If $m < M$, then of course G has value

$$\downarrow^{abc\ldots} = \{*a, *b, *c, \ldots \mid *0, *1, *2, \ldots\}$$

which is the negative of $\uparrow_{abc\ldots}$.

In a sense to be discussed later, we say that $\uparrow_{ABC\ldots}$ has *atomic weight* $+1$ and $\downarrow^{ABC\ldots}$ atomic weight -1, while $*n$ has atomic weight 0. For sums of these games it is not hard to see that if the total atomic weight is 2 or more Left can always win, and he can also win.if the atomic weight is 1 and he has the starting move. Moreover, we have the important *translation property*:

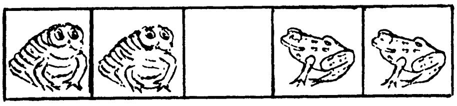
> If the values $*A, *B, *C, \ldots$ are merely $*a + *n, *b + *n, *c + *n, \ldots$ in some order, for *some n*, then we have the equation $\uparrow_{ABC\ldots} = \uparrow_{abc\ldots} + *n$, *for the least such n.*

The particular case $\uparrow_1 = \{0 \mid *\}$ in simplest form is usually called $\uparrow$, and pronounced 'up', with negative $\downarrow$, pronounced 'down'. From the translation property, and the formulas for $*a + *b$, we have also the equations

$\uparrow_0 = \uparrow + * = \{0, * \mid 0\}$ in simplest form
$\uparrow_2 = \uparrow + *3 = \{0 \mid *2\}$ in simplest form
$\uparrow_3 = \uparrow + *2 = \{0 \mid *3\}$ in simplest form, and so on.

The value $\uparrow$ is interesting because it is strictly positive, and so in favor of Left, but it is also strictly less than every positive number, even than every positive infinitesimal number. In the sense of [3] we may call it *small.*

Linear combinations of $\uparrow$ and $*$ arise in many games. For example in Toads and Frogs the starting position



in a lane of length five, has value $*$, while the position

which arises after one move from this, has value $\uparrow$. A rather unexpected equality is the equation

$$\{0 \mid \uparrow\} = \uparrow + \uparrow + *,$$

which has been called *the upstart identity.*

Thermography and the Mean Value Theorem. We already know that the general game G need not have a numerical value. However, it turns out that there is a best numerical approximation m to G, which has the property that when we play the sum of many copies of G, the result is approximately equal to the same number of copies of the number m, which is called the *mean value* of G. For instance, the game $G = \{\{7 \mid 5\} \mid \{4 \mid 1\}\}$ has mean value $4\frac{1}{4}$, and temperature $1\frac{3}{4}$, and this statement allows us to say that $1000G$ lies between any number strictly less than $4250 - 1\frac{3}{4}$ and any number strictly greater than $4250 + 1\frac{3}{4}$.

The *thermograph* is a device for calculating mean values and temperatures. We draw the number scale as the horizontal axis, but with positive numbers to the left and negative ones to the right, instead of the more usual opposite convention. We then draw the thermographs, supposed already computed, of all the options G^L and G^R of G. The thermograph of a number x is a vertical straight line originating at the point x of the axis.

We then, at the height corresponding to any *temperature*, t, take the Leftmost Right boundary of any G^L, and move it a distance t (equal to the height above the axis) to the *right*, and the Rightmost Left boundary of any G^R, and move it distance t to the *left*. The resulting curves define the boundaries of the thermograph of G itself until they meet, at a height t_0 (called the temperature of G) above the axis, the thermograph above t_0 being a single vertical line called the *mast*.

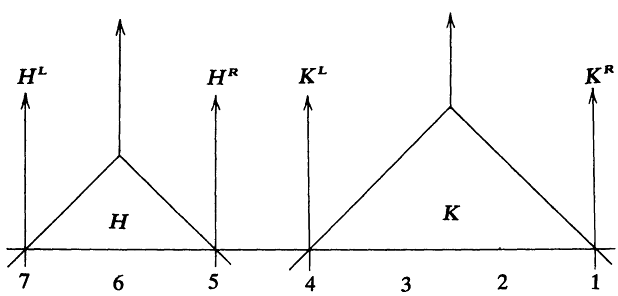


FIG. 9. Thermographs of $H = \{7 \mid 5\}$ and $K = \{4 \mid 1\}$.

A few examples will make the process much clearer (see Figure 9). For the game $H = \{7 \mid 5\}$, the thermograph of $H^L = 7$ is a vertical line through the point 7 on the axis, and that of H^R is a similar vertical through the point 5. So the Left boundary of the thermograph of H starts as a line slanting diagonally up and right through the point 7, and the Right boundary starts as a similar line through 5, slanting diagonally up and left. These meet at a point whose height is 1 above the point 6, and so the temperature of H is 1, and it has mean value 6. Similarly the game $K = \{4 \mid 1\}$ has mean value $2\frac{1}{2}$, and temperature $1\frac{1}{2}$.

Now for the game $G = \{H \mid K\}$, the Right boundary of H starts at 5 and slants up and left to the point 1 above 6, and is vertical thereafter. The Left boundary starts at 4 and slopes diagonally up and right to a point at height $1\frac{1}{2}$ above the point $4\frac{1}{4}$, and is vertical thereafter. The Left and Right boundaries of G, obtained by pushing these inward towards each other, will therefore be a line starting vertically at 5 before turning to slant up and right at height 1, and another starting vertically at 4 before turning to slant up and left at height $1\frac{1}{2}$, as in Figure 10. These meet at height $1\frac{3}{4}$ above the

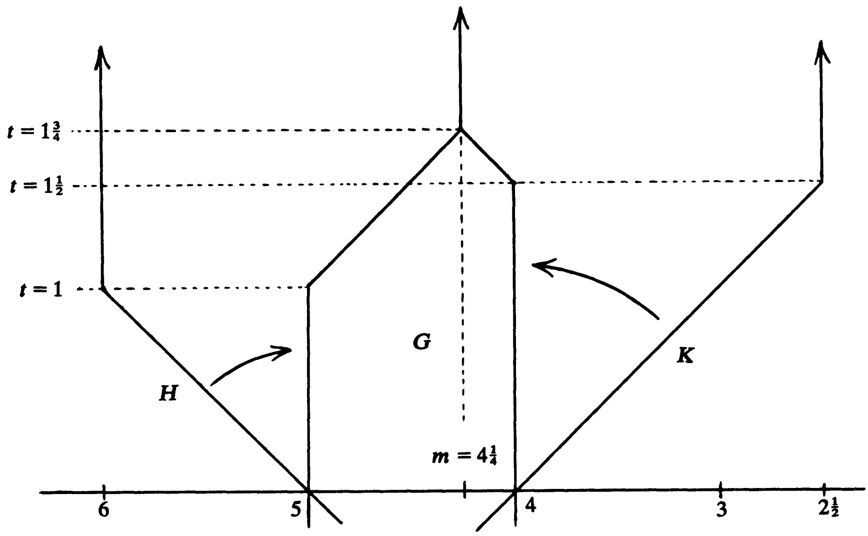


FIG. 10. How to find thermographs.

point $4\frac{1}{4}$ on the axis, showing that the temperature of G is $1\frac{3}{4}$ and its mean value $4\frac{1}{4}$. The same algorithm works for any G for which the Leftmost right boundary of any G^L starts to the left of the Rightmost left boundary of any G^R—in the other cases G is a number, namely that given by the simplicity principle. If t' is any number greater than the temperature of G and m is the mean value of G, we have

$$nm - t' < n \cdot G < nm + t'$$

for any positive integer n, justifying the terminology.

More generally, if games $G_0, G_1, G_2, \ldots$ have the respective mean values $m_0, m_1, m_2, \ldots$, then we can say that

$$m_0 + m_1 + m_2 + \cdots - t' < G_0 + G_1 + G_2 + \cdots < m_0 + m_1 + m_2 + \cdots + t'$$

for any number t' greater than the temperature of every G_i. This often helps us to win by showing that a complicated sum of games is positive without making a detailed analysis.

Atomic weights. For small games like $\uparrow$ we need a more delicate scale (both mean value and temperature of $\uparrow$ are 0) This is provided by the **atomic weight calculus**, which obtains for such small games some of the benefits that thermography has given for large ones. The details are unexpectedly subtle.

There is an atomic weight G'' defined for any game G for which no position has value a non-zero number. This may be computed by the formula

$$G'' = \{(G^L)'' - 2 \mid (G^R)'' + 2\},$$

except that when this defines an integer, it might not be the correct integer. The correct integer in such a case is

the *largest* n with $(G^L)'' - 2 \lhd| n \lhd| (G^R)'' + 2$, if G exceeds remote stars,
the *least* such n if G is exceeded by remote stars, and
the integer *zero*, if G is incomparable with remote stars.

The *remote stars* for a game G are those Nim-heaps $*N$ which do *not* occur as values of positions of G. It can be shown that G has the same order-relations with all the remote stars for G.

As an example of the atomic weight calculus, we take $G = \{0 \mid \uparrow\}$. Since 0 has atomic weight 0 and $\uparrow$ atomic weight 1, the formula would give

$$G'' = \{0-2 \mid 1+2\} = \{-2 \mid 3\}$$

which defines the integer 0. Since this is an integer, we cannot immediately assert that it is the atomic weight, but must first compare G with the remote stars. The smallest star that is remote for G is $*2 = \{0, * \mid 0, *\}$, and we find, by playing the game, that $G + *2$ is positive, so that G exceeds the remote stars and so is the greatest integer $\lhd| 3$, namely 2. Of course the value 2 found this way for the atomic weight is consistent with our earlier evaluation of this G as $\uparrow + \uparrow + *$.

As another example, we take $H = \{\uparrow + \uparrow + * \mid \downarrow + *\}$. The atomic weight formula now gives $\{2-2 \mid -1+2\} = \{0 \mid 1\} = \frac{1}{2}$, which therefore really is the atomic weight. In fact this game is, in a definite sense, one-half of $\uparrow$, and is usually written $\frac{1}{2} \cdot \uparrow$. We can define $x \cdot \uparrow$ for all *non-integral* x by the formula

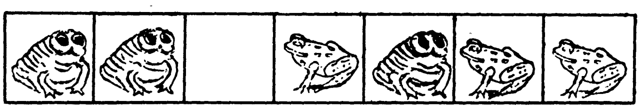
$$x \cdot \uparrow = \{(x^L + 2) \cdot \uparrow + * \mid (x^R - 2) \cdot \uparrow + *\},$$

and this, taken together with the obvious definition for integral x, satisfies the distributive law

$$(x+y) \cdot \uparrow = (x \cdot \uparrow) + (y \cdot \uparrow).$$

The particular cases $\frac{1}{2} \cdot \uparrow, \frac{1}{4} \cdot \uparrow, * \cdot \uparrow$ arise in the twisted form of Bynum's game [**4**, 199–200] which we do not have time to describe here. An even more interesting sequence of infinitesimals turns up in the untwisted form. The theory of these and many other games is made much easier by the most useful property of atomic weights asserting that if a game has atomic weight 2 or more, it is positive.

For example the Toads and Frogs position



can be computed to have the value $\{\Uparrow * \mid 0\}$ (where $\Uparrow * = \uparrow + \uparrow + *$) whose atomic weight is $\{2-2 \mid 0+2\} = 1$. So the position of Figure 11 which has atomic weight 2, must be positive and Left

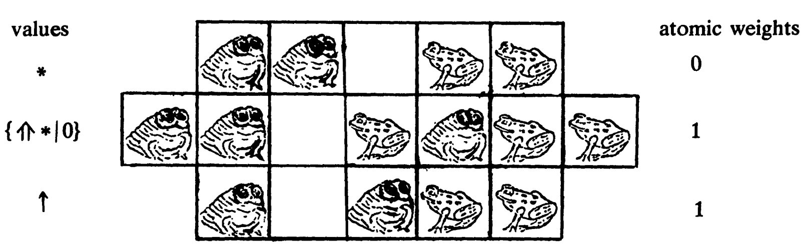


FIG. 11. A Toads and Frogs position of Atomic Weight 2.

should win no matter who starts. In general Left can be sure of winning any game whose atomic weight is 2 or more, but if he has the move, he can win provided only that the atomic weight is fuzzy or positive. (Atomic weights *can* be fuzzy, for example $\{\Uparrow \mid \Downarrow\}$ has atomic weight $\{2-2 \mid -2+2\} = *$, but fortunately they are usually integers.)

The work of Milnor and Hanner. This theory is related to that expounded by John Milnor [9]. Although there are a number of similarities, there are also some important differences. Milnor introduces numbers in an *ad hoc* fashion, as *payoffs*, but for us they arise naturally from the theory. Finite games lead only to the dyadic rationals, but since we allow infinite games satisfying the finishing condition we do in fact get all real numbers as well as the vast array of infinite and infinitesimal numbers like

$$\omega, \omega+1, \omega-1, \omega/2, \sqrt{\omega}, 1/\omega, 1/\omega^2, 1/\omega^{\omega},$$

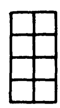
of [3].

Milnor's *incentive* corresponds roughly to our *temperature* and Olof Hanner [8] has given a rather complicated proof of the Mean Value Theorem for Milnor's games. But for Milnor the *only* games of incentive zero are the real numbers, so he has no analogs of our games

$$*, *2, \uparrow, \text{etc.}$$

The games covered by his theory are just those in which every position either has positive temperature or is a *real* number.

We close with a few valedictory remarks. Firstly it seems that the logically most natural theory is that allowing infinite games, subject to the finishing condition. Most of the theory makes no distinction between the finite and infinite case, two notable exceptions being the mean value theorem and Simon Norton's theorem that games with only finitely many positions cannot have odd additive order. (Norton has constructed infinite games of all orders.) Seen in this light, the theory includes the theory of infinite and infinitesimal numbers developed in [3], and greatly extends that theory.

The world of all games has many interesting highways and byways of which the examples here (numbers, values $\{x \mid y\}$, values $*n$, values $\uparrow_{abc\ldots}$) are only a few. There are powers of $\uparrow$ (for example $\uparrow^2 = \{0 \mid \downarrow *\}$) as well as multiples of $\uparrow$, and some extremely small values, for example the value $+_2$ (pronounced "tiny-two") $= \{0 \mid \{0 \mid -2\}\}$ of the Domineering position  . Fortunately, it seems that the most illuminating way to examine this vast wealth of structure is to investigate various partizan games that are naturally suggested as interesting to play. Why don't *you* help to investigate these games and their values, weird and wonderful? It's good fun!

This paper was written in haste. That I have not repented it at leisure is entirely due to the efforts of Richard Guy and Karen McDermid.

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