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Clifford Analysis on Spheres and Hyperbolae

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We develop aspects of Clifford analysis over the sphere and hyperbolae. We focus primarily on the hyperbola lying in the Minkowski type space $\mathbb{R}^{n,1}$. We show that in order to give a proper extension of basic results on Clifford analysis in Euclidean space to this context one needs to consider both hyperbolae lying in $\mathbb{R}^{n,1}$. We also introduce Bergman spaces of L^p left monogenic sections in this context and consider the decomposition of square integrable sections over suitable bundles constructed over subdomains of spheres and hyperbolae. The results presented here cover the necessary background to enable one to set up and solve boundary value problems for field-type equations over hyperbolae. In particular, one can study analogues of the Dirichlet problem for analogues of the Laplacian over hyperbolae and spheres.

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Introduction

In this paper we continue the investigation of function theoretic properties of Dirac operators over spheres and hyperbolae. This builds on results developed in [10–12, 15]. The fact that one can treat function theory associated to Dirac type operators over general manifolds in a similar fashion to the function theory for Dirac operators over Euclidean space is pointed out in [4], see also [2]. However, in the cases of the sphere and the hyperbola one can use Cayley transformations to explicitly carry over results, structures and formulae directly from the Euclidean setting. In this paper we continue to investigate this theme.

Here we primarily focus on the hyperbola. We show that in order to obtain as full an extension of Clifford analysis as possible in this context we need to consider not one hyperbola, but two separate hyperbolae lying in a Minkowski-type space. We set up a Cauchy integral formula in this context and establish the completeness of the Bergman spaces of sections satisfying a generalized Cauchy Riemann equation. These sections are defined on bundles over domains of the hyperbolae, and each fibre of the bundles are isomorphic as vector spaces to a Clifford algebra. This leads us to extend other results from [7] and give a decomposition of the L^2 spaces of sections on these bundles. Parallel arguments over the sphere are indicated. We also indicate how to

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construct Dirac operators over spheres lying in a null space of a Minkowski-type space.

Preliminaries

Here we shall introduce the background material that we shall need for the rest of the paper.

We shall consider the real Clifford algebra, Cl_n, generated from $\mathbb{R}^n$ endowed with a negative definite inner product. So Cl_n has as basis the elements 1, $e_1, \dots, e_n \dots, e_{j_1} \dots e_{j_r}, \dots, e_1 \dots e_n$, where $e_1, \dots e_n$ is an orthonormal basis for $\mathbb{R}^n$, and $1 \leqslant r \leqslant n$. So the basis vectors $e_1, \dots, e_n$ satisfy the anticommutation relationship $e_i e_j + e_j e_i = -2\delta_{i,j}$, where $\delta_{i,j}$ is the Kroneker delta function. It may be observed that the algebra Cl_n has dimension 2^n. For each vector $x \in \mathbb{R}^n \backslash \{0\}$ there is a multiplicative inverse $x^{-1} = -x/\|x\|^2 \in \mathbb{R}^n \backslash \{0\}$. Up to the minus sign this transformation corresponds to the usual Kelvin inversion of non-zero vectors in $\mathbb{R}^n$. For a general element $a = a_0 + \cdots + a_{1,\dots,n} e_1 \dots e_n \in Cl_n$ the norm, $\|a\|$ of a is defined to be $(a_0^2 + \cdots + a_{1,\dots,n}^2)^{1/2}$.

Kelvin inversion is a particular example of a Moebius transformation over $\mathbb{R}^n \cup \{\infty\}$. The group of Moebius transformations over $\mathbb{R}^n \cup \{\infty\}$ is generated by translations, rotations, dilations and Kelvin inversion. In [1, 14] it is shown that each Moebius transformation can be expressed as $y = \psi(x) = (ax + b)(cx + d)^{-1}$, where a, b, c and d belong to Cl_n and satisfy

(i) a, b, c and d are all products of vectors in $\mathbb{R}^n$.

(ii) $a\tilde{c}$, $c\tilde{d}$, $d\tilde{b}$, $b\tilde{a} \in \mathbb{R}^n$, where $\sim$ is the anti-automorphism

$$\sim : Cl_n \to Cl_n : e_{j_1} \dots e_{j_r} \to e_{j_r} \dots e_{j_1}.$$

(iii) $a\tilde{d} - b\tilde{c} = \pm 1$.

The matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is called a Vahlen matrix. The set of all such matrices over $\mathbb{R}^n$ form a group under matrix multiplication, [1], which is called the Vahlen group.

For each pair of positive integers m and n with $n < m$ we have that $Cl_n \subset Cl_m$, and Cl_n is a subalgebra of Cl_m. Moreover, we may consider the complexification, $Cl_n(C)$, of Cl_n. This is a complex algebra of complex dimension 2^n, and Cl_n is a real subalgebra of $Cl_n(C)$.

The Cayley transformation $K_1(x) = (x - e_{n+1})(-e_{n+1} + 1)^{-1}$ transforms $\mathbb{R}^n$ onto $S^n \backslash \{e_{n+1}\}$, where S^n is the unit sphere in $\mathbb{R}^{n+1}$.

For each point $x \in S^n$ we denote the open ball $B(x, r) \cap S^n$ by $B_{S^n}(x, r)$ where $B(x, r) = \{y \in \mathbb{R}^{n+1} : \|x - y\| < r \in \mathbb{R}^+\}$. For $x_1, x_2 \in S^n$ one can find an orthogonal transformation $O : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that $O(x_1) = x_2$ and $O(B_{S^n}(x_1, r)) = B_{S^n}(x_2, r)$. Similarly, one can introduce the spherical shell $A_{S^n}(x, r_1, r_2) = A(x, r_1, r_2) \cap S^n$ where $A(x, r_1, r_2) = \{y \in \mathbb{R}^{n+1} : r_1 < \|x - y\| < r_2, r_1, r_2 \in \mathbb{R}^+\}$ and $x \in S^n$. Again it may be observed that $O(A_{S^n}(x_1, r_1, r_2)) = A_{S^n}(x_2, r_1, r_2)$.

Besides the algebra Cl_n we also need the algebra $Cl_{n,1}$. This Clifford algebra is generated from the Minkowski-type space $\mathbb{R}^{n,1}$ which is spanned by $e_1, \dots, e_n, f_{n+1}$. So $Cl_{n,1} = Cl_n + Cl_n f_{n+1}$ where f_{n+1} satisfies the relationship $f_{n+1}^2 = 1$ and

$f_{n+1}e_j = -e_j f_{n+1}$ for $1 \leqslant j \leqslant n$. On placing $f_{n+1} = ie_{n+1}$ it may be observed that $Cl_{n,1}$ is a real subalgebra of $Cl_{n+1}(C)$. Following [16] it may be observed that the Cayley transformation $K_2(x) = (-x + f_{n+1})(f_{n+1}x + 1)^{-1}$ transforms the unit disc $D_n = \{x \in \mathbb{R}^n\colon \|x\|^2 < 1\}$ onto the hyperbola H_n^+ which is the component of $H_n = \{x \in \mathbb{R}^{n,1}\colon x^2 = 1\}$ which contains f_{n+1}. We denote the component of H_n which contains $-f_{n+1}$ by H_n^-. It may easily be noted that $H_n = H_n^+ \cup H_n^-$, and that the Cayley transformation K_2 extends to a Cayley transformation

$$K_2\colon \mathbb{R}^n \backslash S^{n-1} \to H_n \backslash \{-f_{n+1}\}\colon K_2(x) = (-x + f_{n+1})(f_{n+1}x + 1)^{-1}.$$

It may also easily be noted that $H_n^+ = -H_n^-$. Also, for each $x \in S^{n-1} \subset \mathbb{R}^n$ it may be observed that $(\cosh\frac{\theta}{2} f_{n+1} + \sinh\frac{\theta}{2} x)\, f_{n+1}(\cosh\frac{\theta}{2} f_{n+1} + \sinh\frac{\theta}{2} x)$ describes the action of a Lorentz boost over H_n^+ on f_{n+1}. By varying x and θ one can linearly transform f_{n+1} to any point on H_n^+. We denote this Lorentz boost by $O_{x,\theta}$. For each $x \in H_n^+$ and $r \in \mathbb{R}^+$ let $B_{H_n^+}(x, r)$ denote the ball $\{y \in H_n^+\colon |(x-y)^2| < r^2\}$ and let $A_{H_n^+}(x, r_1, r_2)$ denote the spherical shell $\{y \in H_n^+\colon r_1^2 < |(x-y)^2| < r_2^2\}$. It may be observed that spherical shells are transformed to spherical shells with the same radii under the Lorentz boosts described here. By multiplying such a spherical shell by -1 one can also transform a spherical shell on H_n^+ to a spherical shell on H_n^- with the same radii.

For further properties of Clifford algebras see [8].

We now proceed to introduce the Dirac operator over Euclidean space. This is the differential operator $D = \sum_{j=1}^n e_j(\partial/\partial x_j)$. The properties and applications of this operator provide the basis of Clifford analysis, see for instance [3, 7], and elsewhere. One simple and extremely important property is that $D^2 = -\Delta_n$, where Δ_n is the Laplacian in $\mathbb{R}^n$. For U a domain in $\mathbb{R}^n$ a differentiable function $f\colon U \to Cl_n(C)$ is said to be left monogenic if $Df = 0$, while a differentiable function $g\colon U \to Cl_n(C)$ is said to be right monogenic if $gD = 0$, where $gD = \sum_{j=1}^n (\partial g/\partial x_j)e_j$. It may be observed that if f is left monogenic then $\tilde{f}$ is right monogenic, and if g is right monogenic then $\tilde{g}$ is left monogenic.

Theorem 1. *Suppose that f is a left-monogenic function defined on a domain U and g is a right-monogenic function on U. Suppose also that V is a bounded subdomain of U with a Lipschitz continuous boundary, $S \subset U$. Then $\int_S g(x)n(x)f(x)\,\mathrm{d}\sigma(x) = 0$, where $n(x)$ is the outward pointing vector normal to S at x, and σ is the Hausdorff measure on S.*

The function $G(x) = x/\|x\|^n$ is an example of a function which is both left and right monogenic.

Theorem 2. *Suppose that f is left monogenic on the domain U, and that V is a bounded subdomain of U with Lipschitz continuous boundary $S \subset U$. Then for each $y \in V$*

$$f(y) = \frac{1}{\omega_n}\int_S G(x-y)n(x)f(x)\,\mathrm{d}\sigma(x),$$

where ω_n is the surface area of the unit sphere S^{n-1}.

Theorems 1 and 2 seem to have been first established in the case $n = 3$, and for smooth S, by Dixon, [6].

From [9] and elsewhere we have:

Theorem 3. *Suppose that* $y = \psi(x) = (ax + b)(cx + d)^{-1}$ *is a Moebius transformation and* $f(y)$ *is a left-monogenic function. Then* $J(\psi, x) f(\psi(x))$ *is left monogenic with respect to the variable* x, *where* $J(\psi, x) = \widetilde{(cx + d)}/\| cx + d \|^n$.

From [9] we have:

Definition 1. *Suppose that* S_1 *and* S_2 *are orientable, locally Lipschitz surfaces locally embedded in* $\mathbb{R}^n \cup \{\infty\}$ *and of co-dimension* 1. *Suppose also that there is a Moebius transformation* $y = \psi(x) = (ax + b)(cx + d)^{-1}$ *such that* $\psi(S_1) = S_2$. *Then these surfaces are said to be conformally equivalent.*

Surfaces of co-dimension 1 locally embedded in $\mathbb{R}^n \cup \{\infty\}$ will be refered to as hypersurfaces.

If the hypersurfaces S_1 and S_2 are conformally equivalent then the L^2 spaces $L^2(S_1, Cl_n)$ and $L^2(S_2, Cl_n(C))$ of square integrable, Cl_n valued functions over S_1 and S_2, respectively, are isometric. In fact if $v(y) \in L^2(S_2, Cl_n)$, then $J(\psi, x) v(\psi(x)) \in L^2(S_1, Cl_n)$, and this correspondence gives the isometry, see [5, 9].

It follows from the previous arguments that the Hardy 2-spaces of left, or right, monogenic functions associated to Lipschitz surfaces are transformed isometrically under Moebius transformations.

In [11] it is observed that the Cayley transformation K_1 can be used to re-establish most results from Clifford analysis over $\mathbb{R}^n$ to S^n. This includes analogues of Theorems 1 and 2. An analogue, D_{S^n}, of the Euclidean Dirac operator is set up over S^n, see also [15]. In particular, in [11] it is shown that if $f(y)$ is left monogenic on a domain in $\mathbb{R}^n$, then $D_{S^n} J(K_1^{-1}, x) f(K_1^{-1}(x)) = 0$. The function $J(K_1^{-1}, x) f(K_1^{-1}(x))$ can be seen as a section on the bundle $(S^n \backslash \{e_{n+1}\})(Cl_n(C)) = \{(x, J(K_1^{-1}, x) y)$: $x \in S^{n-1} \backslash \{e_{n+1}\}, y \in Cl_n(C)\}$. This is an example of one of the bundles set up in [10], see also [12]. Strictly speaking this bundle is not a Clifford bundle as each fibre is a multiplication on the left of Cl_n by $J(K_1^{-1}, x)$. By using the other Cayley transformation $K'_1(x)$, $= (x + e_{n+1})(e_{n+1} - 1)^{-1}$, from $\mathbb{R}^n$ onto $S^n \backslash \{-e_{n+1}\}$ one can extend this bundle to a bundle $S^n(Cl_n)$ over S^n. Again this bundle is an example of the type of bundles set up in [10], see also [12]. Following arguments presented in [10–12] one can now talk of left-monogenic sections in $S^n(Cl_n)$. One can also similarly set up right-monogenic sections, but these sections take their values in the bundle obtained by multiplying Cl_n on the right by $\tilde{J}(K_1^{-1})$, and proceeding as has just been outlined for the left-monogenic sections case. As described in [12] one can set up a similar bundle, $H_n^+(Cl_n)$, with left-monogenic sections over H_n^+. By noting that $H_n^+ = -H_n^-$ one can set up a similar bundle, $H_n^-(Cl_n)$, over H_n^-. By using the Cayley transformation $K_2 : \mathbb{R}^n \backslash S^{n-1} \to H_n$, one also has a similar bundle over $H_n \backslash \{-f_{n+1}\}$. Identifying this bundle with the bundle $H_n^-(Cl_n)$ one now gets a bundle $H_n(Cl_n)$ over H_n.

Clifford analysis on H_n

By noting that $H_n^+ = -H_n^-$ one can take the Dirac operator $D_{H_n^+}$ introduced in [11] over H_n^+, and re-introduce it as a Dirac operator, $D_{H_n^-}$ over H_n^-. It may easily be deduced that the function theory described for the operator $D_{H_n^+}$ in [11] can now be set up over H_n^- for the operator $D_{H_n^-}$. So we have a uniquely defined Dirac operator D_{H_n} defined over H_n, such that D_{H_n} restricted to act on smooth sections of $H_n^{\pm}(Cl_n)$ is the operator $D_{H_n^{\pm}}$ acting on these respective sections. By using the Cayley transformation $K_2 : \mathbb{R}^n \backslash S^{n-1} \to H_n$ one can mimick arguments given in [11] to set up the function theory for D_{H_n} analogous to the function theory set up in [11] for $D_{H_n^+}$ over H_n^+.

For each domain U lying in $\mathbb{R}^n$ the open set $U \backslash S^{n-1}$ is conformally equivalent via K_2 to an open set $V = K_2(U \backslash S^{n-1})$ lying in H_n. It should be noted that V might have one component lying on H_n^+ and another component lying on H_n^-. Moreover, if f is a left-monogenic function defined on U then $J(K_2^{-1}, x) f(K_2^{-1}(x))$ is a left monogenic section on $V(Cl_n)$, where $V(Cl_n)$ is the restriction to V of $H_n(Cl_n)$.

Using the Cayley transformation K_2 and arguments in [11] we have:

Theorem 4. *Suppose that U is a domain in $\mathbb{R}^n$ and f is a left-monogenic function on U, and that W is a bounded subdomain of U with a Lipschitz continuous boundary ∂W lying in U. Furthermore, suppose that $\partial W \cap S^{n-1}$ is a set of measure zero when seen as a subset of S^{n-1}. Then for each $y \in K_2(W \backslash S^{n-1})$ we have*

$$g(y) = \frac{1}{\omega_n} \int_{\partial K_2(W)} G(x-y) n(x) g(x) \, \mathrm{d}\sigma(x),$$

where $g(y) = J(K_n^{-1}, y) f(K_2^{-1}(y))$, and $G(x-y) = (x-y)/((x-y)^2)^{n/2}$, with $\partial K_2(W \backslash S^{n-1})$ the boundary of $K_2(W \backslash S^{n-1})$. Moreover, $n(x)$ is the unit vector in $TH_n(x)$, the tangent space to H_n at x, normal to $\partial K_2(W)$, and σ is the Hausdorff measure of $\partial K_2(W)$ seen as a submanifold of H_n.

It should be noted that in the previous theorem we can regard both $K_2(W \backslash S^{n-1})$ and $\partial K_2(W \backslash S^{n-1})$ as having components in both H_n^+ and H_n^-.

Using the Clifford analysis version of Morera's theorem, see [3, 13], one can deduce the following:

Theorem 5. *Suppose that U is a domain in $\mathbb{R}^n$ and that $f(x)$ is continuous on U. Moreover, suppose that $f|_{U \backslash S^{n-1}}$ is left monogenic. Then f is a left-monogenic function on U.*

Using this result and the Cayley transformation K_2 one may obtain:

Theorem 6. *Suppose that $V = K_2(U \backslash S^{n-1})$ for some domain U in $\mathbb{R}^n$, and that g is a left-monogenic section on $V(Cl_n)$. Suppose also that the left-monogenic function $f(u) = J(K_2, u) g(K_2(u))$ has a continuous extension to U. Then for each bounded subdomain W of U with Lipschitz boundary $\partial W \subset U$ and each $y \in K_2(W \backslash S^{n-1})$ we have*

$$g(y) = \frac{1}{\omega_n} \int_{\partial K_2(W)} G(x-y) n(x) g(x) \, \mathrm{d}\sigma(x),$$

provided that $\partial W \cap S^n$ is a set of measure zero when seen as a subset of S^{n-1}.

Following from arguments outlined in the previous section on the conformal covariance of the L^2 spaces of hypersurfaces in $\mathbb{R}^n \cup \{\infty\}$ and their associated H^2 spaces of left-monogenic functions we may easily obtain the following.

Theorem 7. *Let W be a bounded domain in $\mathbb{R}^n$ whose boundary is a Liapunov hypersurface. Moreover, suppose that when considered as a subset of S^{n-1} the set $\partial W \cap S^{n-1}$ has measure zero. Let $\partial K_2(W)(Cl_n)$ be the restriction to $K_2(\partial W)$ of the bundle $H_n(Cl_n)$. If $L^2(\partial K_2(W)(Cl_n))$ is the space of all square integrable sections on $\partial K_2(W)(Cl_n)$, then*

$$L(\partial K_2(W)(Cl_n)) = H^2(K_2(\partial W^+)) \oplus H^2(K_2(\partial W^{-1})),$$

where $W^{\pm}$ are the domains in $\mathbb{R}^n \cup \{\infty\}$ which complement ∂W, and $H^2(K_2(\partial W^{\pm}))$ is the space of left-monogenic sections on $K_2(\partial(W^{\pm}))(Cl_n)$ which continuously extend in the L^2 sense to $\partial K_2(W)$.

It should be noted that in the previous theorem that either $W^+ = W$ or $W^- = W$. The previous theorem is also true if we assume that ∂W is a Lipschitz graph whose intersection with S^{n-1} is a set of measure zero when seen as a subset of S^{n-1}. This fact follows from arguments given in [9].

Definition 2. *Suppose that K is a compact subset of H_n and that h is a continuous section on the bundle $K(Cl_n)$ obtained by restricting the bundle $H_n(Cl_n)$ to K. Then we define the supremum norm of h to be*

$$\sup_{y \in K_2^{-1}(K)} (|J(K_2, y)\tilde{J}(K_2, y)|)^{1/2} \| k(y) \|,$$

where $k(K_2^{-1}(x)) = J(K_2^{-1}, x)h(x)$.

Suppose now that V is a domain in H_n then let us denote the space of left-monogenic sections on $V(Cl_n)$ by $M_l(V(Cl_n))$. Using the supremum norm given in the previous definition we may endow the space $M_l(V(Cl_n))$ with a Frechet topology. It is now straightforward to adapt arguments given in [3, 7] to establish an analogue of the Weierstrass convergence theorem, and so deduce that under this topology the space $M_l(V(Cl_n))$ is complete.

We now proceed to describe a mean value-type inequality for left-monogenic sections.

Theorem 8. *Suppose that V is a domain on H_n^+ and that f is a left-monogenic section over V. Then for each $x \in V$*

$$\| f(x) \| \leqslant C(n)(r_2 - r_1)^{-1} \int_{A_{H_n^+}(x, r_1, r_2)} \| G(x-y) n(y) f(y) \, \mathrm{d}\mu(y) \|,$$

where $C(n) \in \mathbb{R}^+$ is a dimensional constant, $A_{H_n^+}(x, r_1, r_2) \subset V$, and μ is the Hausdorff measure on H_n^+. Moreover, $n(y)$ is the unit vector in $TH_n(y)$, normal to the "sphere" $\{w \in H_n^+ : (w-x)^2 = (y-x)^2\}$ at y.

Using the Lorentz boosts described in the previous section it can be seen that for $1 \leqslant p \leqslant \infty$ the L^p norm of $G(x-y)n(y)$ over $A_{H_n^+}(x, r_1, r_2)$ is independent of the choice of $y \in H_n^+$. Here the vector $n(y)$ is as described in Theorem 8. One can construct an analogue of Theorem 8 over the hyperbola H_n^-, and make the same observation on the L^p norm of $G(x-y)n(y)$.

Definition 3. *An open subset V of H_n is called a domain if there is a domain U lying in $\mathbb{R}^n$ and $K_2(U\backslash S^{n-1}) = V$.*

It should be noted from this definition that a domain on H_n does not have to be connected. It could have one component on H_n^+ and the other on H_n^-. However, if a domain on H_n has two components then it will be an unbounded set.

Suppose that V is a domain in H_n then for $1 \leqslant p \leqslant \infty$ we denote the Bergman space of L^p left-monogenic sections over $V(Cl_n)$ by $B^p(V(Cl_n))$.

Using Hoelder's inequality, Theorem 8 and the observations following Theorem 8 we may obtain:

Theorem 9. *For each domain V lying in H_n the space $B^p(V(Cl_n))$ is complete for $1 \leqslant p \leqslant \infty$.*

By adapting arguments given in [7] one can now readily deduce:

Theorem 10. *Suppose that V is a bounded domain on H_n with Liapunov boundary, then*

$$L^2(V(Cl_n)) = B^2(V(Cl_n)) \oplus D_{H_n}(\overset{\circ}{W}{}^1_2(V(Cl_n))),$$

where $L^2(V(Cl_n))$ is the space of square-integrable sections over $V(Cl_n)$, and $\overset{\circ}{W}{}^1_2(V(Cl_n))$ is the L^2 completion of smooth sections on $V(Cl_n)$ with compact support.

One may now readily adapt arguments given in [7] to consider boundary value problems over bounded domains in H_n with Liapunov boundaries for the operator $D^2_{H_n}$.

In [9] certain domains are constructed in $\mathbb{R}^n$ which may be regarded as Lipshitz-type perturbations of the unit ball. We may consider their images in H_n^+ under dilation and the Cayley transformation K_2. One may readily adapt the arguments given in [7] to see that Theorem 10 also holds if we consider L^2 spaces over the images of such domains in H_n^+.

Concluding remarks

We may similarly deduce analogues of the results described in the previous section for the appropriate sections over domains in S^n. Here we would use the Cayley transformation K_1. One advantage of working over S^n is that all domains on S^n are bounded.

In particular, we can use the spherical shells, $A_{S^n}(x, r_1, r_2)$, to establish a mean value type inequality for left-monogenic sections over $V(Cl_n)$ for each domain V on S^n. It follows that one can now deduce the completeness of the Bergman space $B^p(V(Cl_n))$ of L^p integrable left-monogenic sections on $V(Cl_n)$, where V is a domain on S^n and $1 \leqslant p \leqslant \infty$. Again one can adapt argument given in [7] to show that

$$L^2(V(Cl_n)) = B^2(V(Cl_n)) \oplus D_{S^n}(\overset{\circ}{W}{}^1_2(V(Cl_n))),$$

where $L^2(V(Cl_n))$ is the space of square integrable sections on $V(Cl_n)$ and $\overset{\circ}{W}{}^1_2(V(Cl_n))$ is the L^2 completion of the space of smooth sections on $V(Cl_n)$ with compact support.

It is reasonably well known that the L^2 space of the unit sphere in $\mathbb{R}^n$ decomposes into a direct sum of Hardy 2-spaces of left mongenic functions defined over the domain interior to the sphere and the domain exterior to the sphere in $\mathbb{R}^n$. Under the Cayley transformation K_2 these domains are transformed to H_n^+ and $H_n^- \backslash \{-f_{n+1}\}$, respectively. Moreover, the image of a function belonging to the Hardy 2-space over the domain exterior to the unit sphere extends continuously to all of H_n^-. To properly introduce the analogue of the L^2 space over the unit sphere in $\mathbb{R}^n$ in the context of H_n one needs to attach an $(n-1)$-dimensional sphere at infinity to either H_n^+ or H_n^-. As H_n is assymptotic to the null cone $N = \{x \in \mathbb{R}^{n,1}: x^2 = 0\}$ one can percieve this sphere as being an extension of N. By projecting H_n^+ onto real projective space one can percieve this as a subset of N. Another way to do this is to consider $\mathbb{R}^n$ as a subspace of $\mathbb{R}^{n,1}$ and so consider the open unit ball $D_n \subset \mathbb{R}^n \subset \mathbb{R}^{n,1}$. We then consider the translation $D_n + f_{n+1}$. The boundary, S^{n-1}, of this ball lies in N. This translation is a conformal transformation so one can now easily set up a Dirac operator $D_{S^{n-1}}$ over this sphere, and associated bundles and function theory.

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