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On generalized Hilbert transforms and their interaction with the Radon transform in Clifford analysis

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SUMMARY

The present paper deals with the interaction between the multidimensional Hilbert transform and the Radon transform in Clifford analysis, both transforms being protagonists in multidimensional signal analysis theory. In an attempt to complete the picture, we consider in particular the action of the Radon transform on two types of generalized Hilbert operators, which have been constructed in the context of some well-established families of Clifford distributions. Copyright © 2007 John Wiley & Sons, Ltd.

KEY WORDS: Hilbert transform; Radon transform; Clifford analysis

1. INTRODUCTION

Clifford analysis has gradually developed to a comprehensive theory offering a powerful generalization to higher dimensions of the theory of holomorphic functions in the complex plane. It focusses on monogenic functions, i.e. null solutions of the Clifford-vector valued Dirac operator $\underline{\partial} = \sum_{j=1}^{m} e_j \partial_{x_j}$ where $(e_1, \ldots, e_m)$ forms an orthogonal basis for the quadratic space $\mathbb{R}^m$ underlying the construction of the Clifford algebra $\mathbb{R}_{0,m}$. Monogenic functions are actually refining the properties of harmonic functions of several variables, since the rotation-invariant Dirac operator factorizes the m-dimensional Laplace operator (as does the Cauchy–Riemann operator in the complex plane). This has, a.o., allowed for a nice study of Hardy spaces of monogenic functions and the related multidimensional Cauchy and Hilbert transforms, see [1–6].

The Hilbert transform on the real line, given for an appropriate function or distribution f by

$$\mathscr{H}[f](x) = \frac{1}{\pi} Pv \int_{-\infty}^{+\infty} \frac{f(y)}{x-y} \, \mathrm{d}y$$

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—where Pv denotes the Cauchy principal value—was first generalized to m-dimensional Euclidean space by means of the Riesz transforms R_j

$$R_j[f](\underline{x}) = \lim_{\varepsilon \to 0+} \frac{2}{a_{m+1}} \int_{\mathbb{R}^m \setminus B(\underline{x},\varepsilon)} \frac{x_j - y_j}{|\underline{x} - \underline{y}|^{m+1}} f(\underline{y}) \, \mathrm{d}V(\underline{y}), \quad j = 1, \ldots, m$$

where $a_{m+1} = 2\pi^{(m+1)/2}/\Gamma((m+1)/2)$ denotes the area of the unit sphere S^m in $\mathbb{R}^{m+1}$.

It was Horváth who, already in his 1953 paper [7], introduced the Clifford-vector valued Hilbert operator $\mathscr{H} = \sum_{j=1}^m e_j R_j$. The m-dimensional Hilbert transform in the Clifford analysis setting was taken up again in the 1980s and further studied in e.g. [8–12]. More recently, four families of specific distributions in Clifford analysis were introduced and thoroughly studied (see [13, 14]) and it was shown that the Hilbert kernel is one of those distributions acting as a convolution operator (see also [15, 16]). This observation has lead in [17] to two possible generalizations of the Hilbert transform in $\mathbb{R}^m$ based on these Clifford distributions, where the aim was to preserve in these approaches as much properties of the classical Hilbert transform as possible.

The classical Radon transform on the other hand, is the map which assigns to a given function f the totality of its integrals over all (hyper)planes of a given dimension. One of the main problems of integral geometry is to reconstruct the function f from the information contained in these 'sliced profiles'. In particular, in the case of codimension one, the Radon transform is given by

$$\mathscr{R}[f](\underline{n}, s) = \int_{\underline{x} \in \mathbb{R}^m} \delta(\langle \underline{x}, \underline{n} \rangle - s) f(\underline{x}) \, \mathrm{d}V(\underline{x})$$

with $\underline{n}$ a unit vector in $\mathbb{R}^m$, s a real variable and $\langle \cdot, \cdot \rangle$ the standard inner product. For a detailed treatment of the theory of Radon transforms we refer to the classical works [18–20], while applications are extensively treated in [21] and the references therein. More in general, people have also studied integrals of functions over surfaces belonging to a special class, such as spheres (see [22]), quadrics (see [23]), or even over zeroes of higher order homogeneous polynomials (see [24]). The interaction between integral geometry and Clifford analysis leads to interesting extensions of both fields of research. We mention the plane wave decomposition of the Cauchy kernel (see [25]) as well as the factorization of the Darboux equation for spherical means in Clifford analysis (see [22, 26]) which play a crucial role in the theory of Riesz potentials (see [15, 27]).

In one-dimensional signal processing, the Hilbert transform has become an indispensable tool for both global and local descriptions of a signal, yielding information on various independent signal properties. The involved methods are essentially based on the concept of an analytic signal, consisting of a linear combination of a real valued function with its Hilbert transform. Analytic signals form a universal tool which can be applied in a range of fields such as geophysics, astronomy and image reconstruction. In [28], a contribution was made to two-dimensional signal analysis theory by generalizing the analytic signal to two dimensions in order to design appropriate methods. To this end, a generalized two-dimensional Hilbert transform (also referred to as Riesz transform) was used in combination with the Radon transform. Indeed, the latter projects a two-dimensional signal such that it may be decomposed by means of its intrinsically one-dimensional profile components and it allows to express the generalized Hilbert (or Riesz) transform of this signal in terms of the one-dimensional Hilbert transforms of the obtained profiles.

The present paper aims at contributing to the underlying mathematical theory by studying the interaction between the Hilbert and the Radon transforms in general dimension in the context of Clifford analysis. More specifically, the action of the Radon transform is considered on two types of generalized Hilbert operators, which have been constructed in the context of the families of Clifford distributions mentioned above. In order to keep the paper self-contained, the necessary definitions and notations of Clifford analysis and of the considered distributions have been summarized in Sections 2 and 3. Section 4 deals with the properties of Hilbert transforms in Clifford analysis, both the traditional one and the two constructed generalizations. Section 5 contains the main results of the paper: the obtained formulae will in particular allow to compute the multidimensional (generalized) Hilbert transform of a signal and its Fourier spectrum by subsequent application of the Radon transform, the one-dimensional Hilbert transform and the one-dimensional Fourier transform. Finally, in Section 6, an illustrative example is given.

2. CLIFFORD ANALYSIS

In this section we briefly present the basic definitions and some results of Clifford analysis which are necessary for our purpose. For an in-depth study of this higher dimensional function theory we refer to e.g. [9, 27, 29].

Let $\mathbb{R}^{0,m}$ be the real vector space $\mathbb{R}^m$, endowed with a non-degenerate quadratic form of signature $(0, m)$, let $(e_1, \ldots, e_m)$ be an orthonormal basis for $\mathbb{R}^{0,m}$, and let $\mathbb{R}_{0,m}$ be the universal Clifford algebra constructed over $\mathbb{R}^{0,m}$.

The non-commutative multiplication in $\mathbb{R}_{0,m}$ is governed by the rules

$$e_i e_j + e_j e_i = -2\delta_{i,j} \quad \forall i, j \in \{1, \ldots, m\}$$

For a set $A = \{i_1, \ldots, i_h\} \subset \{1, \ldots, m\}$ with $1 \leqslant i_1 < i_2 < \cdots < i_h \leqslant m$, let $e_A = e_{i_1} e_{i_2} \ldots e_{i_h}$. Moreover, we put $e_\phi = 1$, the latter being the identity element. Then $(e_A : A \subset \{1, \ldots, m\})$ is a basis for the Clifford algebra $\mathbb{R}_{0,m}$. Any $a \in \mathbb{R}_{0,m}$ may thus be written as $a = \sum_A a_A e_A$ with $a_A \in \mathbb{R}$ or still as $a = \sum_{k=0}^{m} [a]_k$ where $[a]_k = \sum_{|A|=k} a_A e_A$ is a so-called k-vector ($k = 0, 1, \ldots, m$). If we denote the space of k-vectors by $\mathbb{R}^k_{0,m}$, then $\mathbb{R}_{0,m} = \bigoplus_{k=0}^{m} \mathbb{R}^k_{0,m}$.

We will also identify an element $\underline{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$ with the one-vector (or vector for short) $\underline{x} = \sum_{j=1}^{m} x_j e_j$. The multiplication of any two vectors $\underline{x}$ and $\underline{y}$ is given by

$$\underline{x}\,\underline{y} = -\langle \underline{x}, \underline{y} \rangle + \underline{x} \wedge \underline{y}$$

with

$$\langle \underline{x}, \underline{y} \rangle = \sum_{j=1}^{m} x_j y_j = -\frac{1}{2}(\underline{x}\,\underline{y} + \underline{y}\,\underline{x})$$

$$\underline{x} \wedge \underline{y} = \sum_{i<j} e_{ij}(x_i y_j - x_j y_i) = \frac{1}{2}(\underline{x}\,\underline{y} - \underline{y}\,\underline{x})$$

being a scalar and a 2-vector (also called bivector), respectively. In particular $\underline{x}^2 = -\langle \underline{x}, \underline{x} \rangle = -|\underline{x}|^2 = -\sum_{j=1}^{m} x_j^2$.

Conjugation in $\mathbb{R}_{0,m}$ is defined as the anti-involution for which $\overline{e}_j = -e_j,\ j = 1, \ldots, m$. In particular for a vector $\underline{x}$ we have $\overline{\underline{x}} = -\underline{x}$.

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The Dirac operator in $\mathbb{R}^m$ is the first order Clifford-vector valued differential operator

$$\underline{\partial} = \sum_{j=1}^{m} e_j \partial_{x_j}$$

its fundamental solution being given by

$$E(\underline{x}) = \frac{1}{a_m} \frac{\overline{\underline{x}}}{|\underline{x}|^m}$$

with $a_m = 2\pi^{m/2}/\Gamma(m/2)$ the area of the unit sphere S^{m-1} in $\mathbb{R}^m$. Considering functions defined in $\mathbb{R}^m$ and taking values in $\mathbb{R}_{0,m}$, we say that the function f is left monogenic in the open region Ω of $\mathbb{R}^m$ iff f is continuously differentiable in Ω and satisfies in Ω the equation $\underline{\partial} f = 0$. As $\overline{\underline{\partial} f} = \overline{f}\,\overline{\underline{\partial}} = -\overline{f}\,\underline{\partial}$, a function f is left monogenic in Ω iff $\overline{f}$ is right monogenic in Ω. As moreover the Dirac operator factorizes the Laplace operator Δ, $-\underline{\partial}^2 = \underline{\partial}\overline{\underline{\partial}} = \overline{\underline{\partial}}\underline{\partial} = \Delta$, a monogenic function in Ω (as well as its components) is harmonic and hence C_∞ in Ω.

Introducing spherical co-ordinates $\underline{x} = r\underline{\omega}$, $r = |\underline{x}|$, $\underline{\omega} \in S^{m-1}$, the Dirac operator $\underline{\partial}$ may be rewritten as

$$\underline{\partial} = \underline{\omega}\partial_r + \frac{1}{r}\,\partial_{\underline{\omega}} = \underline{\omega}\left(\partial_r - \frac{1}{r}\,\underline{\omega}\partial_{\underline{\omega}}\right)$$

while the Laplace operator takes the form

$$\Delta = \partial_r^2 + \frac{m-1}{r}\,\partial_r + \frac{1}{r^2}\,\Delta^*$$

Δ^* being the Laplace–Beltrami operator.

In this paper a fundamental role is played by the homogeneous polynomials $P_p(\underline{x})$ of degree $p \in \mathbb{N}$ which we take to be left (and hence also right) monogenic and vector valued. Note that such kind of polynomials are easily obtained by considering

$$P_p(\underline{x}) = \underline{\partial} S_{p+1}(\underline{x})$$

where $S_{p+1}(\underline{x})$ is a scalar valued harmonic polynomial of degree $(p+1)$. Through inversion, the functions

$$Q_p(\underline{x}) = \frac{\overline{\underline{x}}}{|\underline{x}|^{m+2p}}\, P_p(\underline{x})$$

are seen to be left monogenic homogeneous functions of degree $(-m+1-p)$ in the complement of the origin. By taking restrictions to the unit sphere S^{m-1} of $P_p(\underline{x})$ and $Q_p(\underline{x})$ we, respectively, obtain the so-called inner spherical monogenics $P_p(\underline{\omega})$ and the so-called outer spherical monogenics $Q_p(\underline{\omega}) = \underline{\omega} P_p(\underline{\omega})$. For $p = 0$, we put $P_0(\underline{x}) = 1$.

Finally, in this paper we will adopt the following definition of the Fourier transform in $\mathbb{R}^m$:

$$\mathscr{F}[f(\underline{x})](\underline{y}) = \int_{\mathbb{R}^m} f(\underline{x})\ \exp(-2\pi \mathrm{i}\langle \underline{x}, \underline{y}\rangle)\,\mathrm{d}V(\underline{x})$$

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for which some well-known basic formulae and properties hold, as summarized below

$$\mathscr{F}[\underline{\partial} f](\underline{y}) = 2\pi \mathrm{i} \underline{y} \mathscr{F}[f](\underline{y})$$

$$2\pi \mathrm{i} \mathscr{F}[\underline{x} f](\underline{y}) = -\underline{\partial} \mathscr{F}[f](\underline{y})$$

$$2\pi \mathrm{i} \mathscr{F}[f \underline{x}](\underline{y}) = -\mathscr{F}[f](\underline{y}) \underline{\partial}$$

$$\mathscr{F}[\delta(\underline{x})](\underline{y}) = 1$$

$$\mathscr{F}[1](\underline{y}) = \delta(\underline{y})$$

For the one-dimensional Fourier transform we will use following specific notation: $\mathscr{F}_{x \to y}\{f(x)\}(y)$.

3. FOUR FAMILIES OF CLIFFORD DISTRIBUTIONS

In [13, 14] four families of distributions in Euclidean space $\mathbb{R}^m$, denoted $T_{\lambda,p}$, $U_{\lambda,p}$, $V_{\lambda,p}$ and $W_{\lambda,p}$, depending on parameters $\lambda \in \mathbb{C}$ and $p \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, were studied in the framework of Clifford analysis. In this section we recall the definition of their normalizations $T^*_{\lambda,p}$, $U^*_{\lambda,p}$, $V^*_{\lambda,p}$ and $W^*_{\lambda,p}$.

To this end, we start form the original definitions of $T_{\lambda,p}$, $U_{\lambda,p}$, $V_{\lambda,p}$ and $W_{\lambda,p}$, by means of their action on a testing function ϕ, viz.

$$\langle T_{\lambda,p}, \phi \rangle = a_m \langle Fp r_+^{\mu+p_{\mathrm{e}}}, \Sigma_p^{(0)}[\phi] \rangle$$

$$\langle U_{\lambda,p}, \phi \rangle = a_m \langle Fp r_+^{\mu+p_{\mathrm{e}}}, \Sigma_p^{(1)}[\phi] \rangle$$

$$\langle V_{\lambda,p}, \phi \rangle = a_m \langle Fp r_+^{\mu+p_{\mathrm{e}}}, \Sigma_p^{(3)}[\phi] \rangle$$

$$\langle W_{\lambda,p}, \phi \rangle = a_m \langle Fp r_+^{\mu+p_{\mathrm{e}}}, \Sigma_p^{(2)}[\phi] \rangle$$

Obviously, some notations in the above expressions have to be explained. First, the symbol Fp stands for the well-known distribution 'finite parts' on the real line, where $\mu = \lambda + m - 1$ and p_{e} denotes the 'even part of p', defined by $p_{\mathrm{e}} = p$ if p is even and $p_{\mathrm{e}} = p - 1$ if p is odd. For a real variable x and a complex parameter s, the finite parts distribution Fpx_+^s coincides with the regular distribution x_+^s when $\mathbb{R}es > -1$, while for $\mathbb{R}es < -1$ one defines

$$\langle Fpx_+^s, \phi \rangle = \int_0^{+\infty} x^\mu \left(\phi(x) - \phi(0) - \frac{\phi'(0)}{1!} x - \cdots - \frac{\phi^{(n-1)}(0)}{(n-1)!} x^{n-1} \right) \mathrm{d}x$$

$$= \lim_{\substack{\varepsilon \to 0 \\ >}} \left(\int_\varepsilon^{+\infty} x^\mu \phi(x)\, \mathrm{d}x + \phi(0) \frac{\varepsilon^{\mu+1}}{\mu+1} + \cdots + \frac{\phi^{(n-1)}(0)}{(n-1)!} \frac{\varepsilon^{\mu+n}}{\mu+n} \right)$$

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when the parameter s belongs to the strip $-n-1<\mathbb{R}es<-n$. In this way, using moreover analytic continuation, the distribution $Fp x_+^s$ can be defined and is holomorphic in $\mathbb{C}\backslash\{-1,-2,\ldots\}$, with simple poles for $s=-n$, $n\in\mathbb{N}$. Finally, $\Sigma_p^{(0)}$, $\Sigma_p^{(1)}$, $\Sigma_p^{(2)}$ and $\Sigma_p^{(3)}$ are the generalized spherical mean operators defined on scalar valued testing functions ϕ by

$$\Sigma_p^{(0)}[\phi]=r^{p-p_e}\Sigma^{(0)}[P_p(\underline{\omega})\ \ \phi(\underline{x})]=\frac{r^{p-p_e}}{a_m}\int_{S^{m-1}} P_p(\underline{\omega})\phi(\underline{x})\,\mathrm{d}S(\underline{\omega})$$

$$\Sigma_p^{(1)}[\phi]=r^{p-p_e}\Sigma^{(0)}[\underline{\omega}\ \ P_p(\underline{\omega})\ \ \phi(\underline{x})]=\frac{r^{p-p_e}}{a_m}\int_{S^{m-1}} \underline{\omega}P_p(\underline{\omega})\phi(\underline{x})\,\mathrm{d}S(\underline{\omega})$$

$$\Sigma_p^{(2)}[\phi]=r^{p-p_e}\Sigma^{(0)}[\underline{\omega}\ \ P_p(\underline{\omega})\ \ \underline{\omega}\ \ \phi(\underline{x})]=\frac{r^{p-p_e}}{a_m}\int_{S^{m-1}} \underline{\omega}P_p(\underline{\omega})\underline{\omega}\phi(\underline{x})\,\mathrm{d}S(\underline{\omega})$$

$$\Sigma_p^{(3)}[\phi]=r^{p-p_e}\Sigma^{(0)}[P_p(\underline{\omega})\ \ \underline{\omega}\ \ \phi(\underline{x})]=\frac{r^{p-p_e}}{a_m}\int_{S^{m-1}} P_p(\underline{\omega})\underline{\omega}\phi(\underline{x})\,\mathrm{d}S(\underline{\omega})$$

where $P_p(\underline{\omega})$ is an inner spherical monogenic of degree p as defined in the previous section.

Since it was clear from the definition that all of these distributions would inherit a (still infinite) subset of singularities from their generating distribution Fp, a thorough study of their genuine poles has been carried out, eventually resulting in a normalization procedure where the remaining singularities have been removed through division by a deliberately chosen Γ-function, see e.g. [15]. This has lead to the following normalized versions of the original distributions:

$$\begin{cases} T_{\lambda,p}^{*}=\pi^{(\lambda+m)/2+p}\dfrac{T_{\lambda,p}}{\Gamma\left(\dfrac{\lambda+m}{2}+p\right)}, & \lambda\neq -m-2p-2l \\[2ex] T_{-m-2p-2l,p}^{*}=\dfrac{(-1)^p l!\pi^{m/2-l}}{2^{2p+2l}(p+l)!\Gamma\left(\dfrac{m}{2}+p+l\right)}P_p(\underline{x})\underline{\partial}^{2p+2l}\delta(\underline{x}), & l\in\mathbb{N}_0 \end{cases}$$

$$\begin{cases} U_{\lambda,p}^{*}=\pi^{(\lambda+m+1)/2+p}\dfrac{U_{\lambda,p}}{\Gamma\left(\dfrac{\lambda+m+1}{2}+p\right)}, & \lambda\neq -m-2p-2l-1 \\[2ex] U_{-m-2p-2l-1,p}^{*}=\dfrac{(-1)^{p+1} l!\pi^{m/2-l}}{2^{2p+2l+1}(p+l)!\Gamma\left(\dfrac{m}{2}+p+l+1\right)}(\underline{\partial}^{2p+2l+1}\delta(\underline{x}))P_p(\underline{x}), & l\in\mathbb{N}_0 \end{cases}$$

$$\begin{cases} V^*_{\lambda,p} = \pi^{(\lambda+m+1)/2+p} \dfrac{V_{\lambda,p}}{\Gamma\left(\dfrac{\lambda+m+1}{2}+p\right)}, & \lambda \neq -m-2p-2l-1 \\[2ex] V^*_{-m-2p-2l-1,p} = \dfrac{(-1)^{p+1} l! \pi^{m/2-l}}{2^{2p+2l+1}(p+l)!\Gamma\left(\dfrac{m}{2}+p+l+1\right)} P_p(\underline{x})(\underline{\partial}^{2p+2l+1}\delta(\underline{x})) \end{cases}$$

$$\begin{cases} W^*_{\lambda,p} = \pi^{(\lambda+m)/2+p} \dfrac{W_{\lambda,p}}{\Gamma\left(\dfrac{\lambda+m}{2}+p\right)}, & \lambda \neq -m-2p-2l \\[2ex] W^*_{-m-2p-2l,p} = \dfrac{(-1)^{p+1} l! \pi^{m/2-l}}{2^{2p+2l+2}(p+l+1)!\Gamma\left(\dfrac{m}{2}+p+l+1\right)} \underline{x} P_p(\underline{x}) \underline{x}\, \underline{\partial}^{2p+2l+2}\delta(\underline{x}) \end{cases}$$

where now, for any $p \in \mathbb{N}_0$, the mapping $\lambda \mapsto T^*_{\lambda,p}$ is an entire function of λ, and similarly for the other families.

For a detailed study of the intra- and interrelationships between these families of distributions, we refer to—in chronological order—[13–15, 30].

4. HILBERT TRANSFORMS IN CLIFFORD ANALYSIS

4.1. The classical Hilbert transform in Clifford analysis

In this section we recall the definition and some important properties of the Hilbert transform in $\mathbb{R}^m$ in the framework of Clifford analysis, see e.g. [12].

First we pass to $(m+1)$-dimensional space by introducing an additional basis vector e_0 which follows the usual multiplication rules, i.e. $e_0^2 = -1$ and it anti-commutes with the other basis vectors, viz. $e_0 e_j + e_j e_0 = 0$, $j = 1, \ldots, m$. The variable $x = (x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1}$ is then identified with the vector $x = \sum_{j=0}^m e_j x_j$ in the Clifford algebra $\mathbb{R}_{0,m+1}$. Furthermore, the Dirac operator in $\mathbb{R}^{m+1}$ then reads $\partial = e_0 \partial_{x_0} + \underline{\partial}$.

For a suitable function or distribution f, its Hilbert transform in $\mathbb{R}^m$ is defined as

$$\mathscr{H}[f](\underline{x}) = \overline{e}_0 (H(\cdot) * f(\cdot))(\underline{x}) \tag{1}$$

with H the convolution kernel given by

$$H(\underline{x}) = \frac{2}{a_{m+1}} Pv \frac{\overline{\underline{\omega}}}{r^m} = -\frac{2}{a_{m+1}} U^*_{-m,0}$$

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the last equality being shown in [13] and thus embedding the Hilbert kernel into the $\mathscr{U}^*$-family introduced above.

Some important properties of the Hilbert transform (1) are listed below:

- P(1) $\mathscr{H}$ is translation invariant, i.e. $\mathscr{H}[f(\underline{y}-\underline{t})](\underline{x}) = \mathscr{H}[f](\underline{x}-\underline{t})$, $\forall \underline{t} \in \mathbb{R}^m$.
- P(2) H is a homogeneous distribution of degree $(-m)$, which, for a convolution operator is equivalent with its dilation invariance, i.e. $\mathscr{H}[f(a\underline{y})](\underline{x}) = \mathscr{H}[f](a\underline{x})$, $\forall a > 0$.
- P(3) The Fourier symbol $\mathscr{F}[H](\underline{x}) = \mathrm{i}\underline{\omega}$ is a bounded function, which is equivalent with $\mathscr{H}$ being a bounded linear operator on $L_2(\mathbb{R}^m; \mathbb{R}_{0,m+1})$.
- P(4) $\mathscr{H}^2 = 1$.
- P(5) $\mathscr{H}$ is an unitary operator.

An additional property relates the Hilbert transform to the Cauchy transform in $\mathbb{R}^{m+1}$, defined by the convolution

$$\mathscr{C}[f](x_0, \underline{x}) = (C(x_0, \cdot) * f(\cdot))(\underline{x})$$

with the Cauchy kernel

$$C(x) = C(x_0, \underline{x}) = \frac{1}{a_{m+1}} \frac{\overline{x} e_0}{|x|^{m+1}} = \frac{1}{a_{m+1}} \frac{x_0 + e_0\underline{x}}{|x_0 + e_0\underline{x}|^{m+1}}$$

which is the fundamental solution of the Cauchy–Riemann operator $D_x = \bar{e}_0 \partial$ in $\mathbb{R}^{m+1}$. From its definition, this Cauchy transform is a monogenic function in $\mathbb{R}^{m+1} \setminus \mathbb{R}^m$. Moreover, one has

- P(6) the Hilbert transform arises in a natural way as part of the boundary value—in L_2 or in distributional sense—of the Cauchy transform of an appropriate function or distribution in $\mathbb{R}^m$, since

$$C(0\pm, \underline{x}) = \pm\tfrac{1}{2}\delta(\underline{x}) + \bar{e}_0 \tfrac{1}{2} H(\underline{x})$$

Finally, as for the one-dimensional Fourier transform, we denote the one-dimensional Hilbert transform as $\mathscr{H}_{y\to x}[f(y)](x)$. Explicitly, it is defined by

$$\mathscr{H}_{y\to x}[f(y)](x) = \mathrm{i}(H(\cdot) * f(\cdot))(x)$$

with convolution kernel

$$H(t) = \frac{1}{\pi} \, Pv \, \frac{1}{t}$$

Here it is directly seen that $\mathbb{R}$ has to be interpreted as being embedded into the complex plane $\mathbb{C}$, since the imaginary unit i in some sense takes over the role of the additional basis vector e_0

(or its conjugate). Note that this minor difference in the definition for $m=1$ obviously also leads to a different Fourier symbol, which in this case is given by $-i\,\mathrm{sgn}(x) = -ix/|x|$.

4.2. *Generalized Hilbert transforms in Clifford analysis*

In [17] two possible generalizations of the Hilbert transform in $\mathbb{R}^m$ have been constructed, based on the families of Clifford distributions presented in Section 3. In this subsection, we will recall their definitions, as well as explicitly mention the properties which they inherit from the classical multidimensional Hilbert transform.

In the first approach, generalized Hilbert kernels are used for the convolution. They constitute a refinement of the generalized Hilbert transforms introduced by Horváth [31], who considered convolution kernels, homogeneous of degree $(-m)$, of the form

$$K(\underline{x}) = Pv\frac{S(\underline{\omega})}{r^m}, \quad r = |\underline{x}|$$

where $S(\underline{\omega})$, $\underline{\omega}\in S^{m-1}$ is a surface spherical harmonic. In order to introduce convolution kernels which are homogeneous of degree $(-m)$, and moreover involve inner and outer spherical monogenics, we consider the following specific distributions in $\mathbb{R}^m$:

$$\begin{aligned}
T_{-m-p,p} &= Fp\frac{1}{r^m}P_p(\underline{\omega}) = Pv\frac{P_p(\underline{\omega})}{r^m}\\
U_{-m-p,p} &= Fp\frac{1}{r^m}\underline{\omega}P_p(\underline{\omega}) = Pv\frac{\underline{\omega}P_p(\underline{\omega})}{r^m}\\
V_{-m-p,p} &= Fp\frac{1}{r^m}P_p(\underline{\omega})\underline{\omega} = Pv\frac{P_p(\underline{\omega})\underline{\omega}}{r^m}\\
W_{-m-p,p} &= Fp\frac{1}{r^m}\underline{\omega}P_p(\underline{\omega})\underline{\omega} = Pv\frac{\underline{\omega}P_p(\underline{\omega})\underline{\omega}}{r^m}
\end{aligned} \tag{2}$$

for which it holds that

$$Pv\frac{S_{p+1}(\underline{\omega})}{r^m} = -\frac{1}{2(p+1)}(U_{-m-p,p} + V_{-m-p,p})$$

$$Pv\frac{\underline{\omega}S_{p+1}(\underline{\omega})}{r^m} = -\frac{1}{2(p+1)}(W_{-m-p,p} - T_{-m-p,p})$$

where $P_p(\underline{x}) = \underline{\partial}S_{p+1}(\underline{x})$, $S_{p+1}(\underline{x})$ being a scalar valued solid spherical harmonic and hence, $P_p(\underline{x})$ is a vector valued homogeneous monogenic polynomial of degree p. The Fourier symbols of these

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distributions, given by

$$\begin{aligned}
\mathscr{F}[T_{-m-p,p}](\underline{x}) &= \mathrm{i}^{-p}\pi^{m/2}\,\frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)}\,P_p(\underline{\omega})\\
\mathscr{F}[U_{-m-p,p}](\underline{x}) &= \mathrm{i}^{-p-1}\pi^{m/2}\,\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{m+p+1}{2}\right)}\,\underline{\omega}P_p(\underline{\omega})\\
\mathscr{F}[V_{-m-p,p}](\underline{x}) &= \mathrm{i}^{-p-1}\pi^{m/2}\,\frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{m+p+1}{2}\right)}\,P_p(\underline{\omega})\underline{\omega}\\
\mathscr{F}[W_{-m-p,p}](\underline{x}) &= \mathrm{i}^{-p-2}\pi^{m/2}\,\frac{p\,\Gamma\left(\frac{p}{2}\right)}{(m+p)\,\Gamma\left(\frac{m+p}{2}\right)}\,\left(\underline{\omega}P_p(\underline{\omega})\,\underline{\omega}-\frac{m-2}{p}\,P_p(\underline{\omega})\right)
\end{aligned} \tag{3}$$

are homogeneous of degree 0 and moreover are bounded functions, whence

$$T_{-m-p,p} * f, \quad U_{-m-p,p} * f, \quad V_{-m-p,p} * f, \quad W_{-m-p,p} * f$$

are bounded singular integral operators on $L_2(\mathbb{R}^m;\mathbb{R}_{0,m+1})$, which are direct generalizations of the Hilbert transform $\mathscr{H}$, preserving (properly adapted analogues of the) properties P(1)–P(3), but not satisfying P(4)–P(6). In particular note that these generalized Hilbert transforms no longer constitute unitary operators.

The second approach is based on the intimate relationship between the Hilbert and the Cauchy transform and starts with the construction of a generalized Cauchy transform in $\mathbb{R}^{m+1}$ involving a distribution from one of the above mentioned families as a generalized Cauchy kernel. A new generalized Hilbert transform in $\mathbb{R}^m$ is then defined as part of the L_2 or distributional boundary values of the generalized Cauchy transform considered; this method leads us to the following convolution operator in $\mathbb{R}^m$:

$$\mathscr{H}_p[f] = \overline{e}_0 H_p * f \tag{4}$$

with kernel

$$H_p(\underline{x}) = \frac{2}{a_{m+1,p}} Fp\frac{\overline{\underline{\omega}}P_p(\underline{\omega})}{r^{m+p}} = -\frac{2}{a_{m+1,p}}U^*_{-m-2p,p}$$

where

$$a_{m+1,p} = \frac{(-1)^p}{2^p}\frac{2\pi^{(m+1)/2}}{\Gamma\left(\frac{m+1}{2}+p\right)}$$

DOI: 10.1002/mma

The Fourier symbol

$$\mathscr{F}[H_p] = -\frac{2}{a_{m+1,p}} \mathrm{i}^{-p-1} U^*_{0,p}$$

of the kernel H_p not being a bounded function, the operator $\mathscr{H}_p$ will not be a bounded operator on $L_2(\mathbb{R}^m; \mathbb{R}_{0,m+1})$. However, the Fourier symbol is seen to be a polynomial of degree p, implying that $\mathscr{H}_p$ is a bounded operator between the Sobolev spaces $W^n_2(\mathbb{R}^m; \mathbb{R}_{0,m+1}) \to W^{n-p}_2(\mathbb{R}^m; \mathbb{R}_{0,m+1})$, for $n \geqslant p$. Moreover, in [17] it is shown that the generalized Hilbert transform $\mathscr{H}_p$ is related to the classical Hilbert transform $\mathscr{H}$ as follows:

$$\mathscr{H}_p[f] = \mathscr{H}[P_p(\underline{\partial}) f] \tag{5}$$

for each function $f \in W^n_2$, with $n \geqslant p$. The operator $\mathscr{H}_p$ only fulfills properties P(1) and P(6) (see [17]).

5. THE CLIFFORD RADON TRANSFORM AND ITS ACTION ON THE CLASSICAL AND GENERALIZED HILBERT TRANSFORMS

In his 1917 paper [32] Johann Radon posed and solved the problem of reconstructing a function of two variables $f(x, y)$ if its integrals over arbitrary lines are given. In that original manuscript, the integral over the line with equation $x\cos(\phi) + y\sin(\phi) = p$, is written as

$$F(p, \phi) = \int_{s=-\infty}^{\infty} f(p\cos(\phi) - s\sin(\phi), p\sin(\phi) + s\cos(\phi))\,\mathrm{d}s$$

which can be rewritten in the following form:

$$F(p, \phi) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} \delta(x\cos(\phi) + y\sin(\phi) - p) f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

Later on, Radon himself also considered analogues of his transform in higher dimensions. The idea of integrating over arbitrary lines was then translated into the more general concept of integrating over arbitrary hyperplanes, leading to the transform which assigns to a given function f defined in $\mathbb{R}^m$ the totality of its integrals over all hyperplanes in $\mathbb{R}^m$. Nowadays, the integral transform $f \mapsto F$ is called the *Radon transform* and this operator is usually denoted by $\mathscr{R}$.

More generally, for a sufficiently smooth function f defined in $\mathbb{R}^m$ and with values in the Clifford algebra $\mathbb{R}_{0,m}$, its Radon transform is defined as

$$\mathscr{R}[f](\underline{n}, s) = \int_{\underline{x} \in \mathbb{R}^m} \delta(\langle \underline{x}, \underline{n} \rangle - s) f(\underline{x})\,\mathrm{d}V(\underline{x})$$

with $(\underline{n}, s) \in S^{m-1} \times \mathbb{R}$ and where $\delta(t)$ denotes the Dirac delta-function (allowing here for a slight abuse of mathematical language).

This Radon transform is linear in two different, yet related senses. Indeed, for suitable $\mathbb{R}_{0,m}$-valued functions $f = \sum_A e_A f_A$ and g in $\mathbb{R}^m$ and real scalars a and b, it is clear that on the one hand

$$\mathscr{R}[af + bg](\underline{n}, s) = a\mathscr{R}[f](\underline{n}, s) + b\mathscr{R}[g](\underline{n}, s)$$

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while on the other also

$$\mathscr{R}[f](\underline{n},s)=\mathscr{R}\left[\sum_A e_A f_A\right](\underline{n},s)=\sum_A e_A \mathscr{R}[f_A](\underline{n},s) \tag{6}$$

A well-known, but important formula, relating the Radon transform to the Fourier transform, is given in the following lemma.

Lemma 5.1
For a real variable u and a unit vector $\underline{n}$ we have

$$\mathscr{F}\{f(\underline{x})\}(u\underline{n})=\mathscr{F}_{s\to u}\{\mathscr{R}[f](\underline{n},s)\}(u) \tag{7}$$

where f is a suitable, sufficiently smooth function.

Proof
We subsequently have

$$\begin{aligned}
\mathscr{F}_{s\to u}\{\mathscr{R}[f](\underline{n},s)\}(u) &= \int_{s=-\infty}^{\infty} \mathscr{R}[f](\underline{n},s)\exp(-2\pi \mathrm{i} us)\,\mathrm{d}s \\
&= \int_{-\infty}^{\infty}\left[\int_{\mathbb{R}^m}\delta(\langle \underline{x},\underline{n}\rangle - s)\, f(\underline{x})\,\mathrm{d}V(\underline{x})\right]\exp(-2\pi \mathrm{i} us)\,\mathrm{d}s \\
&= \int_{\mathbb{R}^m} f(\underline{x})\,\mathrm{d}V(\underline{x}) \int_{-\infty}^{\infty}\delta(\langle \underline{x},\underline{n}\rangle - s)\exp(-2\pi \mathrm{i} us)\,\mathrm{d}s \\
&= \int_{\mathbb{R}^m} f(\underline{x})\, \exp(-2\pi \mathrm{i} u\langle \underline{x},\underline{n}\rangle)\,\mathrm{d}V(\underline{x}) = \mathscr{F}\{f(\underline{x})\}(u\underline{n})
\end{aligned}$$

yielding the desired result. □

This lemma learns that the m-dimensional Fourier transform of a suitable function may be obtained through subsequent application of the Radon transform and the one-dimensional Fourier transform. The result is known as the *central-slice theorem*.

Some other elementary properties of the Radon transform are listed in Lemma 5.2. The proofs for scalar valued functions can be found e.g. in [33], the proofs for $\mathbb{R}_{0,m}$-valued functions running along similar lines, when we keep linearity (6) of the Radon transform in mind.

Lemma 5.2
Let f, f_1, f_2 be sufficiently smooth $\mathbb{R}_{0,m}$-valued functions in $\mathbb{R}^m$, then

(i) $\mathscr{R}[f(\underline{x}+\underline{t})](\underline{n},s)=\mathscr{R}[f(\underline{x})](\underline{n},s+\langle t,\underline{n}\rangle)$
(ii) $\mathscr{R}[\partial_{x_j} f(\underline{x})](\underline{n},s)=n_j\,\partial_s\mathscr{R}[f(\underline{x})](\underline{n},s),\ j=1,\ldots,m$
(iii) $\mathscr{R}[(f_1 * f_2)(\underline{x})](\underline{n},s)=(\mathscr{R}[f_1](\underline{n},\cdot) * \mathscr{R}[f_2](\underline{n},\cdot))(s)$

The equality (iii) above might need some additional explanation in the sense that in the left-hand side an m-dimensional convolution is involved, while in the right-hand side only a one-dimensional

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convolution has to be applied on the oriented distance u, while the role of the unit vector $\underline{n}$ in that convolution is the one of a parameter.

As a consequence of formula (7), in [28] a link was established between the two-dimensional Hilbert transform (therein referred to as Riesz transform) of a two-dimensional signal and the one-dimensional Hilbert transforms of the intrinsically one-dimensional profiles obtained from the Radon transform (see Equation (4.34) in [28]). We have generalized his result to higher dimension in our Clifford context.

Proposition 5.3

For a suitable function f defined on $\mathbb{R}^m$ $(m>1)$ and with values in the Clifford algebra $\mathbb{R}_{0,m}$, we have

$$\mathscr{R}[\mathscr{H}[f]](\underline{n},s)=\mathrm{i}\overline{e}_0\underline{n}\mathscr{H}_{u\to s}[\mathscr{R}[f](\underline{n},u)](s),\quad (\underline{n},s)\in S^{m-1}\times\mathbb{R} \tag{8}$$

The above formula means that the Radon transform of the m-dimensional Clifford-vector valued Hilbert transform evaluated in some hyperplane equals the one-dimensional Hilbert transform of the Radon transform evaluated in the oriented distance of that hyperplane to the origin, up to some factor involving the normal vector to the hyperplane.

We have proven this proposition in two essentially different ways. The first proof below follows a suitable adaptation of the techniques used in [28].

Proof 1

We will show (8) by moving to one-dimensional frequency space where the unit vector $\underline{n}$ plays the role of a parameter. In a first step we make use of relation (7) between the Fourier and the Radon transform, yielding

$$\begin{aligned}\mathscr{F}_{s\to q}\{\mathscr{R}[\mathscr{H}[f]](\underline{n},s)\}(q)&=\mathscr{F}\{\mathscr{H}[f](\underline{x})\}(q\underline{n})\\&=\overline{e}_0\,\mathscr{F}\{(H*f)(\underline{x})\}(q\underline{n})\\&=\overline{e}_0\,(\mathscr{F}\{H(\underline{x})\}\mathscr{F}\{f(\underline{x})\})(q\underline{n})\\&=\overline{e}_0\left(\mathrm{i}\,\frac{q\underline{n}}{|q\underline{n}|}\right)\mathscr{F}\{f\}(q\underline{n})=\overline{e}_0(\mathrm{i}\underline{n}\,\mathrm{sgn}(q))\mathscr{F}\{f\}(q\underline{n})\end{aligned}$$

We proceed by noticing that $(-\mathrm{i})\,\mathrm{sgn}(q)$ is the Fourier symbol of the one-dimensional Hilbert kernel H, and by applying once more (7), now on the Fourier transform of f, which leads us to

$$\begin{aligned}\mathscr{F}_{s\to q}\{\mathscr{R}[\mathscr{H}[f]](\underline{n},s)\}(q)&=-\overline{e}_0\underline{n}(\mathscr{F}_{s\to q}\{H(s)\}\mathscr{F}_{s\to q}\{\mathscr{R}[f](\underline{n},s)\})(q)\\&=-\overline{e}_0\underline{n}\mathscr{F}_{s\to q}\{(H(u)*\mathscr{R}[f](\underline{n},u))(s)\}(q)\end{aligned}$$

So in frequency space we have

$$\mathscr{F}_{s\to q}\{\mathscr{R}[\mathscr{H}[f]](\underline{n},s)\}(q)=\mathrm{i}\overline{e}_0\underline{n}\mathscr{F}_{s\to q}\{\mathscr{H}_{u\to s}[\mathscr{R}[f](\underline{n},u)](s)\}(q)$$

from which formula (8) immediately can be deduced. □

The second proof of Proposition 5.3 heavily leans upon a formula in [21, Chapter 4.3], for the Radon transform of a function consisting of a radial part $g(r)$, multiplied by an angular part $S_p(\underline{\omega})$

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which is a spherical harmonic of degree p, viz.

$$\mathscr{R}[g(r)S_p(\underline{\omega})](\underline{n}, s) = a_{m-1}\frac{S_p(\underline{n})}{C_p^\nu(1)}\int_{|s|}^{+\infty} r^{2\nu} g(r) C_p^\nu\left(\frac{s}{r}\right)\left(1-\frac{s^2}{r^2}\right)^{\nu-1/2} \mathrm{d}r \tag{9}$$

with $\nu = (m-2)/2$, and where

$$C_1^\nu(t) = -\frac{m-2}{m-1}(1-t^2)^{1/2-\nu}\frac{\mathrm{d}}{\mathrm{d}t}[(1-t^2)^{\nu+1/2}] \tag{10}$$

is a Gegenbauer polynomial of degree 1 and order ν, defined by means of Rodriguez's formula. Notice that the required condition $\nu > -\frac{1}{2}$ on the order ν is fulfilled as we take $m>1$. For more information about Gegenbauer polynomials we can refer to e.g. [21, 34].

Proof 2
Since the Hilbert transform is a convolution operator, employing Lemma 5.2(iii) yields

$$\mathscr{R}[\mathscr{H}[f](\underline{x})](\underline{n}, s) = \overline{e}_0(\mathscr{R}[H(\underline{x})](\underline{n}, u) * \mathscr{R}[f(\underline{x})](\underline{n}, u))(s)$$

The Hilbert convolution kernel

$$H(\underline{x}) = \frac{2}{a_{m+1}} Pv\, \frac{\overline{\underline{\omega}}}{r^m} = \frac{2}{a_{m+1}}\overline{\underline{\omega}} Fp\, r_+^{-m}$$

is exactly the product of a radial part, Fpr^{-m}, multiplied by the angular part $\overline{\underline{\omega}}$ which is a spherical harmonic of degree 1 in $\mathbb{R}^m$. This means that $\mathscr{R}[H(\underline{x})](\underline{n}, u)$ can be calculated by means of formula (9). We get

$$\mathscr{R}[H(\underline{x})](\underline{n}, u) = \frac{2}{a_{m+1}} a_{m-1}\frac{\overline{\underline{n}}}{C_1^\nu(1)}\int_{|u|}^{\infty} r^{-2} C_1^\nu\left(\frac{u}{r}\right)\left(1-\frac{u^2}{r^2}\right)^{\nu-1/2} \mathrm{d}r \tag{11}$$

where we were allowed to omit the Fp-notation, since the integration interval stays away from the origin. We now concentrate on the integral at the right-hand side of (11). Let $u>0$, then the substitution $t = u/r$ converts this integral into

$$I = \int_u^\infty r^{-2} C_1^\nu\left(\frac{u}{r}\right)\left(1-\frac{u^2}{r^2}\right)^{\nu-1/2}\mathrm{d}r = \frac{1}{u}\int_0^1 C_1^\nu(t)(1-t^2)^{\nu-1/2}\,\mathrm{d}t$$

which, making use of (10), further reduces to

$$I = -\frac{m-2}{m-1}\frac{1}{u}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}[(1-t^2)^{(m-1)/2}]\,\mathrm{d}t = \frac{m-2}{m-1}\frac{1}{u}$$

This leads to

$$\mathscr{R}[H(\underline{x})](\underline{n}, u) = \overline{\underline{n}}\,\frac{1}{\pi}\frac{1}{u} \tag{12}$$

for $u>0$. For negative u, we may either use the substitution $t = -u/r$ and invoke the well-known property $C_1^\nu(-t) = -C_1^\nu(t)$, or start from the symmetry relation $\mathscr{R}[g](\underline{n}, u) = \mathscr{R}[g](-\underline{n}, -u)$.

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In both cases, we are directly lead to formula (12) as well. Reinterpreting the result of these calculations as a convolution kernel in u, we arrive at

$$\mathscr{R}[H(\underline{x})](\underline{n}, u) = \overline{\underline{n}} \frac{1}{\pi} Pv \frac{1}{u} \quad \text{for all } (\underline{n}, u) \in S^{m-1} \times \mathbb{R} \tag{13}$$

Finally, the proof is completed since $(1/\pi) Pv(1/u)$ is directly recognized as the convolution kernel of the one-dimensional Hilbert transform, yielding

$$\mathscr{R}[\mathscr{H}[f](\underline{x})](\underline{n}, s) = \mathrm{i}\overline{e}_0\underline{n}\left(\mathrm{i}\frac{1}{\pi} Pv \frac{1}{u} * \mathscr{R}[f(\underline{x})](\underline{n}, u)\right)(s) = \mathrm{i}\overline{e}_0\underline{n}\mathscr{H}_{u\to s}[\mathscr{R}[f](\underline{n}, u)](s)$$

which is exactly formula (8). □

In the following proposition, the Radon transform of the generalized Hilbert transforms (2) is calculated. Again, the proofs may be given in two different ways. However, since the ideas are clear, we restrict ourselves to giving the one which uses similar techniques as the first proof of Proposition 5.3.

Proposition 5.4
For a suitable function f defined on $\mathbb{R}^m$ and with values in the Clifford algebra $\mathbb{R}_{0,m}$, we have for $p = p_{\mathrm{e}}$

$$\mathscr{R}[T_{-m-p,p} * f](\underline{n}, s) = \mathrm{i}^{-p}\pi^{m/2} \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)} P_p(\underline{n})\mathscr{R}[f](\underline{n}, s)$$

$$\mathscr{R}[U_{-m-p,p} * f](\underline{n}, s) = \mathrm{i}^{-p-1}\pi^{m/2} \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{m+p+1}{2}\right)} \underline{n} P_p(\underline{n})\mathscr{H}_{u\to s}[\mathscr{R}[f](\underline{n}, u)](s)$$

$$\mathscr{R}[V_{-m-p,p} * f](\underline{n}, s) = \mathrm{i}^{-p-1}\pi^{m/2} \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{m+p+1}{2}\right)} P_p(\underline{n})\underline{n}\mathscr{H}_{u\to s}[\mathscr{R}[f](\underline{n}, u)](s)$$

$$\mathscr{R}[W_{-m-p,p} * f](\underline{n}, s) = \mathrm{i}^{-p-2}\pi^{m/2} \frac{p\Gamma\left(\frac{p}{2}\right)}{(m+p)\,\Gamma\left(\frac{m+p}{2}\right)}$$

$$\times \left(\underline{n}\, P_p(\underline{n})\, \underline{n} - \frac{m-2}{p} P_p(\underline{n})\right) \mathscr{R}[f](\underline{n}, s)$$

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while for $p = p_{\mathrm{e}} + 1$

$$\mathscr{R}[T_{-m-p,p} * f](\underline{n}, s) = \mathrm{i}^{-p} \pi^{m/2} \frac{\Gamma\left(\dfrac{p}{2}\right)}{\Gamma\left(\dfrac{m+p}{2}\right)} P_p(\underline{n}) \mathscr{H}_{u \to s}[\mathscr{R}[f](\underline{n}, u)](s)$$

$$\mathscr{R}[U_{-m-p,p} * f](\underline{n}, s) = \mathrm{i}^{-p-1} \pi^{m/2} \frac{\Gamma\left(\dfrac{p+1}{2}\right)}{\Gamma\left(\dfrac{m+p+1}{2}\right)} \underline{n} P_p(\underline{n}) \mathscr{R}[f](\underline{n}, s)$$

$$\mathscr{R}[V_{-m-p,p} * f](\underline{n}, s) = \mathrm{i}^{-p-1} \pi^{m/2} \frac{\Gamma\left(\dfrac{p+1}{2}\right)}{\Gamma\left(\dfrac{m+p+1}{2}\right)} P_p(\underline{n}) \underline{n} \mathscr{R}[f](\underline{n}, s)$$

$$\mathscr{R}[W_{-m-p,p} * f](\underline{n}, s) = \mathrm{i}^{-p-2} \pi^{m/2} \frac{p\, \Gamma\left(\dfrac{p}{2}\right)}{(m+p)\, \Gamma\left(\dfrac{m+p}{2}\right)} \times \left(\underline{n} P_p(\underline{n}) \underline{n} - \frac{m-2}{p} P_p(\underline{n}) \right) \mathscr{H}_{u \to s}[\mathscr{R}[f](\underline{n}, u)](s)$$

Before giving the proof, we first make some preliminary remarks on the structure of the obtained formulae. First of all, it is worth noticing that the last factor in the above results is affected by the parity of p, which is the degree of homogeneity of the spherical monogenic involved; more specifically, this parity dependence is reflected in the fact that the outcome involves either the Radon transform of the considered function, or the Hilbert transform of that Radon transform. Secondly, if we isolate the parity-dependent part in all formulae, then we easily recognize the remaining factor at the right-hand side to be the Fourier symbol of the convolution kernel of the considered operator (see (3)).

Proof

We only calculate $\mathscr{R}[T_{-m-p,p} * f](\underline{n}, s)$ as the other computations run along similar lines. In frequency space we have

$$\begin{aligned}
\mathscr{F}_{s \to q}\{\mathscr{R}[T_{-m-p,p} * f](\underline{n}, s)\}(q) &= \mathscr{F}\{T_{-m-p,p} * f\}(q\underline{n}) \\
&= (\mathscr{F}\{T_{-m-p,p}\} \mathscr{F}\{f\})(q\underline{n}) \\
&= \mathrm{i}^{-p} \pi^{m/2} \frac{\Gamma\left(\dfrac{p}{2}\right)}{\Gamma\left(\dfrac{m+p}{2}\right)} Fp|q|_{+}^{-p} P_p(q\underline{n}) \mathscr{F}\{f\}(q\underline{n})
\end{aligned}$$

Since the degree of homogeneity of P_p equals p, we easily get

$$Fp|q|_+^{-p}P_p(q\underline{n}) = (\mathrm{sgn}(q))^p P_p(\underline{n})$$

leading to

$$\begin{aligned}
&\mathscr{F}_{s\to q}\{\mathscr{R}[T_{-m-p,p} * f](\underline{n}, s)\}(q)\\
&\quad = \mathrm{i}^{-p}\pi^{m/2}\frac{\Gamma\left(\dfrac{p}{2}\right)}{\Gamma\left(\dfrac{m+p}{2}\right)} P_p(\underline{n})(\mathrm{sgn}(q))^p \mathscr{F}_{s\to q}\{\mathscr{R}[f](\underline{n}, s)\}(q\underline{n})
\end{aligned}$$

As

$$(\mathrm{sgn}(q))^p = \begin{cases} 1 & \text{if } p = p_{\mathrm{e}} \\ \mathrm{sgn}(q) & \text{if } p = p_{\mathrm{e}} + 1 \end{cases}$$

we have to distinguish two cases.

(i) If p is even, then

$$\mathscr{F}_{s\to q}\{\mathscr{R}[T_{-m-p,p} * f](\underline{n}, s)\}(q) = \mathrm{i}^{-p}\pi^{m/2}\frac{\Gamma\left(\dfrac{p}{2}\right)}{\Gamma\left(\dfrac{m+p}{2}\right)} P_p(\underline{n})\mathscr{F}_{s\to q}\{\mathscr{R}[f](\underline{n}, s)\}(q)$$

from which the desired formula follows, taking the one-dimensional inverse Fourier transform, viz.

$$\mathscr{R}[T_{-m-p,p} * f](\underline{n}, s) = \mathrm{i}^{-p}\pi^{m/2}\frac{\Gamma\left(\dfrac{p}{2}\right)}{\Gamma\left(\dfrac{m+p}{2}\right)} P_p(\underline{n})\mathscr{R}[f](\underline{n}, s)$$

(ii) If p is odd, then

$$\begin{aligned}
&\mathscr{F}_{s\to q}\{\mathscr{R}[T_{-m-p,p} * f](\underline{n}, s)\}(q)\\
&\quad = \mathrm{i}^{-p}\pi^{m/2}\frac{\Gamma\left(\dfrac{p}{2}\right)}{\Gamma\left(\dfrac{m+p}{2}\right)} P_p(\underline{n})\,\mathrm{sgn}(q)\mathscr{F}_{s\to q}\{\mathscr{R}[f](\underline{n}, s)\}(q)\\
&\quad = \mathrm{i}^{-p+1}\pi^{m/2}\frac{\Gamma\left(\dfrac{p}{2}\right)}{\Gamma\left(\dfrac{m+p}{2}\right)} P_p(\underline{n})\mathscr{F}_{s\to q}\{H(s)\}(q)\mathscr{F}_{s\to q}\{\mathscr{R}[f](\underline{n}, s)\}(q)\\
&\quad = \mathrm{i}^{-p}\pi^{m/2}\frac{\Gamma\left(\dfrac{p}{2}\right)}{\Gamma\left(\dfrac{m+p}{2}\right)} P_p(\underline{n})\mathscr{F}_{s\to q}\{\mathscr{H}_{u\to s}[\mathscr{R}[f](\underline{n}, u)](s)\}(q)
\end{aligned}$$

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and thus

$$\mathscr{R}[T_{-m-p,p} * f](\underline{n}, s) = \mathrm{i}^{-p}\pi^{m/2} \frac{\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{m+p}{2}\right)} P_p(\underline{n})\mathscr{H}_{u\to s}[\mathscr{R}[f](\underline{n}, u)](s)$$

also yielding the desired result. □

In the following proposition the Radon transform of the generalized Hilbert transforms (4) is calculated.

Proposition 5.5
For a suitable function f defined on $\mathbb{R}^m$ and with values in the Clifford algebra $\mathbb{R}_{0,m}$, we have

$$\mathscr{R}[\mathscr{H}_p[f]](\underline{n}, s) = \mathrm{i}\overline{e}_0\underline{n}P_p(\underline{n})\partial_s^p \mathscr{H}_{u\to s}[\mathscr{R}[f](\underline{n}, u)](s) \tag{14}$$

Proof
We will prove the formula in two different ways.

The first way to prove it, is by means of similar techniques as in the previous proposition. So again, the calculations are done in one-dimensional frequency space

$$\begin{aligned}\mathscr{F}_{s\to q}\{\mathscr{R}[\mathscr{H}_p[f]](\underline{n}, s)\}(q) &= \overline{e}_0\mathscr{F}\{(H_p * f)(\underline{x})\}(q\underline{n}) \\ &= \mathrm{i}^{p+1}(2\pi)^p\overline{e}_0 U_{0,p}(q\underline{n})\mathscr{F}\{f(\underline{x})\}(q\underline{n})\end{aligned} \tag{15}$$

Note that we may write

$$\begin{aligned}U_{0,p}(q\underline{n}) &= \underline{n}P_p(\underline{n})q^p\,\mathrm{sgn}(q) \\ &= \mathrm{i}\underline{n}P_p(\underline{n})q^p\mathscr{F}_{s\to q}\{H(s)\}(q) \\ &= \mathrm{i}^{-p+1}(2\pi)^{-p}\underline{n}P_p(\underline{n})\mathscr{F}_{s\to q}\left\{\frac{\mathrm{d}^p}{\mathrm{d}s^p}H(s)\right\}(q)\end{aligned} \tag{16}$$

Substituting (16) into (15) then yields

$$\begin{aligned}&\mathscr{F}_{s\to q}\left\{\mathscr{R}[\mathscr{H}_p[f]](\underline{n}, s)\right\}(q) \\ &\quad = \mathrm{i}^2\overline{e}_0\underline{n}P_p(\underline{n})\left(\mathscr{F}_{s\to q}\left\{\frac{\mathrm{d}^p}{\mathrm{d}s^p}H(s)\right\}\mathscr{F}_{s\to q}\{\mathscr{R}[f](\underline{n}, s)\}\right)(q) \\ &\quad = \mathrm{i}\overline{e}_0\underline{n}P_p(\underline{n})\mathscr{F}_{s\to q}\{\partial_s^p\mathscr{H}_{u\to s}[\mathscr{R}[f](\underline{n}, u)](s)\}(q)\end{aligned}$$

from which formula (14) can be deduced taking the one-dimensional inverse Fourier transform.

The second way to prove the proposition is through a direct calculation, leaning on the one hand on the relation (5) between the generalized Hilbert transform $\mathscr{H}_p$ and the classical one $\mathscr{H}$, and on the other hand on Lemma 5.2(ii). For a function $f \in W_2^n(\mathbb{R}^m; \mathbb{R}_{0,m+1})$ we have

$$\mathscr{R}[\mathscr{H}_p[f]](\underline{n}, s) = \mathscr{R}[\mathscr{H}[P_p(\underline{\partial})f]](\underline{n}, s) = \mathrm{i}\overline{e}_0\underline{n}\mathscr{H}_{u\to s}[\mathscr{R}[P_p(\underline{\partial})f](\underline{n}, u)](s) \tag{17}$$

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where in the last step we made use of (8). Now invoking the linearity (6) of the Radon transform, the fact that $P_p(\underline{x})$ is a vector valued homogeneous polynomial of degree p and Lemma 5.2(ii), we are lead to

$$\mathscr{R}[P_p(\underline{\partial})f](\underline{n},u) = P_p(\underline{n})\partial_u^p \mathscr{R}[f](\underline{n},u) \tag{18}$$

Substituting (18) in (17) gives

$$\begin{aligned}\mathscr{R}[\mathscr{H}_p[f]](\underline{n},s) &= \mathrm{i}\overline{e}_0\underline{n}\mathscr{H}_{u\to s}[P_p(\underline{n})\partial_u^p\mathscr{R}[f](\underline{n},u)](s)\\ &= \mathrm{i}\overline{e}_0\underline{n}P_p(\underline{n})\partial_s^p\mathscr{H}_{u\to s}[\mathscr{R}[f](\underline{n},u)](s)\end{aligned}$$

□

6. AN ILLUSTRATIVE EXAMPLE

A classical way to calculate the Hilbert transform is to determine first the Fourier transform of that Hilbert transform. As the Hilbert transform is a convolution operator whose Fourier symbol is known, one only needs to compute the Fourier transform of the function involved, leading to

$$\mathscr{F}\{\mathscr{H}[f]\}(\underline{x}) = \mathrm{i}\overline{e}_0\underline{\omega}\mathscr{F}\{f\}(\underline{x}) \quad \text{if } m>1 \tag{19}$$

$$\mathscr{F}\{\mathscr{H}[f]\}(t) = \mathrm{sgn}(t)\mathscr{F}\{f\}(t) \quad \text{if } m=1 \tag{20}$$

Then of course, the Hilbert transform itself can be recovered by applying the inverse Fourier transform on both sides of either (19) or (20), depending on the dimension m. At first sight this construction seems laborious, but it can really save you a lot of time in numerous cases.

In the following example we calculate the Hilbert transform, employing this classical method. However, we now also possess an alternative way to determine the Hilbert transform, making use of its connection (8) with the Radon transform. The results of both approaches being identical, this confirms the theoretical results of the previous sections.

Example 6.1
Consider the following function f defined in $\mathbb{R}^m$ $(m>1)$:

$$f(\underline{x}) = \exp(2\pi\mathrm{i}\langle\underline{a},\underline{x}\rangle) = \cos(2\pi\langle\underline{a},\underline{x}\rangle) + \mathrm{i}\sin(2\pi\langle\underline{a},\underline{x}\rangle)$$

with $\underline{a}$ a constant non-zero vector in $\mathbb{R}^m$.

First we calculate the Hilbert transform $\mathscr{H}[f]$ of f, making use of the classical method described above. In this specific example the tensorial character of both the Fourier kernel and the function f, reduces the calculations of the Fourier transform $\mathscr{F}\{f\}$ of the function f to the one-dimensional level. The outcome is then that $\mathscr{F}\{f\}$ has a peak in $\underline{a}$ and is zero elsewhere

$$\mathscr{F}\{f(\underline{x})\}(\underline{y}) = \delta(\underline{y}-\underline{a}) \tag{21}$$

Next, following (19), the Fourier transform of the Hilbert transform gives

$$\mathscr{F}\{\mathscr{H}[f](\underline{x})\}(\underline{y}) = \mathrm{i}\overline{e}_0\frac{\underline{a}}{|\underline{a}|}\delta(\underline{y}-\underline{a}) \tag{22}$$

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Finally, applying the inverse Fourier transform on both sides of (22), leads to the desired Hilbert transform

$$
\begin{aligned}
\mathcal{H}[f](\underline{x}) &= \mathcal{F}^{-1}\{\mathcal{F}\{\mathcal{H}[f]\}(\underline{y})\}(\underline{x}) \\
&= \mathrm{i}\overline{e}_0 \frac{\underline{a}}{|\underline{a}|} \mathcal{F}^{-1}\{\delta(\underline{y}-\underline{a})\}(\underline{x}) \\
&= \mathrm{i}\overline{e}_0 \frac{\underline{a}}{|\underline{a}|} \exp(2\pi\mathrm{i}\langle\underline{a},\underline{x}\rangle)
\end{aligned}
$$

Next, we pass to the alternative approach, where the Hilbert transform is computed by means of the relation (8)

$$\mathcal{R}[\mathcal{H}[f]](\underline{n},s) = \mathrm{i}\overline{e}_0\underline{n}\mathcal{H}_{u\to s}[\mathcal{R}[f](\underline{n},u)](s)$$

Indeed, exploiting the central-slice theorem (7) twice and leaning upon (20), the Fourier transform of the Hilbert transform can be determined as follows:

$$
\begin{aligned}
\mathcal{F}\{\mathcal{H}[f](\underline{x})\}(v\underline{n}) &= \mathcal{F}_{s\to v}\{\mathcal{R}[\mathcal{H}[f]](\underline{n},s)\}(v) \\
&= \mathrm{i}\overline{e}_0\underline{n}\,\mathrm{sgn}(v)\mathcal{F}_{s\to v}\{\mathcal{R}[f](\underline{n},s)\}(v) \\
&= \mathrm{i}\overline{e}_0\underline{n}\,\mathrm{sgn}(v)\mathcal{F}\{f(\underline{x})\}(v\underline{n}) \\
&= \mathrm{i}\overline{e}_0\underline{n}\,\mathrm{sgn}(v)\delta(v\underline{n}-\underline{a})
\end{aligned}
\tag{23}
$$

where in the last line we used (21).

So $\mathcal{F}\{\mathcal{H}[f](\underline{x})\}(v\underline{n})=0$ if $v\underline{n}-\underline{a}\neq\underline{0}$. From the geometrical point of view, this means that if the vectors $\underline{a}$ and $\underline{n}$ have a different direction, then $\mathcal{F}\{\mathcal{H}[f](\underline{x})\}(v\underline{n})=0$. On the other hand $\mathcal{F}\{\mathcal{H}[f](\underline{x})\}(v\underline{n})$ has a peak if $v\underline{n}-\underline{a}=\underline{0}$. We can summarize this as follows:

$$\mathcal{F}\{\mathcal{H}[f](\underline{x})\}(v\underline{n})\neq 0 \Longleftrightarrow v\underline{n}-\underline{a}=\underline{0} \Longleftrightarrow v=\langle\underline{a},\underline{n}\rangle \quad \text{and} \quad \underline{a}\wedge\underline{n}=0$$

Hence, the factor $\mathrm{sgn}(v)$ can be rewritten as

$$\mathrm{sgn}(v)=\mathrm{sgn}(\langle\underline{a},\underline{n}\rangle)=\frac{\langle\underline{a},\underline{n}\rangle}{|\langle\underline{a},\underline{n}\rangle|}=-\underline{n}\frac{\underline{a}}{|\underline{a}|} \tag{24}$$

Eventually, substitution of (24) in (23) gives

$$\mathcal{F}\{\mathcal{H}[f](\underline{x})\}(v\underline{n})=\mathrm{i}\overline{e}_0\frac{\underline{a}}{|\underline{a}|}\,\delta(\underline{y}-\underline{a})$$

which corresponds with (22).

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