A Multidimensional Version of Rolle's Theorem

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In this paper we obtain for functions $f\colon \mathbf{R}^n \to \mathbf{R}^p$ a version of Rolle's Theorem which we hope the readers will find useful and interesting for the following reasons. Three fundamental results from Calculus: namely Rolle's Theorem, the Mean Value Theorem and the Cauchy Generalized Mean Value Theorem can be easily derived from it. The version has intuitive geometrical applications and the proof is very simple.

Teachers may find it appropriate to incorporate our result in a course on Multivariable Calculus, since it provides an example of how certain one-dimensional theorems can be rephrased in higher dimensional spaces, and it shows that by expanding our mathematical horizon we frequently gain in organization and unity. Professional mathematicians are all familiar with these facts, but students will surely derive from them a motivation to learn more.

The basic idea of our result is to assume a certain behavior of f on the boundary ∂R of a n-dimensional region R (in the real line this behavior reduces to the familiar condition $f(a) = f(b)$) to obtain information on the derivative of f at an interior point of R. Of particular relevance to the result is the Mean Value Theorem of Sanderson **[10]** for a function $\mathbf{v}\colon [a, b] \to \mathbf{R}^p$. We extend his theorem to functions of several variables.

The paper ends with an additional, more general version of Rolle's Theorem, and with an open problem and a conjecture which will hopefully stimulate the reader's mathematical curiosity.

We now list the terminology used and the results needed in the sequel. $\mathbf{0}(m \times n)$ stands for the zero matrix with m rows and n columns. $\mathbf{x} \cdot \mathbf{y}$ denotes the Euclidean inner product between $\mathbf{x}$ and $\mathbf{y}$ and the norm of $\mathbf{x}$ is $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$. We repeatedly make reference to the following sets:

$$D(\mathbf{x}_0, r) = \{\mathbf{x} \in \mathbf{R}^n\colon \|\mathbf{x} - \mathbf{x}_0\| \le r\},\ B(\mathbf{x}_0, r) = \{\mathbf{x} \in \mathbf{R}^n\colon \|\mathbf{x} - \mathbf{x}_0\| < r\},$$

$$\text{and } S(\mathbf{x}_0, r) = \{\mathbf{x} \in \mathbf{R}^n\colon \|\mathbf{x} - \mathbf{x}_0\| = r\} = \partial D(\mathbf{x}_0, r).$$

The two propositions below play a key role in the proof of our multi-dimensional version of Rolle's Theorem.

Propostion 1. *Let* $f\colon D(\mathbf{x}_0, r) \subset \mathbf{R}^n \to \mathbf{R}$ *and let* $\mathbf{c} \in B(\mathbf{x}_0, r)$ *be an extremum point of* f. *Assume that* f *is differentiable at* $\mathbf{c}$. *Then* $f'(\mathbf{c}) = \mathbf{0}(1 \times n)$.

Proposition 2. *Let* $f\colon D(\mathbf{x}_0, r) \subset \mathbf{R}^n \to \mathbf{R}$ *be continuous. Then the image of* f *is a closed and bounded interval* $[m, M]$.

We point out that the proof of Rolle's Theorem in $\mathbf{R}$ is based on the one-dimensional version of the two propositions.

Results. The following simple example shows that a straightforward reformulation of Rolle's Theorem in $\mathbf{R}^n$, $n \geq 2$, fails.

Example 1. Let $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ be defined by

$$f(x, y) = \left(x(x^2 + y^2 - 1), y(x^2 + y^2 - 1)\right).$$

The function f is continuous on $D(\mathbf{0}, 1)$, is differentiable on $B(\mathbf{0}, 1)$ and $f(\mathbf{x}) = \mathbf{0}$ for every $\mathbf{x} \in S(\mathbf{0}, 1)$. However, $f'(\mathbf{x}) \neq \mathbf{0}(2 \times 2)$ for all $\mathbf{x} \in B(\mathbf{0}, 1)$.

We are now ready to state and prove our main result.

Theorem 1. *Let $f\colon D(\mathbf{x}_0, r) \subset \mathbf{R}^n \to \mathbf{R}^p$ be continuous on $D(\mathbf{x}_0, r)$ and differentiable on $B(\mathbf{x}_0, r)$. Assume that there exists a vector $\mathbf{v} \in \mathbf{R}^p$ such that*

i) *$\mathbf{v}$ is orthogonal to $f(\mathbf{x})$ for every $\mathbf{x} \in S(\mathbf{x}_0, r)$.*

Then there exists a vector $\mathbf{c} \in B(\mathbf{x}_0, r)$ such that $\mathbf{v} \cdot f'(\mathbf{c})\mathbf{u} = 0$ for every $\mathbf{u} \in \mathbf{R}^n$.

Proof: Let $k\colon \mathbf{R}^p \to \mathbf{R}$ be defined by $k(\mathbf{x}) = \mathbf{v} \cdot \mathbf{x}$. Set $g(\mathbf{x}) = k(f(\mathbf{x}))$. By Proposition 2 the image of g is a bounded and closed interval $[m, M]$. Assumption i) implies that g is 0 on $S(\mathbf{x}_0, r)$. Hence we may assume, without loss of generality, that g reaches its maximum value, M, at a point $\mathbf{c} \in B(\mathbf{x}_0, r)$, namely $M = g(\mathbf{c})$. By Proposition 1 $g'(\mathbf{c}) = \mathbf{0}(1 \times n)$, i.e. $\mathbf{v} \cdot f'(\mathbf{c})\mathbf{u} = 0$ for every $\mathbf{u} \in \mathbf{R}^n$. QED.

Remark 1. Assumption i) can be replaced by the equivalent statement

"ii) $\mathbf{v} \cdot f(\mathbf{x})$ *is constant on* $S(\mathbf{x}_0, r)$";

and the conclusion of the theorem can be expressed in the equivalent but geometrically more intuitive way

"$\mathbf{v}$ *is orthogonal to the vectors* $\dfrac{\partial f}{\partial x_1}(\mathbf{c}), \dfrac{\partial f}{\partial x_2}(\mathbf{c}), \ldots, \dfrac{\partial f}{\partial x_n}(\mathbf{c})$".

Remark 2. $D(\mathbf{x}_0, r)$ can be replaced by the closure of any open, bounded and connected set R of $\mathbf{R}^n$.

Rolle's Theorem, the Mean Value Theorem and the Cauchy Generalized Mean Value Theorem are easily derived from Theorem 1.

Corollary 1 (Cauchy). *Let $a < b$ and $f, g\colon [a, b] \to \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$[f(b) - f(a)]g'(c) = f'(c)[g(b) - g(a)].$$

Proof: If $f(a) = f(b)$ and $g(a) = g(b)$ there is nothing to prove.

Assume $[f(b) - f(a)]^2 + [g(b) - g(a)]^2 > 0$. Define $S\colon [a, b] \to \mathbf{R}^2$ by $S(t) = (g(t), f(t))$. Let $\mathbf{v} = (f(b) - f(a), g(a) - g(b))$. Then $\mathbf{v} \cdot T(a) = \mathbf{v} \cdot T(b) = f(b)g(a) - f(a)g(b)$. Hence, according to Theorem 1 (see Remark 1), there is a point $c \in (a, b)$ such that $\mathbf{v} \cdot T'(c)t = 0$ for every $t \in \mathbf{R}$. With $t \neq 0$ we obtain $[f(b) - f(a)]g'(c) = f'(c)[g(b) - g(a)]$.

Setting $g(x) = x$ gives the Mean Value Theorem. If, in addition $f(b) = f(a)$, then we have Rolle's Theorem. QED

The next Corollary is the Mean Value Theorem of Sanderson **[10]** mentioned in the Introduction.

Corollary 2. *Let $a < b$ and $\mathbf{v}$: $[a, b] \to \mathbf{R}^p$ be k times differentiable. Assume that $\mathbf{v}(a), \mathbf{v}(b)$ and the $k - 1$ derivatives of $\mathbf{v}$ at a are orthogonal to a non-zero vector $\mathbf{v}_0$. Then, for some $c \in (a, b)$, $\mathbf{v}^{(k)}(c)$ is orthogonal to $\mathbf{v}_0$.*

Proof: From Theorem 1 we derive the existence of a point $c_1 \in (a, b)$ such that $\mathbf{v}_0$ is orthogonal to $\mathbf{v}'(c_1)$. The theorem can now be applied to $\mathbf{v}'$ in the interval $[a, c_1]$ to yield a point $c_2 < c_1$ such that $\mathbf{v}_0$ is orthogonal to $\mathbf{v}''(c_2)$. This procedure can be repeated $k - 1$ times to obtain $c = c_k < c_{k-1}$ such that $\mathbf{v}_0 \cdot \mathbf{v}^{(k)}(c) = 0$. QED

A recent result of Evard and Jafari **[4]** (see also **[7]**) follows from Theorem 1.

Corollary 3. *Let $\mathbf{C}$ be the field of complex numbers and f: $\mathbf{C} \to \mathbf{C}$ be a holomorphic function. Assume that there are points $\mathbf{a} \neq \mathbf{b}$ such that $f(\mathbf{a}) = f(\mathbf{b})$. Then there exist $\mathbf{z}_1, \mathbf{z}_2$ in the open line segment joining $\mathbf{a}$ with $\mathbf{b}$ such that* $\mathrm{Re}(f'(\mathbf{z}_1)) = \mathrm{Im}(f'(\mathbf{z}_2)) = 0$.

Proof: Let $f(\mathbf{z}) = f(x + iy) = u(x, y) + iv(x, y)$ and $\mathbf{p} \in \mathbf{R}^2$, $\mathbf{p} = (p_1, p_2) = (\mathrm{Re}(\mathbf{a}), \mathrm{Im}(\mathbf{a}))$, $\mathbf{q} \in \mathbf{R}^2$, $\mathbf{q} = (q_1, q_2) = (\mathrm{Re}(\mathbf{b}), \mathrm{Im}(\mathbf{b}))$. Define $g(t) = (u(\mathbf{q} + t(\mathbf{p} - \mathbf{q})), v(\mathbf{q} + t(\mathbf{p} - \mathbf{q})))$, $t \in [0, 1]$. Notice that $g(0) = g(1)$. According to Theorem 1, for every $\mathbf{x} \in \mathbf{R}^2$, $\mathbf{x} \neq \mathbf{0}$, there exists $t_0 \in (0, 1)$ such that $\mathbf{x} \cdot g'(t_0)t = 0$, for every $t \in \mathbf{R}$. Let $t = 1$ and choose the vector $\mathbf{x}_1 = (p_1 - q_1, p_2 - q_2)$. Then

$$0 = \mathbf{x}_1 \cdot g'(t_0) = \frac{\partial u}{\partial x}(g(t_0))(p_1 - q_1)^2 + \frac{\partial u}{\partial y}(g(t_0))(p_1 - q_1)(p_2 - q_2)$$

$$+ \frac{\partial v}{\partial x}(g(t_0))(p_1 - q_1)(p_2 - q_2) + \frac{\partial v}{\partial y}(g(t_0))(p_2 - q_2)^2.$$

Since f is holomorphic, its real and imaginary part satisfy the Cauchy-Riemann equations (see **[1]**), i.e. $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$. Hence

$$\frac{\partial u}{\partial x}(g(t_0))\left[(p_1 - q_1)^2 + (p_2 - q_2)^2\right] = 0.$$

This implies $\partial u/\partial x(g(t_0)) = \mathrm{Re}(f'(\mathbf{z}_1)) = 0$, where $\mathbf{z}_1 = \mathbf{q} + t_0(\mathbf{p} - \mathbf{q})$.
To obtain the other equality use the vector $\mathbf{x}_2 = (q_2 - p_2, p_1 - q_1)$. QED

Theorem 1 can be given a slightly more general form.

Theorem 2. (Second version of Rolle's Theorem in $\mathbf{R}^n$). *Let f: $D(\mathbf{x}_0, r) \subset \mathbf{R}^n \to \mathbf{R}^p$ be continuous on $D(\mathbf{x}_0, r)$ and differentiable on $B(\mathbf{x}_0, r)$. Let $\mathbf{v} \in \mathbf{R}^p$, $\mathbf{z}_0 \in B(\mathbf{x}_0, r)$ be such that*

ii) $\mathbf{v} \cdot (f(\mathbf{x}) - f(\mathbf{z}_0))$ *does not change sign on $S(\mathbf{x}_0, r)$.*

Then there exists a vector $\mathbf{c} \in B(\mathbf{x}_0, r)$ such that $\mathbf{v} \cdot f'(\mathbf{c})\mathbf{u} = 0$ for every $\mathbf{u} \in \mathbf{R}^n$.

Proof: We may assume, without loss of generality, that $\mathbf{v} \cdot (f(\mathbf{x}) - f(\mathbf{z}_0)) \leq 0$ for all $\mathbf{x} \in S(\mathbf{x}_0, r)$. This implies the existence of a point $\mathbf{c} \in B(\mathbf{x}_0, r)$ such that $\mathbf{v} \cdot f(\mathbf{c}) = M$, where $M = \max\{\mathbf{v} \cdot f(\mathbf{x}): \mathbf{x} \in D(\mathbf{x}_0, r)\}$. Consequently, $\mathbf{v} \cdot f'(\mathbf{c})\mathbf{u} = 0$ for all $\mathbf{u} \in \mathbf{R}^n$. QED

Remark 3. In the case when $n = p = 1$ Theorem 2 says that if for some $z \in (a, b)$ we have

j) either $f(z) \geq \max\{f(a), f(b)\}$ jj) or $f(z) \leq \min\{f(a), f(b)\}$,

then there exists $c \in (a, b)$ such that $f'(c) = 0$. Notice that every $z \in (a, b)$ satisfies either j) or jj) when $f(a) = f(b)$.

The following result (see Boas [3]) is an easy consequence of the above remark.

Corollary 4. *Let $a < b$ and $f: [a, b] \to \mathbf{R}$ be continuous on $[a, b]$ and differentiable on $[a, b]$. Assume that $f'(a) = f'(b)$. Then there exists a point $c \in (a, b)$ such that*

$$f'(c)(c - a) = f(c) - f(a).$$

Proof: A straightforward computation shows that Corollary 4 is true for f if and only if it is true for $g(x) = f(x) - xf'(a)$. Therefore we may assume, without loss of generality, that $f'(a) = f'(b) = 0$. Define

$$h(x) = \begin{cases} \dfrac{f(x) - f(a)}{x - a} & x \neq a \\ 0 & x = a \end{cases}.$$

The function h is continuous on $[a, b]$, differentiable on $(a, b]$ and $h'(b) = -h(b)/(b - a)$.

Assume that $h(b) \neq 0$. From $h(b)h'(b) < 0$ and $h(a) = 0$ we derive the existence of $z \in (a, b)$ which satisfies either i) or ii). In the case when $h(b) = 0$ $(= h(a))$ every point $z \in (a, b)$ will do the job. Hence, by Theorem 2 (Remark 3), there exists $c \in (a, b)$ such that $h'(c) = 0$, and this implies the stated result. QED

Geometrical Applications of Theorem 1 and Theorem 2. We present three geometrical applications. To allow for a visual representation of the results we do not state them in their full generality.

Application 1. Let f: $D(\mathbf{0}, 1) \subset \mathbf{R}^2 \to \mathbf{R}^3$, $f(u, v) = (x(u, v), y(u, v), z(u, v))$ be continuous on $D(\mathbf{0}, 1)$ and differentiable on $B(\mathbf{0}, 1)$ and let $G = \text{Imf}$. Assume that there exists a plane p: $ax + by + cz + d = 0$, such that $(x(u, v), y(u, v), z(u, v)) \in p$ for every $(u, v) \in S(\mathbf{0}, 1)$. Then there is a point $(u_0, v_0) \in B(\mathbf{0}, 1)$ such that the tangent plane to the surface G at the point $f(u_0, v_0)$ is parallel to p.

Justification. By Theorem 1 (see Remark 1) the vector $\mathbf{v}_0 = (a, b, c)$ is orthogonal to

$$\frac{\partial f}{\partial u}(\mathbf{u}_0) = \mathbf{p} \quad \text{and} \quad \frac{\partial f}{\partial v}(H\mathbf{u}_0) = \mathbf{q},$$

for some $\mathbf{u}_0 \in B(\mathbf{0}, 1)$, $\mathbf{u}_0 = (u_0, v_0)$. The tangent plane to G at $f(\mathbf{u}_0)$ is $\{f(\mathbf{u}_0) + m\mathbf{p} + n\mathbf{q}: m, n \in \mathbf{R}\}$, which is obviously parallel to p.

Application 2. Let f: $D(\mathbf{0}, 1) \subset \mathbf{R}^2 \to \mathbf{R}^3$, $f(u, v) = (x(u, v), y(u, v), z(u, v))$ be continuous on $D(\mathbf{0}, 1)$ and differentiable on $B(\mathbf{0}, 1)$. Denote by G the surface

$G = \mathrm{Im} f$ and let $G_0 = f(\partial D(0,1))$. Assume that there is a plane p: $ax + by + cz + d = 0$, such that G_0 is on one side of p and there is a point of S on the other side of p. Then the tangent plane to G at some point $P \in S$ is parallel to p.

Justification. Let $\mathbf{u}_i = (u_i, v_i) \in B(\mathbf{0}, 1)$ be such that $f(\mathbf{u}_i)$ is on the other side of p with respect to G_0. Then $(a, b, c) \cdot (f(\mathbf{u}) - f(\mathbf{u}_i))$ does not change sign on $\partial D(\mathbf{0}, 1)$. The conclusion follows from Theorem 2.

We illustrate this situation with an example

Example 2. Let $f(u, v) = (u^2 + v^2 - u, u^2 + v, u^2 - v)$. Then $G_0 = \{(1 - u, u^2 + v, u^2 - v)\colon u^2 + v^2 = 1\}$ and $f(0,0) = (0,0,0)$ are on opposite sides of the plane p: $x + y + z = 1/2$. Hence there is a point P on $G = \mathrm{Im} f$ where the tangent plane is parallel to p. The point is $P = f(1/6, 0)$.

Application 3. Let $\mathbf{x}$: $[a, b] \to \mathbf{R}^3$ be continuous on $[a, b]$ and differentiable on $[a, b]$, and let $P = \mathbf{x}(a) = (x(a), y(a), z(a))$, $Q = \mathbf{x}(b) = (x(b), y(b), z(b))$. Then for every plane p passing through the line L joining P with Q there is a point $c \in (a, b)$ such that the vector $\mathbf{x}'(c)$ is parallel to p. In particular, when the plane p is the one containing the origin, we obtain that $\mathbf{x}'(c)$ satisfies the equality

$$\text{i)} \quad x(a)[y(b)z'(c) - z(b)y'(c)] + y(a)[z(b)x'(c) - x(b)z'(c)] + z(a)[x(b)y'(c) - y(b)x'(c)] = 0.$$

Justification. The first part is an immediate consequence of Theorem 1, since for every plane passing through L there is a vector $\mathbf{u}$ orthogonal to L and to the plane. For the second part observe that the direction $\mathbf{v}$ of a line orthogonal to p is given by the cross product of the two vectors $\mathbf{x}(a)$ and $\mathbf{x}(b)$, i.e. $\mathbf{v} = \mathbf{x}(a) \times \mathbf{x}(b)$. Thus there exists $c \in (a, b)$ such that $\mathbf{x}(a) \times \mathbf{x}(\mathrm{b}) \cdot \mathbf{x}'(c) = 0$, which implies i). For a different justification of the result presented in Application 3 see [**2**].

Open problem and conjecture. We conclude the paper with an open problem and a conjecture. Theorem 1 and Theorem 2 remain valid if $\mathbf{R}^p$ is replaced by a Hilbert space $\mathbf{H}$. No changes are needed in the proof. They are also true when $\mathbf{R}^p$ is replaced by a Banach space $\mathbf{F}$ with the vector $\mathbf{v}$ substituted by a linear continuous functional ϕ.

We conjecture that the theorems are false when $\mathbf{R}^n$ is replaced by an infinite-dimensional Banach space $\mathbf{E}$, because Proposition 2, which plays a key role in both proofs, fails in $\mathbf{E}$. In fact, the unit closed ball $D(\mathbf{0}, 1)$ of $\mathbf{E}$ is not compact. Consequently, there exists continuous functions f: $D(\mathbf{0}, 1) \to \mathbf{R}$ such that $\mathrm{Im} f$ is an open interval, as illustrated by the following example.

Example 3. Let $\mathbf{H}$ be the Hilbert space of square summable sequences of real numbers and let D be the disk of $\mathbf{H}$ centered at the origin and with radius 1, $D = D(\mathbf{0}, 1)$. Define

$$T\colon D \to \mathbf{H},\ T(\mathbf{x}) = T(x_1, x_2, \dots) = \left(\sqrt{1 - \|\mathbf{x}\|^2}, x_1, x_2, \dots\right).$$

The map T does not have any fixed point on D. In fact, since $\|T(\mathbf{x})\| = 1$ for all $\mathbf{x} \in D$, every potential fixed point $\mathbf{x}$ must be located on the boundary of D, i.e. $\mathbf{x}$ is fixed for T only if $\|\mathbf{x}\| = 1$. This implies $T(\mathbf{x}) = (0, x_1, \dots)$. Combining this result with the equality $T(\mathbf{x}) = \mathbf{x}$ gives $\mathbf{x} = \mathbf{0}$, against the assumption $\|\mathbf{x}\| = 1$. The fixed-point free map T allows us to define the continuous function

$$f\colon D \to \mathbf{R},\, f(\mathbf{x}) = \frac{1}{\|\mathbf{x} - T(\mathbf{x})\|}.$$

Let us show that the image of f is the open half-line $(0.5, \infty)$.

We already know that $\|\mathbf{x} - T(\mathbf{x})\| > 0$ for every $\mathbf{x} \in D$. To verify that the greatest lower bound (glb) of $\{\|\mathbf{x} - T(\mathbf{x})\|\colon \mathbf{x} \in D\}$ is 0 consider the elements $\mathbf{x}_n \in D(\mathbf{0}, 1)$ whose entries after the n position are all 0, while the first n are all equal to $1/\sqrt{n}$:

$$\mathbf{x}_n = \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}, 0, \dots\right).$$

Clearly $\|\mathbf{x}_n\| = 1$ and $\|\mathbf{x}_n - T(\mathbf{x}_n)\| = \sqrt{2/n}$. Hence the greatest lower bound is 0.

To see that $\|\mathbf{x} - T(\mathbf{x})(\| < 2$ for every $\mathbf{x} \in D$, notice that $\|\mathbf{x} - T(\mathbf{x})\| = 2$ requires $\|\mathbf{x}\| = 1$ and $\mathbf{x} = -T(\mathbf{x})$, i.e.

$$(x_1, x_2, \dots) = (0, -x_1, -x_2, \dots).$$

The above equality implies $\mathbf{x} = \mathbf{0}$, a contradiction with $\|\mathbf{x}\| = 1$. To verify that the least upper bound (lub) of $\{\|\mathbf{x} - T(\mathbf{x})\|\colon \mathbf{x} \in D\}$ is 2 consider the elements $\mathbf{y}_n$ whose entries after the n position are all 0, while the first n are alternatively equal to $\pm 1/\sqrt{n}$:

$$\mathbf{x}_n = \left(\frac{1}{\sqrt{n}}, -\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, -\frac{1}{\sqrt{n}}, \dots, (-1)^{n+1}\frac{1}{\sqrt{n}}, 0, \dots\right).$$

Then $\|\mathbf{x}_n - T(\mathbf{x}_n)\| = \sqrt{4 - (2/n)}$, which implies that the least upper bound is 2. Hence the image of f is the open half-line $(0.5, \infty)$.

It would be nice to have an example which shows that Theorems 1 and 2 fail in infinite dimension. So far we have been unable to construct it.

In the References we mention other contributions (see [**5**] pg. 19, [**6**], [**8**], [**9**], [**11**]) regarding Rolle's Theorem, the Mean Value Theorem and the Cauchy Generalized Mean Value Theorem. They are not directly related to this paper, but the reader may find them useful to get a better overview of the work done in this area.

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Reply to CD's

"These are indeed exciting times in the world of Mathematics." I would like to respond to the "Tale of Two CD's" by Dan Kennedy. "The winds of change are blowing through ... the curriculum" and some of us feel like the French citizens in the late 1930's that we might be better off without some of the coming changes. I am a practicing mathematician of a dozen years experience writing simulations, optimizations, and analyses in wireless and landline telephony, printed circuit board production, airline fleet assignment, yield management, and maintenance delays. I also have considerable exposure to Mathematics education as consumer and producer.

It is apt that he chooses the compact (CD) as his analog (pun intended) for the newest New Math. The CD is truly a triumph of marketing over technology. It is quiet and cute, shiny and high-tech. If the medium were truly digital, then the sound wouldn't be dramatically altered by putting a rubber mat on top, painting the rim green, or reversing the prongs of the AC cord. By stuffing thousands of dollars of digital signal processors (DSPs) into the signal path, clever engineers have surpassed cheap turntables to the point where the best $10,000 CD players outperform $1000 turntables. But, of course, you're listening to the DSPs rather than the CD.

Those of us who keep concert seats year after year in spite of the surface noise (audience rustling) and clicks and pops (coughs and sneezes) tend also to find ourselves labelled as "collectors" and "Luddites" as we continue to purchase records. I have over two thousand phonograph records and a Linn, LOCI, and EK-1 to play them. The huge advantage of CD over record is the low manufacturing cost which should have brought the consumer cheap recordings, but somehow this never happened.

The educational analogy to "compact" sound is a simplified curriculm relying on technology to replace the drudgery of traditional teaching methods. We are offering better high school mathematics programs than before, alas, to college students and, occasionally, to graudate students. Reducing student involvement in math courses has failed to attract better or more motivated students to our classrooms; did we really expect it to do so?

In our *Brave New World* (Aldous Huxley, 1932) of post-Modern education, the emphasis is on maintaining the students' willingness to enroll in our courses and come to our classes. We must entertain them and we mustn't scare them away so machines do their "timeses and gazintas" and solve equations for them and invert matrices for them and even graph functions for them. Being able to balance a checkbook without a machine is *A Sense of Power* (Asimov, 1957) in today's Mathematics classroom.

Mathematics is not a spectator sport; we learn it by doing it. While my Linear Programming students this fall will learn to use AMPL modeling language, they also will graph polytopes and crank out Simplex optimizations by hand.

Do I suppose Newton would be flattered to see our students walking a road to discovery essentially the same as his? *I certainly do.* I know I'm flattered to see my own discovery process (including my software) used ten years later to teach new students in cellular mobile telephone system engineering as its success in pedegogy affirms my own confidence in my knowledge. Isaac Newton's results are certainly more than thirty times as worthy of posterity as mine and Calculus and Physics students should see them his way.

Our Mathematics education and curriculum certainly could use a dose of enthusiasm and support from both teacher and students, but I doubt it requires much revision. Computational and display tools can enhance and deepen our insights and our delight, but we must remember that students learn by traveling the road to discovery with their own eyes, ears, and limbs and not by watching machines (or professors) do it for them.

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