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Theory of bianisotropic crystal lattices

E. O. Kamenetskii
Faculty of Engineering, Department of Electrical Engineering–Physical Electronics, Tel Aviv University, Tel-Aviv 69978, Israel
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Bianisotropic media may be characterized as a general class of linear media which exhibit so-called magnetoelectric coupling between the electric and magnetic fields. Bianisotropic composites are attracting considerable attention in view of their potential usefulness and new fundamental problems. A bianisotropic crystal lattice is one of the interesting structures of such materials. In this paper, we developed approaches for an analysis of static and dynamical models of bianisotropic crystal lattices. The static model is based on the Lorenz-Lorentz theory. The dynamical theory is based on the use of the so-called sampling theorem similar to the approach developed by the author for dielectric crystals. [S1063-651X(98)08303-2]

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I. INTRODUCTION

Currently, in electromagnetics and material sciences, the macroscopic magnetoelectric interaction of fields and materials is one of the most interesting topics. Bianisotropic materials may be characterized as a general class of media which exhibit magnetoelectric coupling between the electric and magnetic fields. This class of materials usually includes chiral (or reciprocal biisotropic) media, the so-called Tellegen (or nonreciprocal biisotropic) medium, and natural magnetoelectric crystals [1–4]. Artificial chiral materials have been developed to demonstrate the phenomenon of electromagnetic activity at microwave frequencies, analogous to optical activity. Together with artificial chiral materials (based on the composition of small helices), an idea of microwave bianisotropic materials based on Ω and chiro-Ω particles has been introduced [5,6].

General properties of a large variety of known bianisotropic materials may be analyzed on the basis of effective constitutive parameters. A special feature of continuous chiral media is that we introduce constitutive relationships which connect the macroscopic field vectors, but not separately from macroscopic Maxwell equations. It becomes clear if we consider the so-called Drude-Born-Fedorov constitutive relations for chiral media

$$\vec{\mathcal{D}}=\epsilon[\vec{E}+\beta\vec{\nabla}\times\vec{E}],\quad \vec{B}=\mu[\vec{H}+\beta\vec{\nabla}\times\vec{H}]. \tag{1}$$

One can see that the polarization depends not only on $\vec{E}$ but also on $\vec{\nabla}\times\vec{E}$, likewise, the magnetization depends on $\vec{H}$ as well as on $\vec{\nabla}\times\vec{H}$. The chirality parameter β is, at the same time, a measure of nonlocal effects. Obviously, constitutive parameters of chiral media are not described by quasistatic functions [1–3].

To analyze particulate chiral or Ω composites, it is supposed that the maximum scale of material nonhomogeneity has to be much less than distances of macroscopic field variations. Thus, the medium resembles a continuum rather than a diffraction grating. Let us consider a separate chiral particle immersed into a dielectric host material. It was shown in [7] that for the far-zone scattered fields and low frequency approximation, a small chiral sphere may be considered as a combination of an electric dipole and a magnetic dipole. But, what do we have in the near field zone? Let us now take such a chiral particle as a helix [2,3]. Can one separate and show where are the regions of electric polarization and magnetization? It is impossible, since a chiral particle is described by electrodynamical (not quasistatic) models. An analysis of tiny point-polarizable spheres strung on helical strands, made in [8] shows that ''the microstructure size should be large enough that the electromagnetic (EM) wave in the matrix can appreciate the handedness of the microstructure.'' We have the same situation for Ω particles. To analyze small Ω scatterers, wire-and-loop antenna models can be used. Such particles demonstrate properties of polarizabilities only for the far-zone fields [9]. In the quasistatic limit, chiral and Ω particle composites do not exhibit magnetoelectric coupling between the electric and magnetic fields and demonstrate properties of dielectric or magnetodielectric media.

Macroscopic electrodynamics of dielectric or magnetic continuous media is based on the fact that phenomenologically defined specific properties of a medium may be described separately from macroscopic Maxwell's equations. There are constitutive relations based on, for example, the motion equations of charges in plasma, the motion equations of magnetization in ferromagnetics, and the Lorenz-Lorentz model for dielectrics [1,10,11]. These relations, in fact, are quasistatic relations which may be described as integral-form constitutive relations. For dielectric media, we have [1]

$$\mathcal{D}_i(\vec{r},t)=\int_{-\infty}^{t}dt'\int d\vec{r}'\,\epsilon_{ij}(t,\vec{r},t',\vec{r}')E_j(t',\vec{r}'), \tag{2}$$

Here, only the causality principle (that the electric displacement $\vec{\mathcal{D}}$ at the time t is defined by the electric field $\vec{E}$ at the time $t'<t$) is taken into account. For a time-invariant and spatially homogeneous medium, the kernel $\epsilon_{ij}(t',\vec{r}')$ may be interpreted as a ''response'' of a medium to the field action described by the Dirac δ function.

Integral-form constitutive relations, similar to relation (2), were introduced recently for bianisotropic media [12,13]. These integral-form relations have, however, a formal char-

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acter, until one can show that bianisotropic media with quasistatic constitutive relations really do exist.

A class of bianisotropic composites with quasistatic constitutive relations was recently conceptualized in Refs. [14, 15] (Lakhtakia suggested naming media described in [14] *magnetostatically controlled bianisotropic materials* (MCBMs) [16]). The MCBMs are particulate composites based on magnetostatic-wave (MSW) resonators. Each MSW resonator with surface metallization may be considered as a bianisotropic particle. Induced electric and magnetic dipole moments $\vec{p}$ and $\vec{m}$ of the particle are related to the external electric and magnetic fields. A mutual orientation of vectors $\vec{p}$ and $\vec{m}$ depends on geometries of the ferromagnetic resonator and the region of metallization and depends on orientation of the bias magnetic field as well [14,15]. The main point is that dyadic polarizabilities of a bianisotropic particle in MCBMs are obtained on the basis of the solution of quasistatic problems. Each particle may be considered as a glued pair of two (magnetic and electric) dipoles: the magnetic dipole is due to the ferrite body and the electric dipole is due to the metallization region.

In macroscopic electrodynamics of dielectric crystal lattices, the quasistatic Lorenz-Lorentz theory is used. The main idea of this theory is that the actual Coulomb field in the crystal lattice is different than the macroscopic field [10]. The Lorenz-Lorentz theory is also used to describe artificial dielectrics which are modeled as a triple infinite periodic array of identical dipolar scattering elements in some homogeneous and isotropic host media. Every dipolar scatterer is considered as a δ-function source [17]. Now a question arises: may one use the Lorenz-Lorentz theory in macroscopic electrodynamics of artificial bianisotropic crystals? Obviously, it may be possible when every bianisotropic particle is described quasistatically and is considered as a δ-functional dipolar scatterer. One can see that in the MCBMs we have bianisotropic particles which satisfy these conditions.

It is usually supposed that to use macroscopic Maxwell equations, one has to get over the discrete structure of a medium by the averaging procedure. In their dynamical theory of dielectric crystal lattices, Born and Huang used Ewald's method in providing a way of separating the macroscopic field from the actual Coulomb field [18]. It was supposed that due to dipole-dipole interaction, the lattice could be imagined as a polarized continuum with a small perturbation of dielectric polarization. In this paper, we will develop another approach for the dynamical model of bianisotropic crystal lattices.

Because of the limiting cutoff wave numbers, all variables in macroscopic electrodynamics are finite-spectrum functions [19]. This makes it possible to discretize the fields and to use the so-called sampling theorem for a dielectric medium modeled as a triple infinite periodic array of identical δ-functional scattering elements [20]. Taking into account the Lorenz-Lorentz theory, the dynamical theory of strong field fluctuations in dielectric crystal lattices was developed in the preceding paper [20]. The method used in [20] also becomes important for the dynamical model of bianisotropic crystal lattices when every bianisotropic particle may be considered as δ-functional dipolar scatterer. To develop the dynamical theory of bianisotropic crystal lattices, we will reject the continuum model of bianisotropic materials. The theory will be based on field discretization taking into account the discrete structure of media.

The paper is organized as follows. In Sec. II, we give a brief description of MCBMs with the objective of showing that this kind of bianisotropic material may be described *quasistatically*. We also describe some examples of bianisotropic particles based on MSW resonators and show what kind of constitutive dyadics may be obtained for particulate bianisotropic composites. In Sec. III, we use the Lorenz-Lorentz model for bianisotropic crystal lattices as a material continuum. In Secs. IV, V, and VI we show how the theory in [20] may be extended to bianisotropic materials. These sections are devoted to the use of the sampling theorem in macroscopic electrodynamics of bianisotropic crystal lattices. Section VII contains concluding remarks.

II. MAGNETOSTATISICALLY CONTROLLED BIANISOTROPIC MATERIALS

Electromagnetics of bianisotropic materials hold the key to many important technologies. On microwaves, these bianisotropic materials are composite materials. The main feature of the known bianisotropic composites (based on helices or Ω particles) is the first order role played by the size parameters qa in the emergence of the magnetoelectric properties (here a is the particle size and q is the wave number in the host material). For this reason, the electric and magnetic fields are not curl free away from the particle and the quasistatic effective-medium theories may be applicable only for dilute composites. In other words, such media are modeled as a gas of scatterers. Much needs to be done, however, before these bianisotropic composites come to be used in microwave applications [2,3,5].

Since in a class of bianisotropic composite materials—the MCBMs—every particle is described quasistatically, the effective-medium theories for dense homogenized materials may be successfully used. In this case, we have ''solid state matter'' in comparison with ''gas matter'' based on a composition of helices or Ω particles. A vast number of applications is emerging from future theoretical and experimental works based on these composites. Manufacturing composite materials is based on modern planar technology of ferromagnetic devices [14,15].

The MCBMs are particulate composites based on MSW resonators with surface metallization. Because of essential temporal dispersion of the permeability in ferromagnetics, quasistatic oscillations of magnetization take place in a certain frequency region. One can use a quasimagnetostatic approximation ($\vec{h} \simeq -\vec{\nabla}\psi$) to describe these oscillations. In order to obtain magnetostatic oscillations at certain frequencies in the microwave regime, the characteristic linear dimensions of a ferromagnetic body have to be much less than the free-space wavelength at the same frequency [1,11,21].

Magnetostatic oscillations may be described by two first-order-differential-operator equations [22]. The first equation is

$$\mathcal{L}V = O, \tag{3}$$

where

$$\mathcal{L}=\begin{pmatrix} (\mu_0\overleftrightarrow{\mu})^{-1} & \vec{\nabla} \\ -\vec{\nabla}\cdot & 0 \end{pmatrix} \tag{4}$$

is the differential-matrix operator,

$$V=\begin{pmatrix} \vec{b} \\ \psi \end{pmatrix} \tag{5}$$

is the vector-function of magnetic flux density

$$\vec{b}=-\mu_0\overleftrightarrow{\mu}\vec{\nabla}\psi \tag{6}$$

and magnetostatic potential ψ, and $\overleftrightarrow{\mu}$ is the tensor of the permeability.

The second equation has the form

$$\mathcal{M}\vec{U}=0, \tag{7}$$

where

$$\mathcal{M}=\begin{pmatrix} 0 & -\vec{\nabla}\times \\ \vec{\nabla}\times & i\mu_0\overleftrightarrow{\mu} \end{pmatrix} \tag{8}$$

is Maxwell's operator for the magnetostatic limit,

$$\vec{U}=\begin{pmatrix} \vec{e} \\ \vec{h} \end{pmatrix} \tag{9}$$

is the vector-function of the electric and magnetic fields (here $\vec{h}$ is the potential magnetic field and $\vec{e}$ is the vortex electric field).

It was shown in [22] that Eqs. (3) and (7) describe eigenvalue problems for the magnetostatic waves and, on the basis of these equations, one obtains the excitation equations for magnetostatic modes in ferromagnetic films. In particular, the inhomogeneous differential equation based on operator $\mathcal{L}$ gives the excitation of the MSW by the external magnetic field, and the excitation equation based on differential operator $\mathcal{M}$ gives the excitation (in ferromagnetic films with surface metallization) by the external electric field.

Each MSW resonator with surface metallization may be considered as a quasistatic bianisotropic particle. Induced electric and magnetic dipole moments of the particle are related to the external electric $\vec{E}$ and magnetic $\vec{H}$ fields as follows:

$$\vec{p}=\overleftrightarrow{\alpha}_{ee}\cdot\vec{E}+\overleftrightarrow{\alpha}_{em}\cdot\vec{H};\quad \vec{m}=\overleftrightarrow{\alpha}_{me}\cdot\vec{E}+\overleftrightarrow{\alpha}_{mm}\cdot\vec{H}. \tag{10}$$

One obtains the dyadic polarizabilities $\overleftrightarrow{\alpha}_{me}$ and $\overleftrightarrow{\alpha}_{mm}$ (and, thus, the magnetic dipole moment $\vec{m}$) as a result of integration of magnetization of ferrite $\vec{m}_F$ over the volume of the ferrite body of the resonator. The magnetization of ferrite is related to the magnetostatic potential as [11,21]

$$\vec{m}_F=-(\overleftrightarrow{\mu}-\overleftrightarrow{I})\vec{\nabla}\psi, \tag{11}$$

where $\overleftrightarrow{I}$ is the unit matrix. The dyadic polarizabilities $\overleftrightarrow{\alpha}_{ee}$ and $\overleftrightarrow{\alpha}_{em}$ and, as a result, the electric dipole moment $\vec{p}$, are found on the basis of an integration of the surface charge density τ_S over the surface of the region of metallization. The electric charge density is related to the surface electric current as

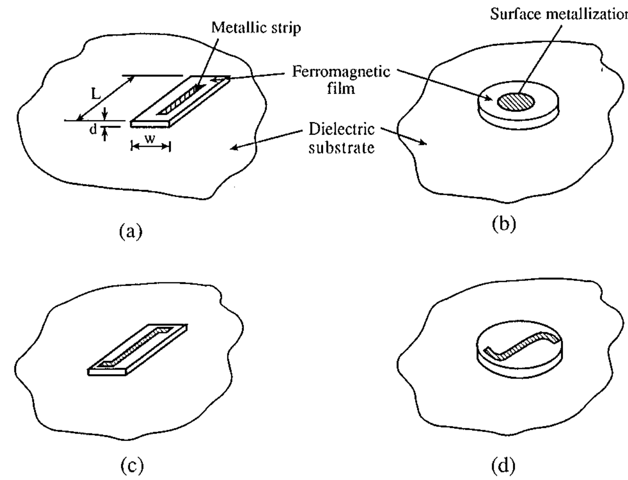

FIG. 1. MSW resonators with surface metallization regions.

$$\vec{\nabla}_S\cdot\vec{j}^{\,e}_S=-i\omega\tau_S, \tag{12}$$

where $\vec{\nabla}_S\cdot$ is the two-dimensional (on the plane of surface metallization) divergence. The surface current density is defined by the equation [22]

$$\vec{n}_C\times[(\vec{\nabla}\psi)^{(+)}-(\vec{\nabla}\psi)^{(-)}]=\vec{j}^{\,e}_S, \tag{13}$$

where $\vec{n}_C$ is the unit normal vector to the contour $\mathcal{C}$—the contour of the region of metallization on the resonator cross section—and $(\vec{\nabla}\psi)^{(\pm)}$ are gradients of the potential ψ above and below the contour C. The main point is that the dyadic polarizabilities in Eq. (10) are obtained on the basis of the solution of quasistatic problems: the magnetization $\vec{m}$ is a function of the magnetostatic potential ψ and the surface electric charge density τ_S is also controlled by the MSW process in a resonator.

A mutual orientation of the dipole moments $\vec{p}$ and $\vec{m}$ depends on the geometry of a ferromagnetic resonator, geometry of the region of metallization, and also on the orientation of the bias magnetic field $\vec{H}_0$. When a parallelepiped-form MSW resonator with $L\gg d$, w and a narrow metallic strip [Fig. 1(a)] is used, we have a parallel orientation of induced electric and magnetic dipole moments $(\vec{p}\|\vec{m})$. For a cylindrical MSW resonator with a circular form of the metallization region [Fig. 1(b)], one obtains a perpendicular orientation of the vectors $(\vec{p}\perp\vec{m})$ if the bias field $\vec{H}_0$ is oriented normally with respect to the plane of a ferromagnetic film. MSW resonators shown in Figs. 1(c) and 1(d) exhibit structures with a lack of symmetry due to the geometry of metallization. These resonators may be obtained with two enantiomorphic forms of metallization [15].

Artificial materials based on thin-film MSW resonators have planar structures exhibiting magnetoelectric properties. Since components of the permeability tensor $\overleftrightarrow{\mu}$ in ferrites depend on the field $\vec{H}_0$, magnetoelectric properties of bi-

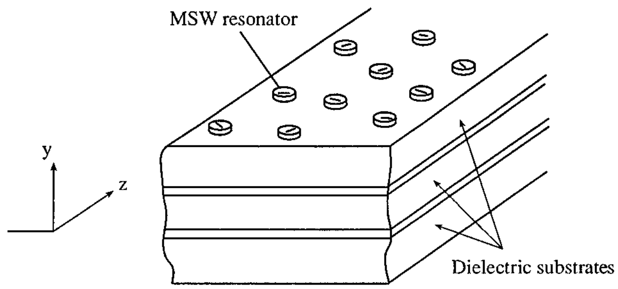

FIG. 2. Bianisotropic composite with randomly distributed MSW resonators.

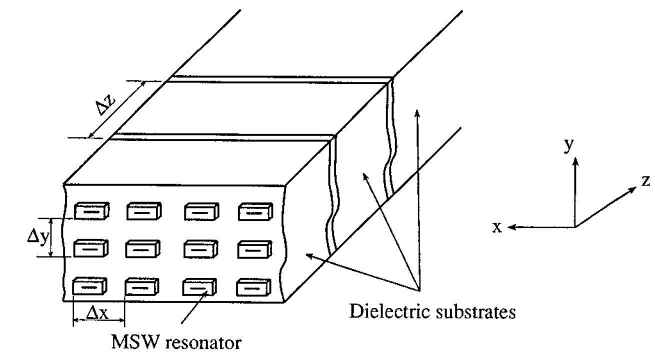

FIG. 3. Bianisotropic composite with aligned MSW resonators (bianisotropic crystal lattice).

anisotropic materials may be tuned via the bias field variation. Figure 2 shows an example of the proposed MCBMs with randomly distributed MSW resonators. The MCBMs are conceived as a pile of dielectric substrates with thin-film ferromagnetic objects. The MSW resonators are depicted schematically without concrete delineation of the resonator shape and the metallization configuration. The particulate composite shown can be homogenized into a material continuum. When noninteracting MSW resonators in a material are assumed, the Maxwell-Garnett approach can be used for estimating the effective constitutive parameters of the MCBM [23–25]:

$$\vec{\mathcal{D}}=\overleftrightarrow{\epsilon}\cdot\vec{E}+\overleftrightarrow{\xi}\cdot\vec{H},$$

$$\vec{B}=\overleftrightarrow{\mu}\cdot\vec{H}+\overleftrightarrow{\zeta}\cdot\vec{E}. \tag{14}$$

The constitutive dyadics are frequency dependent. Let us restrict ourselves here to the qualitative description of some special MCBMs with randomly distributed resonators [15].

(i) Suppose that the MCBM is made by randomly distributing very elongated parallelepiped resonators [Fig. 1(a)], with $\vec{H}_0$ parallel to the y axis (Fig. 2). Assuming the MCBM to be nondissipative and reciprocal bianisotropic media [26], we get

$$\overleftrightarrow{\epsilon}=\begin{bmatrix}\epsilon_1 & 0 & 0\\ 0 & \epsilon_2 & 0\\ 0 & 0 & \epsilon_1\end{bmatrix},\quad \overleftrightarrow{\mu}=\begin{bmatrix}\mu_1 & 0 & 0\\ 0 & \mu_2 & 0\\ 0 & 0 & \mu_1\end{bmatrix},$$

$$\overleftrightarrow{\xi}=i\begin{bmatrix}\xi_1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \xi_1\end{bmatrix},\quad \overleftrightarrow{\zeta}=-\overleftrightarrow{\xi}. \tag{15}$$

(ii) Let us consider the cylindrical MSW resonators of Fig. 1(b) instead, with $\vec{H}_0$ still parallel to the y axis. When nonhomogeneous magnetostatic oscillations take place in the resonators [11,21], the constitutive dyadics of the MCBM take the form

$$\overleftrightarrow{\xi}=i\begin{bmatrix}0 & 0 & \xi_c\\ 0 & 0 & 0\\ \xi_c & 0 & 0\end{bmatrix},\quad \overleftrightarrow{\zeta}=-\overleftrightarrow{\xi}. \tag{16}$$

$\overleftrightarrow{\epsilon}$ and $\overleftrightarrow{\mu}$ have the same form as shown in Eq. (15).

(iii) Finally, suppose that anisotropic resonators depicted in Figs. 1(c) and 1(d) are randomly distributed in the xz planes with $\vec{H}_0$ parallel to the y axis (Fig. 2). Let the MCBM be nondissipative. The constitutive dyadics have the following forms:

$$\overleftrightarrow{\epsilon}=\begin{bmatrix}\epsilon_1' & 0 & i\epsilon_a\\ 0 & \epsilon_2' & 0\\ -i\epsilon_a & 0 & \epsilon_1'\end{bmatrix},\quad \overleftrightarrow{\mu}=\begin{bmatrix}\mu_1' & 0 & i\mu_b\\ 0 & \mu_2' & 0\\ -i\mu_b & 0 & \mu_1'\end{bmatrix},$$

$$\overleftrightarrow{\xi}=\begin{bmatrix}\xi_1' & 0 & \xi_d\\ 0 & 0 & 0\\ \xi_d & 0 & \xi_1'\end{bmatrix},\quad \overleftrightarrow{\zeta}=\overleftrightarrow{\xi}^*, \tag{17}$$

where the asterisk denotes the complex conjugate.

MCBMs made of MSW structures with lack of symmetry [Figs. 1(c) and 1(d)] may be characterized as planar *gyrotropic-chiral* materials. In these materials, gyrotropic effects are combined with geometric effects caused by the lack of reflection symmetry of surface metallization. Such structures are essentially different from the chiroferrite or chiroplasma medium conceived by Engheta *et al.* [27].

We have carried out the qualitative description of some special MCBMs with randomly distributed resonators. Our further consideration will be devoted to quasistatic and dynamical field theories of a three-dimensional regular array of quasistatic bianisotropic particles. The analysis may be used for the calculation of the constitutive parameters of MCBMs with crystal-lattice structures. Figure 3 shows an example of such a structure.

III. THE LORENZ-LORENTZ MODEL FOR BIANISOTROPIC CRYSTAL LATTICES

The Lorenz-Lorentz theory is a static field theory which provides a solution that takes into account only the dipole term in the induced field. The results of this theory are valid only for obstacles with dimensions small compared with their spacing. Being well developed for dielectric media [10,17], the Lorenz-Lorentz theory may be successfully extended to bianisotropic crystal lattices when every bianisotropic particle is described quasistatically and is considered as a δ-functional dipolar scatterer. Crystal lattices based on MSW resonators (or, in other words, the MCBM crystals) show an example of such materials.

Consider a three-dimensional regular array of identical dipolar bianisotropic scattering elements in some homogeneous and isotropic host media characterized by the permittivity ϵ (in a particular case $\epsilon=\epsilon_0$) and the permeability μ_0. The spacing between elements are denoted as Δx, Δy, and Δz. The elements are identified by the integer indices $k,n,l[-\infty<(k,n,l)<\infty]$. Every dipolar scatter is considered as a δ source. Let $\vec{E}^{(e)}$ and $\vec{H}^{(e)}$ be the effective electric and magnetic fields, respectively, acting to polarize and magnetize the particle at the origin. For induced electric and magnetic dipole moments one can rewrite Eq. (10) as

$$\vec{p}=\overleftrightarrow{\alpha}_{ee}\cdot\vec{E}^{(e)}+\overleftrightarrow{\alpha}_{em}\cdot\vec{H}^{(e)},$$
$$\vec{m}=\overleftrightarrow{\alpha}_{me}\cdot\vec{E}^{(e)}+\overleftrightarrow{\alpha}_{mm}\cdot\vec{H}^{(e)}. \tag{18}$$

Let $\vec{E}^{(0)}$ and $\vec{H}^{(0)}$ be the electric and magnetic fields in the host medium. These fields are considered as the external electric and magnetic fields applied to a condensed medium quasistatically modeled as a triple periodic array of dipoles. The effective fields $\vec{E}^{(e)}$ and $\vec{H}^{(e)}$ are equal, respectively, to $\vec{E}^{(0)}+\vec{E}^{(i)}$ and $\vec{H}^{(0)}+\vec{H}^{(i)}$, where $\vec{E}^{(i)}$ and $\vec{H}^{(i)}$ are the interaction electric and magnetic fields. These interaction fields are proportional, respectively, to $\vec{p}$ and $\vec{m}$:

$$\vec{E}^{(i)}=\overleftrightarrow{C}_E\cdot\vec{p},$$
$$\vec{H}^{(i)}=\overleftrightarrow{C}_M\cdot\vec{m}, \tag{19}$$

where tensors $\overleftrightarrow{C}_E$ and $\overleftrightarrow{C}_M$ are the so-called interaction constants. Now we can write

$$\vec{p}=\overleftrightarrow{\alpha}_{ee}\cdot(\vec{E}^{(0)}+\overleftrightarrow{C}_E\cdot\vec{p})+\overleftrightarrow{\alpha}_{em}\cdot(\vec{H}^{(0)}+\overleftrightarrow{C}_M\cdot\vec{m}),$$
$$\vec{m}=\overleftrightarrow{\alpha}_{me}\cdot(\vec{E}^{(0)}+\overleftrightarrow{C}_E\cdot\vec{p})+\overleftrightarrow{\alpha}_{mm}\cdot(\vec{H}^{(0)}+\overleftrightarrow{C}_M\cdot\vec{m}). \tag{20}$$

We obtain two vector equations linking induced dipole moments $\vec{p}$ and $\vec{m}$ with the external applied electric and magnetic fields $\vec{E}^{(0)}$ and $\vec{H}^{(0)}$. This linear system of two vector equations may be inverted to yield

$$\vec{p}=\overleftrightarrow{\nu}_{ee}\cdot\vec{E}^{(0)}+\overleftrightarrow{\nu}_{em}\cdot\vec{H}^{(0)},$$
$$\vec{m}=\overleftrightarrow{\nu}_{me}\cdot\vec{E}^{(0)}+\overleftrightarrow{\nu}_{mm}\cdot\vec{H}^{(0)}, \tag{21}$$

where the dyadics $\overleftrightarrow{\nu}_{ee}$, $\overleftrightarrow{\nu}_{em}$, $\overleftrightarrow{\nu}_{me}$, and $\overleftrightarrow{\nu}_{mm}$ are given by

$$\overleftrightarrow{\nu}_{ee}=\overleftrightarrow{A}_1^{-1}\cdot[\overleftrightarrow{\alpha}_{ee}+\overleftrightarrow{\alpha}_{em}\cdot\overleftrightarrow{C}_M\cdot(\overleftrightarrow{I}-\overleftrightarrow{\alpha}_{mm}\cdot\overleftrightarrow{C}_M)^{-1}\cdot\overleftrightarrow{\alpha}_{me}], \tag{22}$$

$$\overleftrightarrow{\nu}_{em}=\overleftrightarrow{A}_1^{-1}\cdot[\overleftrightarrow{\alpha}_{em}+\overleftrightarrow{\alpha}_{em}\cdot\overleftrightarrow{C}_M\cdot(\overleftrightarrow{I}-\overleftrightarrow{\alpha}_{mm}\cdot\overleftrightarrow{C}_M)^{-1}\cdot\overleftrightarrow{\alpha}_{mm}], \tag{23}$$

$$\overleftrightarrow{\nu}_{me}=\overleftrightarrow{A}_2^{-1}\cdot[\overleftrightarrow{\alpha}_{me}+\overleftrightarrow{\alpha}_{me}\cdot\overleftrightarrow{C}_E\cdot(\overleftrightarrow{I}-\overleftrightarrow{\alpha}_{ee}\cdot\overleftrightarrow{C}_E)^{-1}\cdot\overleftrightarrow{\alpha}_{ee}], \tag{24}$$

$$\overleftrightarrow{\nu}_{mm}=\overleftrightarrow{A}_2^{-1}\cdot[\overleftrightarrow{\alpha}_{mm}+\overleftrightarrow{\alpha}_{me}\cdot\overleftrightarrow{C}_E\cdot(\overleftrightarrow{I}-\overleftrightarrow{\alpha}_{ee}\cdot\overleftrightarrow{C}_E)^{-1}\cdot\overleftrightarrow{\alpha}_{em}], \tag{25}$$

and

$$\overleftrightarrow{A}_1=\overleftrightarrow{I}-\overleftrightarrow{\alpha}_{ee}\cdot\overleftrightarrow{C}_E-\overleftrightarrow{\alpha}_{em}\cdot\overleftrightarrow{C}_M\cdot(\overleftrightarrow{I}-\overleftrightarrow{\alpha}_{mm}\cdot\overleftrightarrow{C}_M)^{-1}\cdot\overleftrightarrow{\alpha}_{me}\cdot\overleftrightarrow{C}_E, \tag{26}$$

$$\overleftrightarrow{A}_2=\overleftrightarrow{I}-\overleftrightarrow{\alpha}_{mm}\cdot\overleftrightarrow{C}_M-\overleftrightarrow{\alpha}_{me}\cdot\overleftrightarrow{C}_E\cdot(\overleftrightarrow{I}-\overleftrightarrow{\alpha}_{ee}\cdot\overleftrightarrow{C}_E)^{-1}\cdot\overleftrightarrow{\alpha}_{em}\cdot\overleftrightarrow{C}_M. \tag{27}$$

The interaction constants $\overleftrightarrow{C}_E$ and $\overleftrightarrow{C}_M$ will be defined below. Now we consider a material-continuum approach for a three-dimensional array. Polarization and magnetization per unit volume are defined as

$$\vec{P}=N\vec{p}=N(\overleftrightarrow{\nu}_{ee}\cdot\vec{E}^{(0)}+\overleftrightarrow{\nu}_{em}\cdot\vec{H}^{(0)}),$$
$$\vec{M}=N\vec{m}=N(\overleftrightarrow{\nu}_{me}\cdot\vec{E}^{(0)}+\overleftrightarrow{\nu}_{mm}\cdot\vec{H}^{(0)}), \tag{28}$$

where $N=1/\Delta V=1/(\Delta x)(\Delta y)(\Delta z)$ is the number of particles per unit volume.

The average electric displacement and the average magnetic flux density are defined by

$$\vec{\mathcal{D}}=\epsilon\vec{E}_a^{(t)}+\vec{P},$$
$$\vec{B}=\mu_0\vec{H}_a^{(t)}+\vec{M}, \tag{29}$$

where $\vec{E}_a^{(t)}$ and $\vec{H}_a^{(t)}$ are the average values of the total electric and magnetic fields. These fields may be represented as

$$\vec{E}_a^{(t)}=\vec{E}^{(0)}+\vec{E}_a^{(p)},$$
$$\vec{H}_a^{(t)}=\vec{H}^{(0)}+\vec{H}_a^{(p)}, \tag{30}$$

where $\vec{E}_a^{(p)}$ and $\vec{H}_a^{(p)}$ are the average values of the electric and magnetic dipole field produced, respectively, by all electric and all magnetic dipoles.

For a bianisotropic particle that is symmetrical about the coordinate planes passing through the center of the particle, the average electric and magnetic fields produced by all induced dipoles is zero. It becomes clear from electrostatic and magnetostatic points of view for the three-dimensional bianisotropic lattice. In each unit cell, the induced electrostatic potential φ and the induced magnetostatic potential ψ may be developed into three-dimensional Fourier series. For electric dipoles, such an analysis was made in [17]. Let, for example, an induced electric dipole have the y component. The induced electrostatic potential φ will be an odd function of y because of the symmetry involved. The y component of the induced electric field is given by derivative $\partial\varphi/\partial y$ and is expressed by cosine terms that depend on y. In result, the average induced electric field in the unit cell is zero. A similar analysis will be made for the x and z components of an induced electric dipole. It is clear that for induced magnetic dipoles, one has a similar magnetostatic problem. If, for example, an induced magnetic dipole has the y component, the induced magnetostatic potential ψ will be an odd function of y. The y component of the induced magnetic field is given by derivative $\partial\psi/\partial y$ and the y component of the average induced magnetic field is zero. The same analysis is possible for the x and z components.

In view of the above consideration and taking Eqs. (28) and (29) into account we can characterize the bianisotropic three-dimensional lattice by the effective constitutive relations

$$\vec{\mathcal{D}}=\overleftrightarrow{\epsilon}_{\rm eff}\cdot\vec{E}^{(0)}+\overleftrightarrow{a}_{\rm eff}\cdot\vec{H}^{(0)},$$
$$\vec{B}=\overleftrightarrow{b}_{\rm eff}\cdot\vec{E}^{(0)}+\overleftrightarrow{\mu}_{\rm eff}\cdot\vec{H}^{(0)}, \tag{31}$$

where

$$\overleftrightarrow{\epsilon}_{\rm eff}=\epsilon\overleftrightarrow{I}+N\overleftrightarrow{\nu}_{ee}\,, \tag{32}$$

$$\overleftrightarrow{a}_{\rm eff}=N\overleftrightarrow{\nu}_{em}\,, \tag{33}$$

$$\overleftrightarrow{b}_{\rm eff}=N\overleftrightarrow{\nu}_{me}\,, \tag{34}$$

$$\overleftrightarrow{\mu}_{\rm eff}=\mu_0\overleftrightarrow{I}+N\overleftrightarrow{\nu}_{mm}\,. \tag{35}$$

The condensed matter quasistatic theory based on the fact that the actual (or, in other words, local) field causing polarization of a particle is different from the macroscopic field, is known as the Clausius-Mossotti equation in connection with electrostatics, or as the Lorenz-Lorentz equation in connection with electromagnetic theory [10,18]. The same approach is used in the Maxwell-Garnett model. The Maxwell-Garnett model originally introduced for simple isotropic spherical scatterers was extended for isotropic chiral and biisotropic composites [7,28]. For bianisotropic composites with randomly dispersed electrically small uniaxial bianisotropic inclusions in the isotropic host material, the Maxwell-Garnett model is available in the literature [24,25]. Our model describes the effective constitutive parameters of bianisotropic crystal lattices with (as a general case) nonuniaxial bianisotropic particles. The only assumptions made so far about the structure of the polarizability dyadics of every particle are that all inverses in Eqs. (22)–(27) exist. An analysis of particular cases whether the necessary inverses exist or not, is beyond the scope of the present consideration.

Another important question also arises in our analysis. There is a question about the coefficients $\overleftrightarrow{C}_E$ and $\overleftrightarrow{C}_M$. The calculation of these coefficients is far from straightforward. We will evaluate the interaction constants below in Sec. VI. It will be shown, in particular, that because of symmetry, the coefficients $\overleftrightarrow{C}_E$ and $\overleftrightarrow{C}_M$ are diagonal tensors.

IV. SPATIAL DISPERSION AND FIELD DISCRETIZATION IN BIANISOTROPIC MATERIALS

As a general description, one can formally introduce the integral-form constitutive relations for bianisotropic media, similarly to relations (2) for dielectrics [12,13]:

$$\begin{aligned}D_i(t,\vec{r})&=\int_{-\infty}^{t}dt'\int d\vec{r}'\,\epsilon_{ij}(t,\vec{r},t',\vec{r}')E_j(t',\vec{r}')\\&\quad+\int_{-\infty}^{t}dt'\int d\vec{r}'\,\xi_{ij}(t,\vec{r},t',\vec{r}')H_j(t',\vec{r}'),\\B_i(t,\vec{r})&=\int_{-\infty}^{t}dt'\int d\vec{r}'\,\zeta_{ij}(t,\vec{r},t',\vec{r}')E_j(t',\vec{r}')\\&\quad+\int_{-\infty}^{t}dt'\int d\vec{r}'\,\mu_{ij}(t,\vec{r},t',\vec{r}')H_j(t',\vec{r}').\end{aligned} \tag{36}$$

[It should be noted that here ϵ_{ij} is not the same as the kernel in Eq. (2).] For time-invariant and spatially homogeneous bianisotropic media, we have

$$\tilde{\vec{\mathcal{D}}}(\omega,\vec{k})=\overleftrightarrow{\epsilon}(\omega,\vec{k})\cdot\tilde{\vec{E}}(\omega,\vec{k})+\overleftrightarrow{\xi}(\omega,\vec{k})\cdot\tilde{\vec{H}}(\omega,\vec{k}), \tag{37}$$

$$\tilde{\vec{B}}(\omega,\vec{k})=\overleftrightarrow{\zeta}(\omega,\vec{k})\cdot\tilde{\vec{E}}(\omega,\vec{k})+\overleftrightarrow{\mu}(\omega,\vec{k})\cdot\tilde{\vec{H}}(\omega,\vec{k}), \tag{38}$$

where tildes denote the Fourier images.

Now the question arises of whether one can obtain bianisotropic media which are described by relations (36)–(38) and for which the long-wavelength limit

$$\begin{aligned}&\overleftrightarrow{\epsilon}(\omega,\vec{k})|_{|\vec{k}|\to 0}=\overleftrightarrow{\epsilon}(\omega),\quad \overleftrightarrow{\xi}(\omega,\vec{k})|_{|\vec{k}|\to 0}=\overleftrightarrow{\xi}(\omega),\\&\overleftrightarrow{\zeta}(\omega,\vec{k})|_{|\vec{k}|\to 0}=\overleftrightarrow{\zeta}(\omega),\quad \overleftrightarrow{\mu}(\omega,\vec{k})|_{|\vec{k}|\to 0}=\overleftrightarrow{\mu}(\omega),\end{aligned} \tag{39}$$

exists. In such a case; the time-domain integral relations [29] may be considered as a particular case of Eq. (36) for the long-wavelength limit.

For a time-invariant and spatially homogeneous medium, the kernels $\epsilon_{ij}(t',\vec{r}')$, $\xi_{ij}(t',\vec{r}')$, $\zeta_{ij}(t',\vec{r}')$ and $\mu_{ij}(t',\vec{r}')$ in Eq. (36) may be interpreted as ''responses'' of the medium to the Dirac δ-functional electric and magnetic fields [30]. The local character of these ''response functions'' shows that constitutive parameters in Eq. (36) have to be obtained on the basis of solution of *quasistatic* problems. As was discussed in Sec. I, not one of the known biisotropic and bianisotropic media (chiral, Tellegen, and Ω media) can be described by local quasistatic parameters. There are, however, the MCBMs which are described *quasistatically*. When the integral-form constitutive relations are not introduced formally, but describe real bianisotropic materials, physical effects, based on relations (36), may be analyzed. It was shown [12], for example, that for the quasimonochromatic electromagnetic field, the energy transport in bianisotropic media is possible if components of complex envelopes satisfy certain differential relations.

When bianisotropic constitutive parameters have the local quasistatic limit, a dynamical model of bianisotropic crystal lattices may be realized. For such a purpose, an approach based on the field discretization developed in [20] for dielectric crystal lattices may be successfully used for bianisotropic crystal lattices.

Let us consider spatially homogeneous bianisotropic media as a linear system of space signal processing (supposing that the causality principle is taken into account). We can rewrite Eq. (36) as

$$D_i(\vec{r})=L_1\left[\int d\vec{r}'\,\delta(\vec{r}-\vec{r}')E_j(\vec{r}')\right]+L_2\left[\int d\vec{r}'\,\delta(\vec{r}-\vec{r}')H_j(\vec{r}')\right], \tag{40}$$

$$B_i(\vec{r})=L_3\left[\int d\vec{r}'\,\delta(\vec{r}-\vec{r}')E_j(\vec{r}')\right]+L_4\left[\int d\vec{r}'\,\delta(\vec{r}-\vec{r}')H_j(\vec{r}')\right], \tag{41}$$

where L_p ($p=1, 2, 3, 4$) are linear operators describing transformation of an input signal (E_j,H_j) into an output signal $(\mathcal{D}_i,B_i)$. Since operators L_p are linear operators, one can consider constitutive tensors in Eq. (36) as the so-called apparatus functions or impulse-response functions [31,32]

$$\epsilon_{ij}(\vec{r}-\vec{r}')\equiv L_1[\delta(\vec{r}-\vec{r}')],\quad \xi_{ij}(\vec{r}-\vec{r}')\equiv L_2[\delta(\vec{r}-\vec{r}')],$$
$$\zeta_{ij}(\vec{r}-\vec{r}')\equiv L_3[\delta(\vec{r}-\vec{r}')],\quad \mu_{ij}(\vec{r}-\vec{r}')\equiv L_4[\delta(\vec{r}-\vec{r}')]. \tag{42}$$

When a bianisotropic medium is considered as a linear system of space signal processing and modeled as a triple infinite periodic array, components of continuous fields may be represented as sampling values on the basis of the procedure shown in [20]. One can correlate the polarization and magnetization of every bianisotropic particle with sampling values of the effective electric and magnetic fields and define sampling values of the electric displacement and the magnetic flux density. All variables in the macroscopic electrodynamics are finite-spectrum functions [19,20]. In our case, this means that the Fourier images of all variables are equal to zero for

$$|q_x|>Q_x,\quad |q_y|>Q_y,\quad |q_z|>Q_z, \tag{43}$$

where $\vec{q}$ is the wave vector in the host material and Q_x,Q_y,Q_z are the limiting cutoff numbers. If spacings between nodes of a lattice are satisfied with the conditions

$$\Delta x\leqslant\frac{1}{2Q_x},\quad \Delta y\leqslant\frac{1}{2Q_y},\quad \Delta z\leqslant\frac{1}{2Q_z}, \tag{44}$$

the sampling theorem enables us to reconstruct the full Fourier spectrum of the electromagnetic field on the basis of discrete field samplings [20,33].

One of the main purposes of the further analysis is to develop a *dynamical model* of bianisotropic crystal lattices. The sampling theorem will be a powerful tool for this purpose.

V. FIELD SAMPLINGS IN BIANISOTROPIC CRYSTAL LATTICES

For the i components of the total electric and magnetic fields, the field samplings are defined as [20]

$$E_{s_i}^{(t)}(\vec{r})=\sum_{l=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}E_{a_i}^{(t)}(k\Delta x,n\Delta y,l\Delta z)\times\delta(x-k\Delta x)\,\delta(y-n\Delta y)\,\delta(z-l\Delta z), \tag{45}$$

$$H_{s_i}^{(t)}(\vec{r})=\sum_{l=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}H_{a_i}^{(t)}(k\Delta x,n\Delta y,l\Delta z)\times\delta(x-k\Delta x)\,\delta(y-n\Delta y)\,\delta(z-l\Delta z). \tag{46}$$

For the i components of electric fields $E_i^{(0)},E_{a_i}^{(p)}$ and the magnetic fields $H_i^{(0)}$ and $H_{a_i}^{(p)}$ in Eq. (30) we have similar expressions denoted, respectively, as $E_{s_i}^{(0)},E_{s_i}^{(p)}$ and $H_{s_i}^{(0)}$, $H_{s_i}^{(p)}$. Because of the linearity of relationships, one has

$$E_{s_i}^{(t)}=E_{s_i}^{(0)}+E_{s_i}^{(p)},\qquad H_{s_i}^{(t)}=H_{s_i}^{(0)}+H_{s_i}^{(p)}. \tag{47}$$

The microscopic electric and magnetic dipole moment densities $\vec{p}_{\mathrm{mic}}$ and $\vec{m}_{\mathrm{mic}}$ are expressed as sequences of δ functions [34] and, therefore, are represented as a series of samplings. For the i component we have

$$(p_{\mathrm{mic}})_i=\sum_{l=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}p_i(k\Delta x,n\Delta y,l\Delta z)\,\delta(x-k\Delta x)\times\delta(y-n\Delta y)\,\delta(z-l\Delta z)\equiv p_{s_i}, \tag{48}$$

$$(m_{\mathrm{mic}})_i=\sum_{l=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}m_i(k\Delta x,n\Delta y,l\Delta z)\,\delta(x-k\Delta x)\times\delta(y-n\Delta y)\,\delta(z-l\Delta z)\equiv m_{s_i}, \tag{49}$$

where $\vec{p}(k\Delta x,n\Delta y,l\Delta z)$ and $\vec{m}(k\Delta x,n\Delta y,l\Delta z)$ are the electric and magnetic dipole moments of the particle characterized by the numbers k,n,l.

Now we define the sampling vectors $\vec{D}_s$ and $\vec{B}_s$ as

$$\vec{\mathcal{D}}_s=\epsilon\vec{E}_s^{(t)}+\vec{P}_s, \tag{50}$$

$$\vec{B}_s=\mu_0\vec{H}_s^{(t)}+\vec{M}_s, \tag{51}$$

where

$$\vec{P}_s=\frac{\vec{p}_s}{\Delta V}, \tag{52}$$

$$\vec{M}_s=\frac{\vec{m}_s}{\Delta V}. \tag{53}$$

All components of vectors in Eqs. (50), (51) are defined similarly to Eqs. (45), (46), and (48), (49). Taking Eq. (47) into account, one obtains

$$\vec{\mathcal{D}}_s=\epsilon[\vec{E}_s^{(0)}+\vec{E}_s^{(p)}]+\vec{P}_s, \tag{54}$$

$$\vec{B}_s=\mu_0[\vec{H}_s^{(0)}+\vec{H}_s^{(p)}]+\vec{M}_s. \tag{55}$$

The induced dipole moments of every particle are expressed by Eq. (18). For the sampling functions we have

$$\vec{p}_s = \overleftrightarrow{\alpha}_{ee}\cdot(\vec{E}_s^{(0)}+\vec{E}_s^{(i)})+\overleftrightarrow{\alpha}_{em}\cdot(\vec{H}_s^{(0)}+\vec{H}_s^{(i)}),$$
$$\vec{m}_s = \overleftrightarrow{\alpha}_{me}\cdot(\vec{E}_s^{(0)}+\vec{E}_s^{(i)})+\overleftrightarrow{\alpha}_{mm}\cdot(\vec{H}_s^{(0)}+\vec{H}_s^{(i)}). \tag{56}$$

In a static model of bianisotropic crystals, we supposed that the interaction fields $\vec{E}^{(i)}$ and $\vec{H}^{(i)}$ are proportional, respectively, to induced dipole moments $\vec{p}$ and $\vec{m}$ [see Eq. (19)]. In a dynamical model, we have to suggest that a more general type of relationships between the interaction fields and induced dipole moments, takes place:

$$\vec{E}^{(i)}=\overleftrightarrow{C}_{ee}\cdot\vec{p}+\overleftrightarrow{C}_{em}\cdot\vec{m},$$
$$\vec{H}^{(i)}=\overleftrightarrow{C}_{me}\cdot\vec{p}+\overleftrightarrow{C}_{mm}\cdot\vec{m}, \tag{57}$$

where coefficients $\overleftrightarrow{C}_{ee}$, $\overleftrightarrow{C}_{em}$, $\overleftrightarrow{C}_{me}$, and $\overleftrightarrow{C}_{mm}$ are dependable on the wave vector $\vec{q}$. For the long-wavelength limit, one has

$$\overleftrightarrow{C}_{ee}\big|_{q\to 0}=\overleftrightarrow{C}_E, \quad \overleftrightarrow{C}_{em}\big|_{q\to 0}=0,$$
$$C_{me}\big|_{q\to 0}=0, \quad C_{mm}\big|_{q\to 0}=\overleftrightarrow{C}_M.$$

It will be shown, however, (see the next section of the paper) that because of symmetry, $\overleftrightarrow{C}_{em}=\overleftrightarrow{C}_{me}=0$ for all values of the wave vector $\vec{q}$. Taking this into account, we can rewrite Eq. (57) for the sampling functions

$$\vec{E}_s^{(i)}=\overleftrightarrow{C}_{ee}\cdot\vec{p}_s,$$
$$\vec{H}_s^{(i)}=\overleftrightarrow{C}_{mm}\cdot\vec{m}_s. \tag{58}$$

Substitution of Eq. (58) into Eq. (56) gives a system of two vector equations linking the sampling functions of induced dipole moments with the sampling functions of the external applied fields:

$$\vec{p}_s=\overleftrightarrow{\alpha}_{ee}\cdot(\vec{E}_s^{(0)}+\overleftrightarrow{C}_{ee}\cdot\vec{p}_s)+\overleftrightarrow{\alpha}_{em}\cdot(\vec{H}_s^{(0)}+\overleftrightarrow{C}_{mm}\cdot\vec{m}_s),$$
$$\vec{m}_s=\overleftrightarrow{\alpha}_{me}\cdot(\vec{E}_s^{(0)}+\overleftrightarrow{C}_{ee}\cdot\vec{p}_s)+\overleftrightarrow{\alpha}_{mm}\cdot(\vec{H}_s^{(0)}+\overleftrightarrow{C}_{mm}\cdot\vec{m}_s). \tag{59}$$

This system of equations may be rewritten as

$$\vec{p}_s=\overleftrightarrow{\beta}_{ee}\cdot\vec{E}_s^{(0)}+\overleftrightarrow{\beta}_{em}\cdot\vec{H}_s^{(0)},$$
$$\vec{m}_s=\overleftrightarrow{\beta}_{me}\cdot\vec{E}_s^{(0)}+\overleftrightarrow{\beta}_{mm}\cdot\vec{H}_s^{(0)}, \tag{60}$$

where

$$\overleftrightarrow{\beta}_{ee}=\overleftrightarrow{F}_1^{-1}\cdot[\overleftrightarrow{\alpha}_{ee}+\overleftrightarrow{\alpha}_{em}\cdot\overleftrightarrow{C}_{mm}\cdot(\overleftrightarrow{I}-\overleftrightarrow{\alpha}_{mm}\cdot\overleftrightarrow{C}_{mm})^{-1}\cdot\overleftrightarrow{\alpha}_{me}], \tag{61}$$

$$\overleftrightarrow{\beta}_{em}=\overleftrightarrow{F}_1^{-1}\cdot[\overleftrightarrow{\alpha}_{em}+\overleftrightarrow{\alpha}_{em}\cdot\overleftrightarrow{C}_{mm}\cdot(\overleftrightarrow{I}-\overleftrightarrow{\alpha}_{mm}\cdot\overleftrightarrow{C}_{mm})^{-1}\cdot\overleftrightarrow{\alpha}_{mm}], \tag{62}$$

$$\overleftrightarrow{\beta}_{me}=\overleftrightarrow{F}_2^{-1}\cdot[\overleftrightarrow{\alpha}_{me}+\overleftrightarrow{\alpha}_{me}\cdot\overleftrightarrow{C}_{ee}\cdot(\overleftrightarrow{I}-\overleftrightarrow{\alpha}_{ee}\cdot\overleftrightarrow{C}_{ee})^{-1}\cdot\overleftrightarrow{\alpha}_{ee}], \tag{63}$$

$$\overleftrightarrow{\beta}_{mm}=\overleftrightarrow{F}_2^{-1}\cdot[\overleftrightarrow{\alpha}_{mm}+\overleftrightarrow{\alpha}_{me}\cdot\overleftrightarrow{C}_{ee}\cdot(\overleftrightarrow{I}-\overleftrightarrow{\alpha}_{ee}\cdot\overleftrightarrow{C}_{ee})^{-1}\cdot\overleftrightarrow{\alpha}_{em}], \tag{64}$$

and

$$\overleftrightarrow{F}_1=\overleftrightarrow{I}-\overleftrightarrow{\alpha}_{ee}\cdot\overleftrightarrow{C}_{ee}-\overleftrightarrow{\alpha}_{em}\cdot\overleftrightarrow{C}_{mm}\cdot(\overleftrightarrow{I}-\overleftrightarrow{\alpha}_{mm}\cdot\overleftrightarrow{C}_{mm})^{-1}\cdot\overleftrightarrow{\alpha}_{me}\cdot\overleftrightarrow{C}_{ee}, \tag{65}$$

$$\overleftrightarrow{F}_2=\overleftrightarrow{I}-\overleftrightarrow{\alpha}_{mm}\cdot\overleftrightarrow{C}_{mm}-\overleftrightarrow{\alpha}_{me}\cdot\overleftrightarrow{C}_{ee}\cdot(\overleftrightarrow{I}-\overleftrightarrow{\alpha}_{ee}\cdot\overleftrightarrow{C}_{ee})^{-1}\cdot\overleftrightarrow{\alpha}_{em}\cdot\overleftrightarrow{C}_{mm}. \tag{66}$$

Expressions (59) and (61)–(66) are the same form as expressions (20) and (22)–(27) of the static model. We have, however, different interaction constants for the static and dynamical models.

The sampling functions of the polarization and magnetization per unit volume are

$$\vec{P}_s=N\vec{p}_s=N(\overleftrightarrow{\beta}_{ee}\cdot\vec{E}_s^{(0)}+\overleftrightarrow{\beta}_{em}\cdot\vec{H}_s^{(0)}),$$
$$\vec{M}_s=N\vec{m}_s=N(\overleftrightarrow{\beta}_{me}\cdot\vec{E}_s^{(0)}+\overleftrightarrow{\beta}_{mm}\cdot\vec{H}_s^{(0)}). \tag{67}$$

Now let us define the fields $\vec{E}_s^{(p)}$ and $\vec{H}_s^{(p)}$ in Eqs. (54) and (55). The dipole fields $\vec{E}^{(p)},\vec{H}^{(p)}$ are equal, respectively, to the sums of interaction fields $\vec{E}^{(i)},\vec{H}^{(i)}$ and dipole fields, produced by the particle located at the origin $\vec{E}^{(1)},\vec{H}^{(1)}$. It was shown in [20] that for an electric dipole the sampling function $\vec{E}_s^{(1)}$ is equal to zero. Obviously, for a magnetic dipole, we have the same situation, that is, the sampling function $\vec{H}_s^{(1)}$ is equal to zero as well. So, one can write that $\vec{E}_s^{(p)}$ and $\vec{H}_s^{(p)}$ are, respectively, equal to $\vec{E}_s^{(i)}$ and $\vec{H}_s^{(i)}$. Keeping only radiated (curl) parts of $\vec{E}_s^{(p)}$ and $\vec{H}_s^{(p)}$ we have

$$\vec{E}_s^{(p)}=\Delta V(\overleftrightarrow{C}_{ee}-\overleftrightarrow{C}_E)\cdot\vec{P}_s,$$
$$\vec{H}_s^{(p)}=\Delta V(\overleftrightarrow{C}_{mm}-\overleftrightarrow{C}_M)\cdot\vec{M}_s. \tag{68}$$

At the quasistatic limit $(q\to 0)$, one has $\vec{E}_s^{(p)}=\vec{H}_s^{(p)}=0$.

Taking into account Eqs. (67) and (68), we can represent Eqs. (54) and (55) as

$$\vec{\mathcal{D}}_s(\vec{r})=\overleftrightarrow{\kappa}_{ee}(\vec{q})\cdot\vec{E}_s^{(0)}(\vec{r})+\overleftrightarrow{\kappa}_{em}(\vec{q})\cdot\vec{H}_s^{(0)}(\vec{r}),$$
$$\vec{B}_s(\vec{r})=\overleftrightarrow{\kappa}_{me}(\vec{q})\cdot\vec{E}_s^{(0)}(\vec{r})+\overleftrightarrow{\kappa}_{mm}(\vec{q})\cdot\vec{H}_s^{(0)}(\vec{r}), \tag{69}$$

where

$$\overleftrightarrow{\kappa}_{ee}(\vec{q})=\epsilon[\overleftrightarrow{I}+(\overleftrightarrow{C}_{ee}-\overleftrightarrow{C}_E)\cdot\overleftrightarrow{\beta}_{ee}]+N\overleftrightarrow{\beta}_{ee}, \tag{70}$$

$$\overleftrightarrow{\kappa}_{em}(\vec{q})=\epsilon(\overleftrightarrow{C}_{ee}-\overleftrightarrow{C}_E)\cdot\overleftrightarrow{\beta}_{em}+N\overleftrightarrow{\beta}_{em}, \tag{71}$$

$$\overleftrightarrow{\kappa}_{me}(\vec{q})=\mu_0(\overleftrightarrow{C}_{mm}-\overleftrightarrow{C}_{em})\cdot\overleftrightarrow{\beta}_{me}+N\overleftrightarrow{\beta}_{me}, \tag{72}$$

$$\overleftrightarrow{\kappa}_{mm}(\vec{q})=\mu_0[\overleftrightarrow{I}+(\overleftrightarrow{C}_{mm}-\overleftrightarrow{C}_M)\cdot\overleftrightarrow{\beta}_{mm}]+N\overleftrightarrow{\beta}_{mm}. \tag{73}$$

Here $\vec{q}$ is considered as a parameter. One can see that for $q=0$ tensors $\overleftrightarrow{\kappa}_{ee},\overleftrightarrow{\kappa}_{em},\overleftrightarrow{\kappa}_{me},\overleftrightarrow{\kappa}_{mm}$ correspond, respectively, to tensors $\overleftrightarrow{\epsilon}_{\text{eff}},\overleftrightarrow{a}_{\text{eff}},\overleftrightarrow{b}_{\text{eff}},\overleftrightarrow{\mu}_{\text{eff}}$ in Eq. (31).

When the sampling functions $\vec{\mathcal{D}}_s(\vec{r})$ and $\vec{B}_s(\vec{r})$ are known, one can reconstruct the Fourier spectrum of the electric displacement $\tilde{\tilde{\vec{\mathcal{D}}}}(\vec{q})$ and magnetic flux density $\tilde{\tilde{\vec{B}}}(\vec{q})$. On the basis of the sampling theorem, one has [20,31–33]

$$\tilde{\tilde{\vec{\mathcal{D}}}}(\vec{q})=[\overleftrightarrow{\kappa}_{ee}(\vec{q})\cdot\tilde{\tilde{\vec{E}}}_s^{(0)}+\overleftrightarrow{\kappa}_{em}(\vec{q})\cdot\tilde{\tilde{\vec{H}}}_s^{(0)}(\vec{q})]\Delta V \times\sqcap\left(\frac{q_x}{2Q_x}\right)\sqcap\left(\frac{q_y}{2Q_y}\right)\sqcap\left(\frac{q_z}{2Qz}\right) \equiv\tilde{\vec{\mathcal{D}}}(\vec{q})\Delta V\sqcap\left(\frac{q_x}{2Qx}\right)\sqcap\left(\frac{q_y}{2Qy}\right)\sqcap\left(\frac{q_z}{2Qz}\right), \tag{74}$$

$$\tilde{\tilde{\vec{B}}}(\vec{q})=[\overleftrightarrow{\kappa}_{me}(\vec{q})\cdot\tilde{\tilde{\vec{E}}}_s^{(0)}+\overleftrightarrow{\kappa}_{mm}(\vec{q})\cdot\tilde{\tilde{\vec{H}}}_s^{(0)}(\vec{q})]\Delta V \times\sqcap\left(\frac{q_x}{2Q_x}\right)\sqcap\left(\frac{q_y}{2Q_y}\right)\sqcap\left(\frac{q_z}{2Qz}\right) \equiv\tilde{\vec{B}}(\vec{q})\Delta V\sqcap\left(\frac{q_x}{2Qx}\right)\sqcap\left(\frac{q_y}{2Qy}\right)\sqcap\left(\frac{q_z}{2Qz}\right), \tag{75}$$

where $\sqcap$ denotes the rectangle function.

These expressions may be rewritten as

$$\tilde{\vec{\mathcal{D}}}(\vec{q})=\overleftrightarrow{\kappa}_{ee}(\vec{q})\cdot\tilde{\vec{E}}^{(0)}(\vec{q})+\overleftrightarrow{\kappa}_{em}(\vec{q})\cdot\tilde{\vec{H}}^{(0)}(\vec{q}),$$
$$\tilde{\vec{B}}(\vec{q})=\overleftrightarrow{\kappa}_{ee}(\vec{q})\cdot\tilde{\vec{E}}^{(0)}(\vec{q})+\overleftrightarrow{\kappa}_{em}(\vec{q})\cdot\tilde{\vec{H}}^{(0)}(\vec{q}). \tag{76}$$

The fields $\tilde{\vec{E}}^{(0)}(\vec{q})$, $\tilde{\vec{H}}^{(0)}(\vec{q})$, $\tilde{\tilde{\vec{E}}}_s^{(0)}(\vec{q})$, and $\tilde{\tilde{\vec{H}}}_s^{(0)}(\vec{q})$ in Eqs. (74)–(76) are the Fourier images of the fields $\vec{E}^{(0)}(\vec{r})$, $\vec{H}^{(0)}(\vec{r})$, $\vec{E}_s^{(0)}(\vec{r})$, and $\vec{H}_s^{(0)}(\vec{r})$, respectively.

Expression (76) may be considered as an analog of formulas (37) and (38) used for spatially dispersive continuous bianisotropic media. In our case, however, the wave vector $\vec{q}$ and the fields $\vec{E}^{(0)}$,$\vec{H}^{(0)}$ do not correspond to the wave vector and the fields in the medium. There are the wave vector and the fields in the host material (vacuum, in a particular case).

When we consider tensors $\overleftrightarrow{\kappa}_{ee}(\vec{q})$, $\overleftrightarrow{\kappa}_{em}(\vec{q})$, $\overleftrightarrow{\kappa}_{me}(\vec{q})$, and $\overleftrightarrow{\kappa}_{mm}(\vec{q})$ as the Fourier images of certain original functions, respectively, $\overleftrightarrow{\kappa}_{ee}(\vec{r})$, $\overleftrightarrow{\kappa}_{em}(\vec{r})$, $\overleftrightarrow{\kappa}_{me}(\vec{r})$, and $\overleftrightarrow{\kappa}_{mm}(\vec{r})$, the convolution-form expressions based on Eq. (76), take place. These expressions may be considered as an analog of Eq. (36) for time-invariant and spatially homogeneous media when the causality principle is taken into account.

VI. EVALUATION OF THE INTERACTION CONSTANTS

We have developed the static field theory of bianisotropic crystal lattices supposing that the interaction tensors $\overleftrightarrow{C}_E$ and $\overleftrightarrow{C}_M$ are known. Similarly, we have developed a dynamical model of bianisotropic crystal lattices supposing that the interaction tensors $\overleftrightarrow{C}_{ee}(\vec{q})$, $\overleftrightarrow{C}_{em}(\vec{q})$, $\overleftrightarrow{C}_{me}(\vec{q})$, and $\overleftrightarrow{C}_{mm}(\vec{q})$ are known as well. Now the question about evaluation of the interaction constants arises. In our further analysis, we will evaluate the interaction constants separately for static and dynamical models of bianisotropic crystal lattices.

A. The interaction constants for the static model

The interaction tensors $\overleftrightarrow{C}_E$ and $\overleftrightarrow{C}_M$ are given, respectively, from summations of the static electric and magnetic fields due to the array of bianisotropic particles. Taking into account the electric-dipole and the magnetic-dipole fields in the quasistatic limit [10], one can rewrite Eq. (19) as

$$\overleftrightarrow{C}_E\cdot\vec{p}=\frac{1}{4\pi\epsilon}\sum_{l=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}{\sum_{k=-\infty}^{\infty}}'\frac{3\vec{u}(\vec{u}\cdot\vec{p})-\vec{p}}{r_{kml}^3}, \tag{77}$$

$$\overleftrightarrow{C}_M\cdot\vec{m}=\frac{1}{4\pi}\sum_{l=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}{\sum_{k=-\infty}}'\frac{3\vec{u}(\vec{u}\cdot\vec{m})-\vec{m}}{r_{kml}^3}, \tag{78}$$

where the primes indicate omission of the terms with $k=n=l=0$, $\vec{u}$ is a unit vector directed along the radius-vector $\vec{r}_{knl}$, and r_{knl} is a distance from the particle at the origin to the particle characterized by numbers k,n,l.

For the y component, for example, we can rewrite Eq. (77) as

$$(\overleftrightarrow{C}_E\cdot\vec{p})_y=\frac{1}{4\pi\epsilon}\sum_{l=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}{\sum_{k=-\infty}^{\infty}}'\frac{3[(k\Delta x)(n\Delta y)p_x+(n\Delta y)^2p_y+(n\Delta y)(l\Delta z)p_z]-[(k\Delta x)^2+(n\Delta y)^2+(l\Delta z)^2]p_y}{[(k\Delta x)^2+(n\Delta y)^2+(l\Delta z)^2]^{5/2}}. \tag{79}$$

Since the indices in Eq. (79) run equally over positive and negative values, the cross terms involving $(k\Delta x)(n\Delta y)p_x$ and $(n\Delta y)(l\Delta z)p_z$ vanish. This gives in result the interaction constant as a diagonal tensor. One can rewrite Eq. (79) as

$$(\overleftrightarrow{C}_E)_{yy}=\frac{1}{4\pi\epsilon}\sum_{l=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}{\sum_{k=-\infty}^{\infty}}'\frac{2(n\Delta y)^2-(k\Delta x)^2-(l\Delta z)^2}{[(k\Delta x)^2+(n\Delta y)^2+(l\Delta z)^2]^{5/2}}, \tag{80}$$

The similar series in the right-hand sides, one has for other components of the diagonal tensor $\overleftrightarrow{C}_E$ and also for components of the diagonal tensor $\overleftrightarrow{C}_M$. The calculation of such series is far from straightforward and has been the subject of many works. A detailed consideration and a list of relevant references concerning an analysis of the quasistatic constants can be found in [17].

For magnetostatically controlled bianisotropic media described in Sec. II, a useful approach may be used to avoid the difficulty of handling serieses of type (80). This consists of the assumption that the principal contribution to the interaction fields acting on the particle at the origin comes from those particles which lie in the same plane, i.e., the $z=\text{const}$ in Fig. 3. This assumption may be well satisfied since the MCBMs are conceived as a pile of dielectric substrates with planar ferromagnetic objects [14,15]. In particular cases of very elongated parallelepiped MSW resonators [Fig. 1(a)] or cylindrical MSW resonators [Fig. 1(b)], one can use an analysis of two-dimensional strip-type or disk-type artificial dielectrics [17]. Thus, the calculation of the quasistatic interaction constants in bianisotropic crystal lattice may sufficiently be based on the well-developed static theory of artificial dielectrics. Such a detailed analysis of concrete structures is beyond the scope of the present consideration.

B. The interaction constants for the dynamical model

To obtain the interaction constants in the dynamical model of bianisotropic crystal lattices, one has to take full expressions for the electric and magnetic fields radiated by a combination of electric and magnetic dipole sources. On the basis of known expressions for such fields [10], we can rewrite (57) as follows:

$$\overleftrightarrow{C}_{ee}\cdot\vec{p}+\overleftrightarrow{C}_{em}\cdot\vec{m}=\frac{1}{4\pi\epsilon}\sum_{l=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}{}' \left\{[3\vec{u}(\vec{u}\cdot\vec{p})-\vec{p}]\left(\frac{1}{r_{knl}^3}-\frac{i2\pi g}{r_{knl}}\right)-[\vec{u}(\vec{u}\cdot\vec{p})-\vec{p}]\,\frac{4\pi^2q^2}{r_{knl}}-\sqrt{\mu_0\epsilon}(\vec{u}\times\vec{m})\right.$$
$$\left.\times\left(1+\frac{i}{2\pi q r_{knl}}\right)\frac{4\pi^2q^2}{r_{knl}}\right\}\exp(i2\pi q r_{knl}), \tag{81}$$

$$\overleftrightarrow{C}_{me}\cdot\vec{p}+\overleftrightarrow{C}_{mm}\cdot\vec{m}=\frac{1}{4\pi}\sum_{l=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}{}' \left\{\sqrt{\mu_0\epsilon}(\vec{u}\times\vec{p})\left(1+\frac{i}{2\pi q r_{knl}}\right)\frac{4\pi^2q^2}{r_{knl}}+[3\vec{u}(\vec{u}\cdot\vec{m})-\vec{m}]\left(\frac{1}{r_{knl}^3}-\frac{i2\pi q}{r_{knl}}\right)\right.$$
$$\left.-[\vec{u}(\vec{u}\cdot\vec{m})-\vec{m}]\,\frac{4\pi^2q^2}{r_{knl}}\right\}\exp(i2\pi q r_{knl}), \tag{82}$$

Obviously terms with vector products $\vec{u}\times\vec{m}$ and $\vec{u}\times\vec{p}$ are equal to zero since the indices run equally over positive and negative values. For this reason, the cross terms involving terms $(k\Delta x)(n\Delta y)p_x,(n\Delta y)(l\Delta z)p_z$, etc., vanish. As a result, we have that $\overleftrightarrow{C}_{em}=\overleftrightarrow{C}_{me}=0$ and that $\overleftrightarrow{C}_{ee}$ and $\overleftrightarrow{C}_{mm}$ are diagonal tensors.

In our consideration, the tensors $\overleftrightarrow{C}_{ee}(\vec{q})$ and $\overleftrightarrow{C}_{mm}(\vec{q})$ are the finite-spectrum functions. This means that for $|q_x|>Q_x$, $|q_y|>Q_y$, $|q_z>Q_z$ the interaction constants are equal to zero. Let the limiting cutoff wave numbers are defined as

$$Q_x=\frac{1}{2\Delta x},\quad Q_y=\frac{1}{2\Delta y},\quad Q_z=\frac{1}{2\Delta z}. \tag{83}$$

One can carry out an analysis similar to the analysis made in [20] for dielectric crystal lattices. The components of tensors $\overleftrightarrow{C}_{ee}(\vec{q})$ and $\overleftrightarrow{C}_{mm}(\vec{q})$ are described by expressions of a form similar to Eqs. (43) and (44) in [20]. Therefore, one can see that in the dynamical model of bianisotropic crystal lattices, the components of the interaction tensors are the Fourier images of rapidly convergent series.

VII. CONCLUSION

Among a number of known temporally dispersive bianisotropic media, there are a class of bianisotropic composites—the MCBMs—which may be described *quasistatically*. This makes it possible to realize homogenized dense materials with randomly distributed or aligned bianisotropic particles.

A bianisotropic crystal lattice is one of the interesting structures of such materials. In this paper, we developed approaches for an analysis of static and dynamical models of bianisotropic crystal lattices. The static model is based on the Lorenz-Lorentz theory and describes the effective constitutive parameters of bianisotropic crystal lattices with (as a general case) nonuniaxial bianisotropic particles. In such a general consideration of the static model, however, a question regarding the convergence of series describing the interaction tensors arises. For same particular cases one can use well-known results obtained in the theory of artificial dielectrics.

Two ways may be used for the description of the electromagnetic field-condensed media interaction. One way is to get over a discrete structure of a medium by the averaging procedure and another way may be conceived as follows: to discretize the fields on the basis of discrete structure of a medium. When initial restrictions to the wave number spectrum take place, one can use the so-called sampling theorem for a medium modeled as a triple infinite periodic array of identical δ-functional scattering elements.

The dynamical theory of bianisotropic crystal lattices is based on the use of the sampling theorem similarly to the approach developed in [20] for dielectric crystals. As the main results of our dynamical model, one has the effective constitutive parameters of bianisotropic composites dependent on the wave vector in the host material. This makes it possible to use this constitutive parameters for further analyses of spatially dispersive bianisotropic media.

Here we discussed only the electric and magnetic dipole fields. The vector multipole fields are beyond the scope of the present analysis and may be the subject of future investigations.

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