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A necessary and sufficient condition for a Bedrosian identity

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We present a sufficient and necessary condition for the Bedrosian identity to hold for a large class of mono-components based on a generalized Sinc-function. Copyright © 2009 John Wiley & Sons, Ltd.

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1. Introduction

The Hilbert transform is a well-known and useful concept in harmonic analysis and signal processing. Originally defined for periodic functions, it was soon extended to functions defined on the whole real line. This transform solves the Riemann–Hilbert problem for a real-valued function by providing (up to a constant) its harmonic conjugate. Furthermore, it stands as a standard example of a singular integral operator and it is commonly described on the real line via a convolution with the Cauchy kernel.

Recently, the work of N. E. Huang brought a renewed interest in this transform (for a description of his method and several of its applications, we refer to [1]). Based on a pure empirical method, the so-called Huang–Hilbert transform is capable of analyzing nonlinear and non-stationary signals. Shortly speaking, the method relies on obtaining, from the Hilbert transform

$$y(t) = \frac{1}{\pi} p \cdot v \cdot \int_{-\infty}^{+\infty} \frac{x(\tau)}{\tau - t} \mathrm{d}\tau$$

of a given L_p-function $x = x(t)$, the amplitude and phase of the nonlinear and non-stationary signal x. Hereby, one obtains a (complex-valued) function which can be written as $z(t) = x(t) + \mathrm{i}y(t) = \rho(t)\mathrm{e}^{\mathrm{i}\theta(t)}$, where $\rho = \rho(t)$ represents its amplitude and $\theta = \theta(t)$ its phase function. From this last function, we can obtain (locally) the instantaneous frequency $\omega(t) = \mathrm{d}\theta/\mathrm{d}t$. However, previous works of Cohen [2] showed that such a definition only makes physical sense when the function $x = x(t)$ is a mono-component, that is to say, whenever its (non-negative) analytic instantaneous frequency is well defined. Several authors have studied the problem of admissible mono-components and the corresponding properties of its associated phases (see, for instance, the work of Qian, Chen, Li, among many others [3–5]).

Another problem lies in the question of what functions ρ represent the amplitude of a given function $\mathrm{e}^{\mathrm{i}\theta(t)}$. A key point here is the Bedrosian theorem (1965) [6–9], which states that whenever the respective frequency domains of two functions f, g are non-intersecting and the frequency of g is higher than the one of f, then one has $H(fg) = fH(g)$. Usually, the Bedrosian theorem is used to ensure that the identity

$$H(\rho(t)\cos(\theta(t))) = \rho(t)H(\cos(\theta(t)))$$

holds. This identity, known as Bedrosian identity, characterizes the mono-component $x(t) = \rho(t)\cos(\theta(t))$.

Unfortunately, there exists a large class of functions which are mono-components (or IMF's, in Huang's terminology) for which Bedrosian theorem does not apply since $\rho(t)$ and $\cos(\theta(t))$ have intersecting frequency domains. Therefore, the problem of

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determining under which conditions a Bedrosian identity holds is an important problem. The authors would like to remark that there exist already several such results for some special cases (see, e.g. [10, 11], among others). In this paper, we present conditions for the Bedrosian identity to hold for a large class of mono-components, which we obtain by considering the boundary-value of a single Blaschke product, i.e. a function which is of the form $e^{i\theta_a(t)}=(e^{it}-a)/(1-ae^{it})$.

The paper is organized as follows: the second section is dedicated to establishing the main results necessary for the desired identities: we depart from the one side ladder function filter H_a^+ to construct a Fourier pair (H_a^+, r) which is latter on used in order to obtain the Hilbert transform of a generalized Sinc-function. Moreover, we also study the frequency behavior of each scale of the generalized Sinc-function based on the properties of the Hilbert transform. Finally, in Sections 3 and 4 we give our main results, Theorem 3.4 and Theorem 4.4, where we establish sufficient and necessary conditions, respectively, under which the Bedrosian identity holds with respect to our nonlinear Fourier atoms.

2. Bedrosian identity for nonlinear Fourier atoms

In this section we will resort to the generalized Sinc-function (see [4]) in order to prove Bedrosian identities involving nonlinear Fourier atoms. Shortly speaking, and as stated before, we wish to determine and study solutions of the equation

$$\mathscr{H}(\rho\cos\theta_a)(t)=\rho(t)\mathscr{H}(\cos\theta_a)(t), \quad t\in\mathbb{R} \tag{1}$$

where $\cos\theta_a(t)=\mathrm{Re}((e^{it}-a)/(1-ae^{it}))$, for $-1<a<1$.

To start our study we show that the generalized Sinc-function is a special solution of the above equation.

Theorem 2.1
The generalized Sinc function $\mathrm{Sinc}_a(t):=p_a(t)\sin t/t=\sin\theta_a(t)/t$ satisfies Equation (1), that is to say,

$$\mathscr{H}(\mathrm{Sinc}_a(\cdot)\cos\theta_a(\cdot))(t)=\mathrm{Sinc}_a(t)\mathscr{H}(\cos\theta_a)(t)=\frac{\sin^2\theta_a(t)}{t}, \quad t\in\mathbb{R} \tag{2}$$

Moreover, for any l^2 sequence $c=\{c_k\}$, the function

$$\rho(t)=\sum_{k\in\mathbb{Z}}c_k\frac{\sin\theta_a(t-2k\pi)}{t-2k\pi}=\sum_{k\in\mathbb{Z}}c_k\mathrm{Sinc}_a(t-2k\pi), \quad t\in\mathbb{R}$$

satisfies the Equation (1).

Let us remark that since the Hilbert transform is translation invariant and $\cos\theta_a$, $\sin\theta_a$ are both 2π periodic we get that if f satisfies the Bedrosian identity, then 2π translations of f satisfy the identity as well. Therefore, to prove this theorem, it is enough to prove relation (2). To this end we consider the so-called one-sided ladder shape filter

$$H_a^+(t)=a^{\lfloor|t|\rfloor}\chi_{\mathbb{R}_+}(t):=\sum_{k=0}^{\infty}a^k\chi_{[k,k+1)}(t)=(1-a)\sum_{k=1}^{\infty}a^{k-1}\chi_{[0,k)}(t)$$

This function appears when we calculate the Fourier transform of the generalized Sinc-function. We know that Sinc_a can be represented as the multiplication of periodic Poisson kernel p_a with the usual Sinc-function. Note that p_a has the expansion $p_a(t)=\sum_{k=-\infty}^{\infty}a^{|k|}e^{ikt}$. It is now easy to see that Sinc_a can be represented as a weighted summation of modulations of the usual Sinc, which means the Fourier transform of Sinc_a is a piecewise constant and exponential decaying function. This suggests us to consider the piecewise constant function H_a^+ restricted to the positive real line.

Denoting the Fourier transform of f by $\hat{f}$ or $\mathscr{F}[f]$, as usual, we will show that H_a^+ and the function

$$r(t)=\frac{1}{\sqrt{2\pi}}\frac{1}{1-ae^{it}}\frac{1-e^{it}}{-it}$$

form a pair under the Fourier transform.

Lemma 2.2
The one-sided ladder shape filter satisfies

$$\mathscr{F}[H_a^+](\xi)=r(-\xi)=\frac{1}{\sqrt{2\pi}}\frac{1}{1-ae^{-i\xi}}\frac{1-e^{-i\xi}}{i\xi}$$

and

$$\mathscr{F}[r](\xi)=\mathscr{F}\left(\frac{1}{\sqrt{2\pi}}\frac{1}{1-ae^{i\cdot}}\frac{1-e^{i\cdot}}{-i\cdot}\right)(\xi)=H_a^+(\xi)$$

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Proof
A straightforward calculation leads to

$$\begin{aligned}
\mathscr{F}[H_a^+](\xi) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H_a^+(t)\mathrm{e}^{-\mathrm{i}\xi t}\,\mathrm{d}t = \frac{1-a}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sum_{k=1}^{\infty} a^{k-1}\chi_{[0,k)}(t)e^{-i\xi t}dt \\
&= \frac{1-a}{\sqrt{2\pi}}\sum_{k=1}^{\infty} a^{k-1}\int_0^k \mathrm{e}^{-\mathrm{i}\xi t}\,\mathrm{d}t = \frac{1-a}{\sqrt{2\pi}}\sum_{k=1}^{\infty} a^{k-1}\frac{1-\mathrm{e}^{-\mathrm{i}k\xi}}{\mathrm{i}\xi} \\
&= \frac{1-a}{\sqrt{2\pi}}\left(\frac{1}{1-a}-\frac{\mathrm{e}^{-\mathrm{i}\xi}}{1-a\mathrm{e}^{-\mathrm{i}\xi}}\right)\Big/(\mathrm{i}\xi) = \frac{1}{\sqrt{2\pi}}\frac{1-a\mathrm{e}^{-\mathrm{i}\xi}-(1-a)\mathrm{e}^{-\mathrm{i}\xi}}{\mathrm{i}\xi(1-a\mathrm{e}^{-\mathrm{i}\xi})} \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{1-a\mathrm{e}^{-\mathrm{i}\xi}}\frac{1-\mathrm{e}^{-\mathrm{i}\xi}}{\mathrm{i}\xi}
\end{aligned}$$

This proves the first identity. The second one follows from an application of the inverse Fourier transform. □

Now, we consider the Möbius transform $\tau_a(z)=(z-a)(1-az)^{-1}$, with $a\in(-1,1)$. Its real and imaginary part restricted to the unit circle give us the following representation of the nonlinear Fourier atoms $\cos\theta_a(t)$ and $\sin\theta_a(t)$.

Lemma 2.3
For a real number $a\in(-1,1)$, the following equations hold:

$$\sin\theta_a(t) = p_a(t)\sin t = \frac{(1-a^2)\sin t}{1-2a\cos t+a^2},\quad t\in\mathbb{R}$$

$$\cos\theta_a(t) = \frac{(1+a^2)\cos t-2a}{1-2a\cos t+a^2},\quad t\in\mathbb{R}$$

The above two lemmas allow us to give a formula for the Hilbert transform of the generalized Sinc-function.

Lemma 2.4
The Hilbert transform of Sinc_a is

$$\mathscr{H}(\mathrm{Sinc}_a)(t) = \frac{1-\cos\theta_a(t)}{t},\quad t\in\mathbb{R} \tag{3}$$

Proof
From [4] we have the following result by Qian that

$$(\mathscr{H}\sin\theta_a)(t) = -\cos\theta_a(t)-a$$

This leads to

$$\begin{aligned}
\mathscr{H}(\mathrm{Sinc}_a(\cdot))(t) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\theta_a(y)}{y}\frac{\mathrm{d}y}{t-y} \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty}\sin\theta_a(y)\frac{1}{t}\left(\frac{1}{y}+\frac{1}{t-y}\right)\mathrm{d}y \\
&= \frac{1}{t}\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\theta_a(y)}{y}\,\mathrm{d}y+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\theta_a(y)}{t-y}\,\mathrm{d}y\right) \\
&= \frac{1}{t}(-\mathscr{H}(\sin\theta_a)(0)+\mathscr{H}(\sin\theta_a)(t)) = \frac{1}{t}(-(-\cos\theta_a(0))-\cos\theta_a(t)) \\
&= \frac{1-\cos\theta_a(t)}{t}
\end{aligned}$$

□

In the same way we obtain the Hilbert transform of the function $\sin(2\theta_a(t))/2t$.

Lemma 2.5
The following formula holds:

$$\mathscr{H}\left(\frac{\sin(2\theta_a(\cdot))}{2\cdot}\right)(t) = \frac{\sin^2\theta_a(t)}{t},\quad t\in\mathbb{R} \tag{4}$$

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This leaves us in condition to prove Theorem 2.1.

Proof of Theorem 2.1
Set $\rho=\mathrm{Sinc}_a$. On one hand, using (4) the left-hand side of (1) equals to

$$\mathscr{H}(\mathrm{Sinc}_a(\cdot)\cos\theta_a(\cdot))(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\theta_a(y)}{y}\cos\theta_a(y)\frac{\mathrm{d}y}{t-y}=\mathscr{H}\left(\frac{\sin(2\theta_a(\cdot))}{2\cdot}\right)(t)=\frac{\sin^2\theta_a(t)}{t},\quad t\in\mathbb{R}$$

On the other hand, using again Qian's result (see [4]) that $\mathscr{H}\cos\theta_a(t)=\sin\theta_a(t)$ the right-hand side of (1) gives

$$\mathrm{Sinc}_a(t)\mathscr{H}(\cos\theta_a)(t)=\frac{\sin^2\theta_a(t)}{t},\quad t\in\mathbb{R}$$

□

3. Sufficient condition for Bedrosian identity

In this section we describe a sufficient condition for the Bedrosian identity to hold in case of our non-linear Fourier atoms. To that effect, we remark that since $(f+\mathrm{i}Hf)\mathrm{e}^{\mathrm{i}\theta_a}$ is an analytic signal we have the following theorem.

Theorem 3.1
The following equality holds:

$$\mathscr{H}[(\mathscr{H}\,\mathrm{Sinc}_a)(\cdot)\cos\theta_a(\cdot)](t)=(\mathscr{H}\,\mathrm{Sinc}_a)(t)(\mathscr{H}\cos\theta_a)(t)=(\mathscr{H}\,\mathrm{Sinc}_a)(t)\sin\theta_a(t),\quad t\in\mathbb{R}\tag{5}$$

for $a\in(-1,+1)$.

This theorem together with Theorem 2.1 allows us to state the following sufficient condition for the Bedrosian identity.

Theorem 3.2
For arbitrary sequence pairs $c=\{c_k\}$ and $d=\{d_k\}$ in $l^2(\mathbb{Z})$, the function

$$\rho(t)=\sum_{k\in\mathbb{Z}}c_k\mathrm{Sinc}_a(t-2k\pi)+\sum_{k\in\mathbb{Z}}d_k(\mathscr{H}\,\mathrm{Sinc}_a)(t-2k\pi)\tag{6}$$

satisfies Equation (1).

Let us now characterize the space of such functions. To this end we need the following identity:

$$\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}a^{l-1}a^{m-1}\min\{l,m\}=\frac{1}{(1-a)^3(1+a)}$$

This identity allows us to characterize the space $L_2(\mathbb{R})$ in terms of shifts of our generalized Sinc-function.

Theorem 3.3
The system $\{\sqrt{(1-a)/\pi(1+a)}\mathrm{Sinc}_a(\cdot-n\pi):n\in\mathbb{Z}\}$ is an orthonormal system in $L^2(\mathbb{R})$.

Proof
We know that

$$\mathrm{Sinc}_a(\pi t)=p_a(\pi t)\,\mathrm{Sinc}(\pi t)=(1-a^2)\sum_{l=1}^{\infty}a^{l-1}\frac{\sin l\pi t}{\pi t}$$

as well as the form of its Fourier transform

$$\mathscr{F}[\mathrm{Sinc}_a](\xi)=(1-a^2)\sum_{l=1}^{\infty}a^{l-1}\frac{1}{\sqrt{2\pi}}\chi_{[-l\pi,l\pi]}(\xi)$$

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Therefore, we get that, for any integer n, it holds

$$\int_{-\infty}^{\infty}\sqrt{\frac{1-a}{1+a}}\mathrm{Sinc}_a(\pi t)\sqrt{\frac{1-a}{1+a}}\mathrm{Sinc}_a(\pi t-n)\,\mathrm{d}t$$

$$=\frac{1-a}{1+a}\frac{(1-a^2)^2}{2\pi}\int_{-\infty}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}a^{l-1}a^{m-1}\chi_{[-l\pi,l\pi]}(\xi)\chi_{[-m\pi,m\pi]}(\xi)\mathrm{e}^{\mathrm{i}n\xi}\,\mathrm{d}\xi$$

$$=\frac{(1-a)^3(1+a)}{2\pi}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}a^{l-1}a^{m-1}\int_{[-l\pi,l\pi]\cap[-m\pi,m\pi]}\mathrm{e}^{\mathrm{i}n\xi}\,\mathrm{d}\xi$$

$$=(1-a)^3(1+a)\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}a^{l-1}a^{m-1}\min\{l,m\}\delta_{n,0}=\delta_{n,0} \qquad \square$$

We remark that while in Theorem 3.3 all entire shifts of our generalized Sinc-function appear, in Theorem 3.2 only even shifts are considered. Furthermore, we would like to point out that, while the above system is an orthonormal one, it is not a basis of $L_2(\mathbb{R})$. Thus, Theorem 3.2 will not be true for the whole of $L_2(\mathbb{R})$. This leads us to the following conclusion.

Theorem 3.4
Any function $\rho\in\mathrm{span}\left\{\sqrt{(1-a)/\pi(1+a)}\mathrm{Sinc}_a(\cdot-n\pi):n\in\mathbb{Z}\right\}$ with $\rho((2k+1)\pi)=0$, for all $k\in\mathbb{Z}$, fulfils (1).

4. Necessary condition for the Bedrosian identity

While in the previous section we obtain a sufficient condition we will now show that it also necessary, i.e. we want to show that any function that satisfies (1) is necessarily of type (6). To facilitate our discussion, we introduce two subspaces of $L^2(\mathbb{R})$

$$S_1=\{f\in L^2(\mathbb{R}):f \text{ satisfies } (1)\}$$

$$S_2=\{f\in L^2(\mathbb{R}):f \text{ satisfies } (6)\}$$

From Theorem 3.2, we know that $S_2\subset S_1$. Now we will prove $S_1=S_2$. The outline of our proof is the following: taking a function $f\in S_1$ which satisfies

$$\langle f,\mathrm{Sinc}_a(\cdot-2k\pi)\rangle=0,\quad \langle f,[\mathscr{H}\,\mathrm{Sinc}_a](\cdot-2k\pi)\rangle=0,\quad k\in\mathbb{Z} \tag{7}$$

we show that $f=0$.

First, we need to establish two additional facts.

Lemma 4.1
Suppose that f satisfies (7) and (1). Then $f(2k\pi)=0$ and $[\mathcal{H}f](2k\pi)=0$ for any integer $k\in\mathbb{Z}$.

Proof
From condition (7) and the 2π-periodicity of $\sin\theta_a$ and $\cos\theta_a$ we get that

$$\int_{-\infty}^{\infty}f(t)\frac{\sin(\theta_a(t-2k\pi))}{t-2k\pi}\,\mathrm{d}t=-\int_{-\infty}^{\infty}\frac{f(t)\sin(\theta_a(t))}{2k\pi-t}\,\mathrm{d}t=-\mathscr{H}[f(\cdot)\sin(\theta_a(\cdot))](2k\pi)=0,\quad k\in\mathbb{Z}$$

Using Bedrosian identity (1), we get

$$f(2k\pi)\cos(\theta_a(2k\pi))=-\mathscr{H}[f(\cdot)\sin(\theta_a(\cdot))](2k\pi)=0,\quad k\in\mathbb{Z}$$

From this we conclude that $f(2k\pi)=0$ since by Lemma 2.3 we know that $\cos(\theta_a(2k\pi))\neq 0$.

We would like to remark that $f\perp[\mathscr{H}\,\mathrm{Sinc}_a](\cdot-2k\pi)$ implies $\mathscr{H}f\perp\mathrm{Sinc}_a(\cdot-2k\pi)$. Therefore, similar arguments can be used in order to conclude that $[\mathscr{H}f](2k\pi)=0$. The proof of this lemma is completed. $\square$

Lemma 4.2
Suppose that a real-valued $f\in L^2(\mathbb{R})$ satisfies (1). Then the Fourier transform of f satisfies the equations

$$\hat{f}(\xi-n)=a^n\hat{f}(\xi)\quad \text{a.e. } \xi\in(-\infty,0),\quad n\in\mathbb{Z}_+ \tag{8}$$

and

$$\hat{f}(\xi+n)=a^n\hat{f}(\xi)\quad \text{a.e. } \xi\in(0,+\infty),\quad n\in\mathbb{Z}_+ \tag{9}$$

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Proof

By Theorem 3.1, we know that Equation (1) leads to $\mathscr{H}(f(\cdot)e^{i\theta_a(\cdot)})(t)=-if(t)e^{i\theta_a(t)}$. Writing this equation in Fourier domain we obtain

$$\mathscr{F}[f(\cdot)e^{i\theta_a(\cdot)}](\xi)=0 \quad \text{a.e. } \xi\in(-\infty,0)$$

which implies

$$\sum_{k\in\mathbb{Z}} c_k(e^{i\theta_a(\cdot)})\hat{f}(\xi-k)=0 \quad \text{a.e. } \xi\in(-\infty,0)$$

Here $c_k(e^{i\theta_a(\cdot)})$ denotes the kth Fourier coefficient of $e^{i\theta_a(\cdot)}$, which is, by a direct calculation

$$c_k(e^{i\theta_a(\cdot)})=\begin{cases}-a, & k=0\\ (1-a^2)a^{k-1}, & k\geqslant 1\\ 0, & k<0\end{cases}$$

Therefore, we get

$$a\hat{f}(\xi)=(1-a^2)\sum_{k=1}^{\infty}a^{k-1}\hat{f}(\xi-k) \quad \text{a.e. } \xi\in(-\infty,0) \tag{10}$$

It is easy to check that

$$\hat{f}(\xi-1)=a\hat{f}(\xi) \quad \text{a.e. } \xi\in(-\infty,0)$$

so that, by induction, we arrive at (8). In order to verify (9), we consider the conjugate function $f(\cdot)e^{-i\theta_a(\cdot)}$, which has only negative frequencies, i.e.,

$$\mathscr{F}[f(\cdot)e^{-i\theta_a(\cdot)}](\xi)=0 \quad \text{a.e. } \xi\in(0,+\infty)$$

A similar reasoning in terms of $c_k(e^{-i\theta_a(\cdot)})$ leads to

$$a\hat{f}(\xi)=(1-a^2)\sum_{k=-\infty}^{-1}a^{|k|-1}\hat{f}(\xi-k)=(1-a^2)\sum_{k=1}^{\infty}a^{|k|-1}\hat{f}(\xi+k) \quad \text{a.e. } \xi\in(0,\infty)$$

Using induction, we complete the proof of this lemma. □

From the above two lemmas, we obtain the following theorem.

Theorem 4.3

Suppose that a real-valued $f\in L^2(\mathbb{R})$ satisfies (7) and (1). Then $f=0$.

Proof

Using Theorem 3.1 and applying the Poisson summation formula to the function $\hat{f}$, we get

$$\sum_{k\in Z}\hat{f}(\xi+k)=\sum_{k\in Z}f(-2k\pi)e^{i2k\pi\xi}=0 \quad \text{a.e. } \xi\in\mathbb{R}$$

Now, we suppose that $\xi>0$. Applying (9), we observe that

$$\sum_{k\in Z}\hat{f}(\xi+k)=\hat{f}(\xi)+\sum_{k\geqslant 1}\hat{f}(\xi+k)+\sum_{k\leqslant -1}\hat{f}(\xi+k)=\hat{f}(\xi)+\hat{f}(\xi)\sum_{k\geqslant 1}a^k+\sum_{k\leqslant -1}\hat{f}(\xi+k)=\frac{\hat{f}(\xi)}{1-a}+\sum_{k\leqslant -1}\hat{f}(\xi+k)$$

This means

$$\frac{\hat{f}(\xi)}{1-a}+\sum_{k=1}^{\infty}\hat{f}(\xi-k)=0 \quad \text{a.e. } \xi\in(0,+\infty) \tag{11}$$

Starting from (11), we establish a new iterative relation for $\hat{f}$. For a fixed non-negative integer m, suppose that $\xi\in(m,m+1)$. We can rewrite the above series (11) as

$$\sum_{k=1}^{\infty}\hat{f}(\xi-k)=\sum_{k=1}^{m}\hat{f}(\xi-k)+\sum_{k=m+1}^{\infty}\hat{f}(\xi-k)=\sum_{k=1}^{m}\hat{f}((\xi-m)+(m-k))+\sum_{k=m+1}^{\infty}\hat{f}((\xi-m-1)-(k-m-1))$$

Let us remark that $\xi-m\in(0,1)$ so, applying (9) and (8) to the above identity, we obtain

$$\sum_{k=1}^{\infty}\hat{f}(\xi-k)=\hat{f}(\xi-m)\sum_{k=1}^{m}a^{m-k}+\hat{f}(\xi-m-1)\sum_{k=m+1}^{\infty}a^{k-m-1}=\frac{1-a^m}{1-a}\hat{f}(\xi-m)+\frac{\hat{f}(\xi-m-1)}{1-a}$$

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Substituting this into (11) we arrive at

$$\hat{f}(\xi)+(1-a^m)\hat{f}(\xi-m)+\hat{f}(\xi-m-1)=0, \quad \xi\in(m,m+1) \tag{12}$$

In a similar way we get for $\mathscr{H}f$

$$(\mathscr{H}f)^\wedge(\xi)+(1-a^m)(\mathscr{H}f)^\wedge(\xi-m)+(\mathscr{H}f)^\wedge(\xi-m-1)=0$$

that is to say,

$$\operatorname{sgn}(\xi)\hat{f}(\xi)+(1-a^m)\operatorname{sgn}(\xi-m)\hat{f}(\xi-m)+\operatorname{sgn}(\xi-m-1)\hat{f}(\xi-m-1)=0$$

for all $\xi\in(m,m+1)$. This leads to

$$\hat{f}(\xi)+(1-a^m)\hat{f}(\xi-m)-\hat{f}(\xi-m-1)=0, \quad \xi\in(m,m+1) \tag{13}$$

Now, comparing relations (12) and (13), we immediately obtain

$$\hat{f}(\xi-m-1)=0, \quad \xi\in(m,m+1) \tag{14}$$

and

$$\hat{f}(\xi)+(1-a^m)\hat{f}(\xi-m)=0, \quad \xi\in(m,m+1) \tag{15}$$

Using Lemma 4.2, we get

$$\hat{f}(\xi-m)=a^m\hat{f}(\xi-m)+(1-a^m)\hat{f}(\xi-m)=\hat{f}((\xi-m)+m)+(1-a^m)\hat{f}(\xi-m)=\hat{f}(\xi)+(1-a^m)\hat{f}(\xi-m), \quad \xi\in(m,m+1)$$

so that

$$\hat{f}(\xi-m)=0, \quad \xi\in(m,m+1) \tag{16}$$

Substituting (16) and (14) into (12) or (13), we obtain

$$\hat{f}(\xi)=0, \quad \xi\in(m,m+1) \tag{17}$$

Finally, since m is arbitrary and by the hermitian property of f that $\hat{f}(-\cdot)=\bar{\hat{f}}(\cdot)$, we conclude $\hat{f}(\xi)=0$ for $\xi\in\mathbb{R}$ almost everywhere. This completes the proof of this theorem. □

Summing up the above statements we get the following theorem.

Theorem 4.4
For any real-valued function $f\in L^2(\mathbb{R})$ and $-1<a<1$ the Bedrosian identity

$$\mathscr{H}(f\cos\theta_a)(t)=f(t)\mathscr{H}(\cos\theta_a)(t), \quad t\in\mathbb{R}$$

holds if and only if there exist sequence pairs $c=\{c_k\}$ and $d=\{d_k\}$ in $l^2(\mathbb{Z})$ such that f has the form

$$f(t)=\sum_{k\in\mathbb{Z}}c_k\,\mathrm{Sinc}_a(t-2k\pi)+\sum_{k\in\mathbb{Z}}d_k(\mathscr{H}\,\mathrm{Sinc}_a)(t-2k\pi)$$

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