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Coherence, polarization, and entanglement for classical light fields

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ABSTRACT

We examine a classical version of entanglement between spatial and polarization degrees of freedom for classical light. We examine the relation between classical entanglement, polarization, and several recently introduced measures of coherence for vectorial waves. We show that there is no definite relation between quantum and classical entanglement.

1. Introduction

Coherence of polarized waves is an issue of very recent development with plenty of promising possibilities. The inextricable mixing of coherence and polarization allows the investigation of new concepts and phenomena [1–6].

In this work we elaborate on the idea of a classical counterpart of entanglement between spatial and polarization degrees of freedom for classical waves. More specifically, we focus on the relation between classical entanglement, polarization, and several recently introduced measures of coherence for vectorial waves.

Classical entanglement can be defined as the lack of separability of the cross-spectral density tensor in spatial and polarization components [7–10]. This definition fully mimics the quantum definition in terms of lack of separability of the density matrix.

The properties and implications of classical entanglement are fundamentally different from the ones that follow from quantum entanglement [7–10]. More specifically:

(i) The classical entanglement to be considered here involves different degrees of freedom of the same particle (so to speak, since classically there are no light particles) instead of entanglement between different particles.
(ii) Classically there is no nonlocality since the entangled subsystems (space and polarization) cannot be spatially separated.
(iii) Classically there is no measurement-induced collapse due to particle indivisibility, so that there are no mutually exclusive outcomes.
(iv) Finally, we will show that there is no definite relation between classical and quantum entanglement, since every cross-spectral density tensor can be derived both from entangled and factorized quantum states.

Leaving aside these remarks, the definition of classical entanglement is nevertheless meaningful, clear, and unambiguous. In this regard we show that classical entanglement is clearly related with some features of coherence between vectorial waves. Thus, the analysis addressed in this work may be useful by translating results and concepts between the classical and quantum domains, while providing a fruitful development of coherence for vectorial waves.

The relation between coherence and entanglement has been studied in previous works, such as [11–15], revealing some duality relations similar to the ones we will find in Sections 3 and 4 below. Nevertheless, there is a fundamental difference between these works and the analysis addressed here. This is point (i) above, since such previous works focus on quantum two-particle entanglement, which manifests in two-point coincidence detection. The coherence issues addressed in this work focus exclusively on single-particle interference that manifests on single-point independent

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detection. Moreover, the duality in [11] refers to entanglement in one light source and coherence in a different source, while in Sections 3 and 4 below coherence and entanglement refer to the same light wave.

In this regard we note that also for two-point coincidence detection it has been studied to what extent classical light waves can reproduce the results obtained with quantum entangled light sources [9,10,16,17].

Moreover, the classical-optics analogy of quantum entanglement has triggered some other interesting theoretical and experimental tests of classical versions of quantum entanglement [18–26].

2. Entanglement, factorization, and locality

We consider the spatial-frequency domain for transversal electromagnetic fields with two components $E_{x,y}$ of frequency ω at two definite spatial points $\boldsymbol{r}_{1,2}$. For the sake of illustration we may consider that they represent the electromagnetic field at the two small apertures of a Young interferometer (see Fig. 1).

The cross-spectral density tensor $\boldsymbol{\Gamma}$ reads

$$\boldsymbol{\Gamma}_{j,k;\alpha,\beta} = \boldsymbol{\Gamma}_{j,k}(\boldsymbol{r}_\alpha, \boldsymbol{r}_\beta) = \left\langle E_j(\boldsymbol{r}_\alpha) E_k^*(\boldsymbol{r}_\beta) \right\rangle, \tag{1}$$

where the Latin indices $j,k = x,y$ refer to polarization components, the Greek indices $\alpha, \beta = 1,2$ refer to space points, and the angle brackets denote ensemble averages.

2.1. Quantum-classical analogy

Next we develop a formal analogy between classical optics and quantum mechanics suggesting classical entanglement. For example, a transversal field with nonfluctuating deterministic amplitudes at two spatial points $\varepsilon_j(\boldsymbol{r}_\alpha)$, can be described by the four dimensional complex vector

$$|\psi\rangle \propto \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_x(\boldsymbol{r}_1) \\ \varepsilon_y(\boldsymbol{r}_1) \\ \varepsilon_x(\boldsymbol{r}_2) \\ \varepsilon_y(\boldsymbol{r}_2) \end{pmatrix}, \tag{2}$$

with cross-spectral density tensor

$$\boldsymbol{\Gamma} = \boldsymbol{\varepsilon}\boldsymbol{\varepsilon}^\dagger \propto |\psi\rangle\langle\psi|. \tag{3}$$

This is to say that the classical state space is a four-dimensional complex Hilbert space $\mathscr{H}$.

Since $\boldsymbol{\Gamma}$ is Hermitian and nonnegative a fruitful quantum-classical analogy can be established by identifying $\boldsymbol{\Gamma}$ as a quantum density matrix $\rho = \boldsymbol{\Gamma}/\mathrm{tr}\boldsymbol{\Gamma}$. Deterministic fields such as (2) are represented by pure states, while fluctuating fields correspond to mixed states.

The Hilbert space $\mathscr{H}$ can be described also by the tensor product $\mathscr{H} = \mathscr{H}_s \otimes \mathscr{H}_p$ of a pair of complex two-dimensional vector spaces. The Hilbert space $\mathscr{H}_s$ describes the spatial variable and is spanned by the abstract basis vectors $|\alpha = 1,2\rangle_s$. The Hilbert space $\mathscr{H}_p$ describes the polarization variable and is spanned by the basis vectors $|j = x,y\rangle_p$. In the product basis the same state (2) can be expressed as

$$|\psi\rangle \propto \boldsymbol{\varepsilon} = \sum_{j=x,y} \sum_{\alpha=1,2} \varepsilon_j(\boldsymbol{r}_\alpha)|\alpha\rangle_s \otimes |j\rangle_p. \tag{4}$$

Throughout the paper we will focus on states of the form (2) and (4), since they provide the most clear illustration of entanglement.

Alternatively, the same state space $\mathscr{H}$ can be described by the direct sum of a pair of two-dimensional complex spaces as $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$. The vector space $\mathscr{H}_1$ describes the field state at point $\boldsymbol{r}_1$ and is spanned by the basis vectors $|1\rangle_s \otimes |j = x,y\rangle_p$. The vector space $\mathscr{H}_2$ describes the field state at point $\boldsymbol{r}_2$ and is spanned by the basis vectors $|2\rangle_s \otimes |j = x,y\rangle_p$.

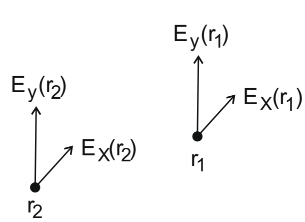

Fig. 1. Transversal electromagnetic field at two spatial points.

2.2. Subsystems

Concerning entanglement it is crucial to identify the two subsystems that may be in separable or entangled states. In this work we focus on the factorization $\mathscr{H} = \mathscr{H}_s \otimes \mathscr{H}_p$ in space and polarization subsystems.

2.2.1. Spatial subsystem

The spatial subsystem is obtained by removing the polarization degrees of freedom, leading to the 2×2 correlation matrix $\boldsymbol{\Upsilon}_s$

$$\boldsymbol{\Upsilon}_{s,\alpha,\beta} = \sum_{j=x,y} \boldsymbol{\Gamma}_{j,j;\alpha,\beta} = \left\langle \boldsymbol{E}_\beta^\dagger \cdot \boldsymbol{E}_\alpha \right\rangle, \quad \boldsymbol{E}_\alpha = \begin{pmatrix} E_x(\boldsymbol{r}_\alpha) \\ E_y(\boldsymbol{r}_\alpha) \end{pmatrix}, \tag{5}$$

or

$$\boldsymbol{\Upsilon}_s = \begin{pmatrix} I_1 & \mu_s\sqrt{I_1 I_2} \\ \mu_s^*\sqrt{I_1 I_2} & I_2 \end{pmatrix}, \tag{6}$$

where $\boldsymbol{E}_{1,2}$ are the complex two-dimensional polarization vectors at each point, $I_{1,2}$ are the corresponding intensities

$$I_\alpha = \langle|\boldsymbol{E}_\alpha|^2\rangle = \langle|E_x(\boldsymbol{r}_\alpha)|^2\rangle + \langle|E_y(\boldsymbol{r}_\alpha)|^2\rangle, \tag{7}$$

and

$$\mu_s = \frac{\left\langle \boldsymbol{E}_2^\dagger \cdot \boldsymbol{E}_1 \right\rangle}{\sqrt{I_1 I_2}} = \frac{\mathrm{tr}\boldsymbol{\Gamma}(\boldsymbol{r}_1, \boldsymbol{r}_2)}{\sqrt{\mathrm{tr}\boldsymbol{\Gamma}(\boldsymbol{r}_1, \boldsymbol{r}_1)\mathrm{tr}\boldsymbol{\Gamma}(\boldsymbol{r}_2, \boldsymbol{r}_2)}} \tag{8}$$

coincides with the degree of coherence for vectorial electromagnetic waves introduced in [1,2].

2.2.2. Polarization subsystem

The polarization subsystem is obtained by removing the spatial degrees of freedom, leading to the 2×2 correlation matrix $\boldsymbol{\Upsilon}_p$

$$\boldsymbol{\Upsilon}_{p,j,k} = \sum_{\alpha=1,2} \boldsymbol{\Gamma}_{j,k;\alpha,\alpha} = \left\langle \boldsymbol{E}_k^\dagger \cdot \boldsymbol{E}_j \right\rangle, \quad \boldsymbol{E}_j = \begin{pmatrix} E_j(\boldsymbol{r}_1) \\ E_j(\boldsymbol{r}_2) \end{pmatrix}, \tag{9}$$

or

$$\boldsymbol{\Upsilon}_p = \begin{pmatrix} I_x & \mu_p\sqrt{I_x I_y} \\ \mu_p^*\sqrt{I_x I_y} & I_y \end{pmatrix} = \boldsymbol{\Gamma}(\boldsymbol{r}_1, \boldsymbol{r}_1) + \boldsymbol{\Gamma}(\boldsymbol{r}_2, \boldsymbol{r}_2), \tag{10}$$

where $\boldsymbol{E}_{x,y}$ are complex two-dimensional vectors associated to each polarization component, $I_{x,y}$ are the corresponding intensities

$$I_j = \langle|\boldsymbol{E}_j|^2\rangle = \langle|E_j(\boldsymbol{r}_1)|^2\rangle + \langle|E_j(\boldsymbol{r}_2)|^2\rangle, \tag{11}$$

and μ_p is the polarization dual of μ_s

$$\mu_p = \frac{\left\langle \boldsymbol{E}_y^\dagger \cdot \boldsymbol{E}_x \right\rangle}{\sqrt{I_x I_y}}. \tag{12}$$

Since $\boldsymbol{\Upsilon}_p = \boldsymbol{\Gamma}(\boldsymbol{r}_1, \boldsymbol{r}_1) + \boldsymbol{\Gamma}(\boldsymbol{r}_2, \boldsymbol{r}_2)$ this is an average polarization state, so that we may refer to it as representing mean polarization.

2.3. Entanglement

Classical entanglement between polarization and space arises when $\mathbf{\Gamma}$ is not separable. The cross-spectral density tensor $\mathbf{\Gamma}$ is separable when it can be expressed in the form

$$\mathbf{\Gamma} = \sum_m \mathbf{\Upsilon}_{s,m} \otimes \mathbf{\Upsilon}_{p,m}, \tag{13}$$

where $\mathbf{\Upsilon}_{s,m}$ and $\mathbf{\Upsilon}_{p,m}$ are 2×2 Hermitian and non negative matrices in the spatial and polarization spaces, respectively.

A simple example of separable state is

$$|\psi\rangle \propto (|1\rangle_s + |2\rangle_s) \otimes \left(|x\rangle_p + |y\rangle_p\right), \tag{14}$$

where the polarization state is the same in both points $\boldsymbol{r}_{1,2}$.

An example of entangled state is

$$|\psi\rangle \propto |1\rangle_s \otimes |x\rangle_p + |2\rangle_s \otimes |y\rangle_p, \tag{15}$$

where the polarization state is different at each point $\boldsymbol{r}_{1,2}$.

We can appreciate that the duality entanglement/separability amounts to be equivalent to inhomogeneous/homogeneous polarization distribution.

2.4. Unitary, local, and nonlocal transformations

In the coherence and polarization context the symmetry properties under the action of transparent devices (i.e., lossless phase plates and beam splitters) play a key role.

In our case, the transformations preserving total intensity are the unitary transformations $U(4) : \mathscr{H} \to \mathscr{H}$ implemented by 4×4 unitary matrices $\boldsymbol{U}\boldsymbol{U}^\dagger = \boldsymbol{U}^\dagger \boldsymbol{U} = \mathscr{I}_4$, where $\mathscr{I}_4$ is the 4×4 identity matrix. This has different interesting subgroups.

2.4.1. Entanglement-local transformations $\boldsymbol{U}_s \otimes \boldsymbol{U}_p$

In the language of entanglement, local transformations are transformations that act on each subsystem independently. They do not alter entanglement so we will refer to them as entanglement-local to distinguish them from the space-local transformations defined below.

In our case the subsystems are described by the space $\mathscr{H}_s$ and polarization $\mathscr{H}_p$ vector spaces. Thus entanglement-local unitary transformations are of the form $\boldsymbol{U}_s \otimes \boldsymbol{U}_p$ where $\boldsymbol{U}_\ell : \mathscr{H}_\ell \to \mathscr{H}_\ell, \ell = s, p$, are 2×2 unitary matrices. These transformations do not modify the entangled/separable nature of the field state, since they produce the same transformation of the polarization state at both spatial points $\boldsymbol{r}_{1,2}$.

2.4.2. Space-local transformations $\boldsymbol{U}_1 \oplus \boldsymbol{U}_2$

In classical optics local transformations usually refer to transformations that change of the polarization state at each spatial point $\boldsymbol{r}_{1,2}$ independently. They are of the form $\boldsymbol{U}_1 \oplus \boldsymbol{U}_2$, where $\boldsymbol{U}_\alpha : \mathscr{H}_\alpha \to \mathscr{H}_\alpha, \alpha = 1, 2$, are 2×2 unitary matrices. We will refer to them as space-local to distinguish them from the entanglement-local transformations defined above.

It is worth noting that in general these are entanglement-nonlocal operations since they can modify the polarization state at each point $\boldsymbol{r}_{1,2}$ differently. For example the entangled state (15) can be produced from the separable state (14) by placing suitable half-wave plates at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$.

2.5. Factorization

In order to proceed further it is convenient to distinguish between two different meanings for the term factorization.

2.5.1. Classical factorization

By classical factorization we will refer to the property

$$\left\langle E_j(\boldsymbol{r}_\alpha) E_k^*(\boldsymbol{r}_\beta) \right\rangle = \varepsilon_j(\boldsymbol{r}_\alpha) \varepsilon_k^*(\boldsymbol{r}_\beta), \tag{16}$$

where $\varepsilon_j(\boldsymbol{r}_\alpha)$ are nonfluctuating deterministic numbers. This is equivalent to $\mathrm{tr}(\mathbf{\Gamma}^2) = (\mathrm{tr}\mathbf{\Gamma})^2$ and to the purity of the corresponding density matrix $\rho = \mathbf{\Gamma}/\mathrm{tr}\mathbf{\Gamma} = |\psi\rangle\langle\psi|$, so that $|\psi\rangle$ is of the form (4).

This means that the field state has complete coherence and is fully polarized according to every criterion assessing the amount of coherence and polarization, with the only exception of the criterion introduced in [1,2].

Throughout the paper we will focus on classically factorized states since they provide the most clear illustration of entanglement.

2.5.2. Quantum factorization

By quantum factorization we will refer to full statistical independence between space and polarization. This is $\mathbf{\Gamma} = \mathbf{\Upsilon}_s \otimes \mathbf{\Upsilon}_p$, which is a particular case of separable $\mathbf{\Gamma}$.

3. Entanglement and coherence

Next we present suitable measures of coherence and entanglement analyzing their relationships. Coherence between vectorial waves can be analyzed from different perspectives and accordingly several measures of coherence have been introduced [1–6]. They are not contradictory as far as they focus on different aspects of the same phenomenon. This is reflected in their different symmetry properties under basic groups of transformations in Section 2.4.

3.1. Entanglement measure

For classically factorized states (4) we can suitably asses the degree of classical entanglement by the linear entropy [27]

$$\epsilon^2 = 2\left[1 - \frac{\mathrm{tr}\left(\mathbf{\Upsilon}_j^2\right)}{(\mathrm{tr}\mathbf{\Upsilon}_j)^2}\right], \quad j = s, p. \tag{17}$$

Both correlation matrices $\mathbf{\Upsilon}_{s,p}$ lead to the same ϵ in the above formula, as it can be seen by using the Schmidt decomposition for instance [28]. Maximum classical entanglement $\epsilon = 1$ occurs when $\mathbf{\Upsilon}_{s,p}$ are proportional to the corresponding identity $\mathbf{\Upsilon}_j \propto \mathscr{I}_j$, such as for the state (15). The minimum $\epsilon = 0$ occurs when the subsystems are in pure states so that $\mathrm{tr}(\mathbf{\Upsilon}_j^2) = (\mathrm{tr}\mathbf{\Upsilon}_j)^2$, such as for the state (14).

This measure of entanglement is invariant under entanglement-local transformations $\boldsymbol{U}_s \otimes \boldsymbol{U}_p$, while lacks invariance under the space-local ones $\boldsymbol{U}_1 \oplus \boldsymbol{U}_2$.

3.2. Entanglement and global coherence

The global amount of coherence μ_g conveyed both by the spatial and polarization degrees of freedom can be measured by the Hilbert–Schmidt distance between $\mathbf{\Gamma}$ and the 4×4 identity matrix $\mathscr{I} = \mathscr{I}_s \otimes \mathscr{I}_p$ representing fully incoherent and fully unpolarized light [6]

$$\mu_g^2 = \frac{4}{3}\mathrm{tr}\left[\left(\frac{1}{\mathrm{tr}\mathbf{\Gamma}}\mathbf{\Gamma} - \frac{1}{4}\mathscr{I}\right)^2\right] = \frac{4}{3}\left[\frac{\mathrm{tr}(\mathbf{\Gamma}^2)}{(\mathrm{tr}\mathbf{\Gamma})^2} - \frac{1}{4}\right]. \tag{18}$$

Maximum coherence $\mu_g = 1$ is equivalent to the classical factorization in (16). Minimum coherence $\mu_g = 0$ is reached exclusively by the quantum factorized separable state $\mathbf{\Gamma} \propto \mathscr{I}_s \otimes \mathscr{I}_p$.

It can be appreciated that this measure of coherence is invariant under the full $U(4)$ group. This includes the space-local transformations $\boldsymbol{U}_1 \oplus \boldsymbol{U}_2$ that alter the amount of entanglement. For example, maximum coherence $\mu_g = 1$ is equally reached by the separable state (14) as well as by the entangled state (15) that are related by a $\boldsymbol{U}_1 \oplus \boldsymbol{U}_2$ transformation. Thus there can be no definite relation between classical entanglement and global coherence.

3.3. Entanglement and Hilbert–Schmidt coherence

Focusing on the interference between two vectorial waves, a convenient measure of coherence is provided by the Hilbert–Schmidt norm of the cross-spectral density tensor

$$\mu_{HS}^2 = \frac{\mathrm{tr}\left[\boldsymbol{\Gamma}(\boldsymbol{r}_1,\boldsymbol{r}_2)\boldsymbol{\Gamma}^\dagger(\boldsymbol{r}_1,\boldsymbol{r}_2)\right]}{\mathrm{tr}\boldsymbol{\Gamma}(\boldsymbol{r}_1,\boldsymbol{r}_1)\mathrm{tr}\boldsymbol{\Gamma}(\boldsymbol{r}_2,\boldsymbol{r}_2)}. \tag{19}$$

This measure is invariant under the space-local $\boldsymbol{U}_1 \oplus \boldsymbol{U}_2$ subgroup of $U(4)$. In particular, we have $\mu_{HS} = 1$ for all classically factorized states, irrespectively of whether they are fully entangled ($\epsilon = 1$) or completely separable ($\epsilon = 0$). Thus, there cannot be definite relation with entanglement.

3.4. Entanglement and trace coherence

We have already noticed after (8) that the spatial-subsystem coherence μ_s coincides with the degree of coherence for vectorial electromagnetic waves μ_t introduced in [1,2] in terms of the trace of the cross-spectral density tensor

$$\mu_s = \mu_t = \frac{\mathrm{tr}\boldsymbol{\Gamma}(\boldsymbol{r}_1,\boldsymbol{r}_2)}{\sqrt{\mathrm{tr}\boldsymbol{\Gamma}(\boldsymbol{r}_1,\boldsymbol{r}_1)\mathrm{tr}\boldsymbol{\Gamma}(\boldsymbol{r}_2,\boldsymbol{r}_2)}}. \tag{20}$$

This is because μ_t is introduced in terms of the visibility of intensity interference fringes fully disregarding polarization, which is equivalent to consider just the reduced spatial subsystem.

At difference with previous coherence measures in this case there is no invariance under space-local transformations $\boldsymbol{U}_1 \oplus \boldsymbol{U}_2$ so there is room for finding meaningful relations between entanglement and coherence. Using (6), (10), and (17) we get

$$\epsilon^2 = \left(1-\delta_j^2\right)\left(1-|\mu_j|^2\right), \quad j = s,p, \tag{21}$$

where

$$\delta_s = \frac{I_1 - I_2}{I_1 + I_2}, \quad \delta_p = \frac{I_x - I_y}{I_x + I_y}. \tag{22}$$

In the quantum domain the quantities $\delta_{s,p}$ represent the a priori predictability of the location and polarization state, respectively, of the photon [13,14,29–38]. Therefore, for classically factorized states classical entanglement equals the product of unpredictability $1-\delta_{s,p}^2$ and incoherence $1-|\mu_{s,p}|^2$. For fixed $\delta_{s,p}$ we have that larger $|\mu_{s,p}|$ is equivalent to lesser classical entanglement and vice versa.

Finally we notice that the maximum of μ_t under $\boldsymbol{U}_1 \oplus \boldsymbol{U}_2$ transformations considered in [40] becomes by construction invariant under $\boldsymbol{U}_1 \oplus \boldsymbol{U}_2$, and thus insensible to entanglement.

4. Entanglement and polarization

We can further develop the above relation between entanglement and subsystem coherence after noting that $\mu_{s,p}$ are not invariant under the entanglement-preserving subgroup $\boldsymbol{U}_s \otimes \boldsymbol{U}_p$. Thus we look for a more universal relation between entanglement and coherence by finding the maximum of $\mu_{s,p}$ under $\boldsymbol{U}_s \otimes \boldsymbol{U}_p$ transformations.

For definiteness let us focus first on the spatial subsystem. In such a case we can show that

$$P_s^2 = 2\mathrm{tr}\left[\left(\frac{1}{\mathrm{tr}\Upsilon_s}\Upsilon_s - \frac{1}{2}\mathscr{I}_s\right)^2\right] = 2\frac{\mathrm{tr}(\Upsilon_s^2)}{(\mathrm{tr}\Upsilon_s)^2} - 1 \tag{23}$$

is the maximum degree of coherence $|\mu_s|$ achievable when arbitrary $\boldsymbol{U}_s \otimes \boldsymbol{U}_p$ transformations are applied to the original fields. This is because Υ_s is transformed as

$$\Upsilon_s(\boldsymbol{U}_s) = \mathrm{tr}_p\left(\boldsymbol{U}\boldsymbol{\Gamma}\boldsymbol{U}^\dagger\right) = \boldsymbol{U}_s\Upsilon_s\boldsymbol{U}_s^\dagger, \tag{24}$$

where tr_p means trace with respect to the polarization space. Then it can be seen that

$$P_s^2 = \frac{[I_1(\boldsymbol{U}_s) - I_2(\boldsymbol{U}_s)]^2 + 4I_1(\boldsymbol{U}_s)I_2(\boldsymbol{U}_s)|\mu_s(\boldsymbol{U}_s)|^2}{[I_1(\boldsymbol{U}_s) + I_2(\boldsymbol{U}_s)]^2} \geqslant |\mu_s(\boldsymbol{U}_s)|^2. \tag{25}$$

The equality is reached for all $\boldsymbol{U}_s$ when $|\mu_s(\boldsymbol{U}_s)| = 1$ (so that $P_s = 1$) or when $I_1(\boldsymbol{U}_s) = I_2(\boldsymbol{U}_s)$ [6]. On the other hand, when $\det\Upsilon_s \neq 0$ we get $\mu_s(\boldsymbol{U}_s) = 0$ for the transformation $\boldsymbol{U}_s$ that diagonalizes Υ_s.

Finally, from (17) and (23) it readily follows that for classically factorized states it always holds:

$$\epsilon^2 + P_s^2 = 1. \tag{26}$$

Note that both ϵ and P_s are invariant under entanglement-local transformations $\boldsymbol{U}_s \otimes \boldsymbol{U}_p$.

A fully equivalent relation $\epsilon^2 + P_p^2 = 1$ can be derived in terms of the standard degree of polarization P_p defined in terms of the coherence matrix Υ_p (10) [39]

$$P_p^2 = 2\mathrm{tr}\left[\left(\frac{1}{\mathrm{tr}\Upsilon_p}\Upsilon_p - \frac{1}{2}\mathscr{I}_p\right)^2\right] = 2\frac{\mathrm{tr}\left(\Upsilon_p^2\right)}{(\mathrm{tr}\Upsilon_p)^2} - 1. \tag{27}$$

We recall that, since $\Upsilon_p = \boldsymbol{\Gamma}(\boldsymbol{r}_1,\boldsymbol{r}_1) + \boldsymbol{\Gamma}(\boldsymbol{r}_2,\boldsymbol{r}_2)$, we have that P_p represents a degree of mean polarization.

Incidentally, the above relations imply $P_s = P_p$, so that after (25) the degree of mean polarization P_p is the maximum trace degree of coherence μ_t achievable under entanglement-local unitary transformations,

$$|\mu_t(\boldsymbol{U}_s)| \leqslant P_p. \tag{28}$$

This provides a new perspective on the idea of coherence maximization by unitary transformations [40,41].

Finally we note that

$$P_{s,p} = \frac{|\boldsymbol{s}_1 + \boldsymbol{s}_2|}{I_1 + I_2} = \sqrt{1-\epsilon^2}, \tag{29}$$

where $\boldsymbol{s}_j, j = 1,2$ are the three Stokes parameters

$$s_{j,k} = \mathrm{tr}\left[\boldsymbol{\sigma}_k\boldsymbol{\Gamma}(\boldsymbol{r}_j,\boldsymbol{r}_j)\right], \tag{30}$$

$\boldsymbol{\sigma}_k, k = 1,2,3$ are the Pauli matrices, and it holds that $|\boldsymbol{s}_j| = I_j$.

The main conclusion of this section is that ϵ and $P_{s,p}$ are complementary features. This has a clear meaning in standard quantum interferometry, since the space-polarization entanglement implies that polarization stores knowledge about the path followed by the photon. Thus, larger entanglement means larger path information, which in turn unavoidably implies lesser visibility according with complementarity [13,14,29–38].

4.1. Entanglement and mean polarization for arbitrary spatial domain

Finally let us generalize these results to the case when we consider transversal electromagnetic fields defined on an arbitrary spatial domain, instead on just two points $\boldsymbol{r}_{1,2}$. In such a case the polarization subsystem becomes described by the 2×2 correlation matrix

$$\Upsilon_p = \int_{-\infty}^{\infty} d^2\boldsymbol{r}\boldsymbol{\Gamma}(\boldsymbol{r},\boldsymbol{r}) = \frac{1}{2}\sum_{j=0}^{3}\langle s_j\rangle\boldsymbol{\sigma}_j \tag{31}$$

where $\boldsymbol{\sigma}_j$ are the Pauli matrices, including $\boldsymbol{\sigma}_0$ as the identity, and $\langle s_j\rangle$ are the spatial averages of the Stokes parameters

$$\langle s_j \rangle = \int_{-\infty}^{\infty} d^2 \boldsymbol{r} s_j(\boldsymbol{r}). \quad (32)$$

Then from (17) it can be easily seen that

$$\epsilon^2 + \mathscr{P}_p^2 = 1, \quad \mathscr{P}_p^2 = \frac{\langle s_1 \rangle^2 + \langle s_2 \rangle^2 + \langle s_3 \rangle^2}{\langle s_0 \rangle^2}, \quad (33)$$

where $\mathscr{P}_p$ is a degree of mean polarization that generalizes P_p in (29) to arbitrary spatial distributions. This average is different from other spatial averages of the degree of polarization [42].

5. Classical versus quantum entanglement

Next we show that there is no definite relation between classical and quantum entanglement. This is that any cross-spectral density tensor $\boldsymbol{\Gamma}$ can equally well correspond to entangled or separable quantum states.

To show this we translate the fields $E_j(\boldsymbol{r}_\alpha)$ to the quantum domain in terms of the corresponding four independent commuting complex-amplitude operators $a_{j,\alpha}$

$$E_j(\boldsymbol{r}_\alpha) \to a_{j,\alpha}, \quad [a_{j,\alpha}, a_{k,\beta}^\dagger] = \delta_{j,k}\delta_{\alpha,\beta}. \quad (34)$$

5.1. Quantum separable

Let us consider the separable product of coherent state

$$|\boldsymbol{\varepsilon}\rangle = |\varepsilon_x(\boldsymbol{r}_1)\rangle_{x,1} |\varepsilon_y(\boldsymbol{r}_1)\rangle_{y,1} |\varepsilon_x(\boldsymbol{r}_2)\rangle_{x,2} |\varepsilon_y(\boldsymbol{r}_2)\rangle_{y,2}, \quad (35)$$

where $|\varepsilon_j(\boldsymbol{r}_\alpha)\rangle_{j,\alpha}$ are coherent states

$$a_{j,\alpha}|\boldsymbol{\varepsilon}\rangle = \varepsilon_j(\boldsymbol{r}_\alpha)|\boldsymbol{\varepsilon}\rangle. \quad (36)$$

The cross-spectral density tensor can be computed as

$$\boldsymbol{\Gamma}_{j,k;\alpha,\beta} = \langle \boldsymbol{\varepsilon}| a_{k,\beta}^\dagger a_{j,\alpha} |\boldsymbol{\varepsilon}\rangle = \varepsilon_k^*(\boldsymbol{r}_\beta)\varepsilon_j(\boldsymbol{r}_\alpha), \quad (37)$$

that reproduces the cross-spectral density tensor of any classically factorized state. For example, for the classically entangled state (15) this is

$$|\boldsymbol{\varepsilon}\rangle = |\varepsilon\rangle_{x,1}|0\rangle_{y,1}|0\rangle_{x,2}|\varepsilon\rangle_{y,2}, \quad (38)$$

where $|0\rangle_{j,\alpha}$ represents the vacuum state, while for the classically separable state (14) this is

$$|\boldsymbol{\varepsilon}\rangle = |\varepsilon\rangle_{x,1}|\varepsilon\rangle_{y,1}|\varepsilon\rangle_{x,2}|\varepsilon\rangle_{y,2}. \quad (39)$$

5.2. Quantum entangled

Let us consider the one-photon entangled quantum states

$$|\psi\rangle \propto \sum_{j,\alpha} \varepsilon_j(\boldsymbol{r}_\alpha)|1_{j,\alpha}\rangle, \quad (40)$$

where

$$|1_{j,\alpha}\rangle = |0\rangle_{x,1} \cdots |1\rangle_{j,\alpha} \cdots |0\rangle_{y,2}, \quad (41)$$

and $|1\rangle_{j,\alpha}$ is a one-photon state in the corresponding mode $a_{k,\beta}^\dagger a_{k,\beta}|1\rangle_{j,\alpha} = \delta_{j,k}\delta_{\alpha,\beta}|1\rangle_{j,\alpha}$. The state (40) is clearly entangled in the sense of lack of separability, although the entanglement between one-photon and the vacuum may be controversial [43].

With these states we can get again the most general classically factorized cross-spectral density tensor as in (37) since

$$a_{j,\alpha}\sum_{k,\beta} \varepsilon_k(\boldsymbol{r}_\beta)|1_{k,\beta}\rangle = \varepsilon_j(\boldsymbol{r}_\alpha)|\text{vacuum}\rangle, \quad (42)$$

with $|\text{vacuum}\rangle = |0\rangle_{x,1}|0\rangle_{y,1}|0\rangle_{x,2}|0\rangle_{y,2}$.

Thus, quantum entangled states can lead both to entangled and separable cross-spectral density tensors. For example for the classically entangled state (15) this is $\varepsilon_x(\boldsymbol{r}_1) = \varepsilon_y(\boldsymbol{r}_2), \varepsilon_y(\boldsymbol{r}_1) = \varepsilon_x(\boldsymbol{r}_2) = 0$, while for the classically separable state (14) this is $\varepsilon_j(\boldsymbol{r}_\alpha) = \varepsilon$ for all j, α.

6. Conclusions

A suitable classical version of entanglement naturally arises in the context of coherence for classical vectorial light. For classically factorized states we have obtained the following results:

(i) There is no definite relation between classical entanglement and global and Hilbert–Schmidt degrees of coherence.
(ii) There is a definite and meaningful relation between classical entanglement and the trace degree of coherence.
(iii) There is a definite and meaningful relation between classical entanglement and mean polarization. This extends also to arbitrary spatial distributions.
(iv) The above relations mimic similar results in the quantum domain although in a very different context.
(v) By examining the reduced spatial and polarization subsystems we have found a suitable polarization counterpart of the trace degree of coherence, and a suitable spatial counterpart of the degree of mean polarization.
(vi) As a byproduct we have found that the degree of mean polarization is the maximum trace degree of coherence achievable via entanglement-local unitary transformations.
(vii) We have shown that there is no definite relation between classical and quantum entanglement since every classically factorized cross-spectral density tensor can correspond either to quantum entangled or quantum separable states.

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