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Stochastic resonance in the Heaviside nonlinearity with white noise and arbitrary periodic signal

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A Heaviside nonlinearity with adjustable threshold is fed by an arbitrarily distributed white noise plus a periodic signal of arbitrary wave form. A general and exact treatment demonstrates that this system is capable of stochastic resonance in a large variety of conditions and offers a complete characterization of this property. In particular it gives the possibility of observing nonzero phase shifts with nondynamic stochastic resonance. [S1063-651X(96)02305-4]

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Stochastic resonance is a property of noise-enhanced signal transmission that occurs in certain nonlinear systems driven by a coherent periodic signal added to a noise. It was introduced some 15 years ago in the context of climate dynamics [1] and has since been observed in a large variety of both natural and model systems [2].

At the origin stochastic resonance was essentially observed in dynamic nonlinear systems of a bistable type. A general theory has been proposed for this case [3], which is based on a rate equation that determines the probability of occupation of the two stable states. In the limit of a small modulation by the coherent input, this theory derives approximate expressions for the characteristics of the output that stochastically resonates. Also, for complete applicability this theory requires an explicit expression for the transition rate between states, which is usually obtainable only within the approximation of a slow modulation. The theory of [3] received experimental groundings from experiments performed on a ring laser with an acousto-optic modulator to induce switching between two stable states [4]. Recently, stochastic resonance has been extended to dynamic systems of a monostable or excitable type [5] and here also the associated theoretical treatments are derived in the limit of a small or slow modulation.

A common character of the above-mentioned systems is their *dynamic* nature, i.e., the nonlinear process that stochastically resonates involves both the signals and their time derivatives to determine the output. This situation complicates the calculation of the output autocorrelation function, usually the first step in the theoretical analysis, because the dynamic system broadens the correlation of the input noise and mixes it, in a complicated way in the output response, with correlation originating from the coherent input.

Very recently, a simple example of a nondynamic (static) system that stochastically resonates has been proposed [6], under the form of a unidirectional level crossing by a sine wave plus a Gaussian noise that triggers output spikes. A theoretical description of this system is given in the limit of a slow and small modulation.

Additionally, in the above-mentioned theoretical treatments, the hypothesis of a Gaussian noise is often crucial and the periodic input is restricted to a sine wave.

In the following we consider an even simpler static nonlinear system, driven by a white but arbitrarily distributed noise plus a periodic input of arbitrary wave form. To date, this system appears to us as the conceptually simplest system that brings together the ingredients for stochastic resonance. We present an exact theory that provides a complete description of the ability of this system to stochastically resonate.

Let $s(t)$ represent a periodic signal with period T_s and $\eta(t)$ a stationary white noise, with the complementary distribution function $F_c(u)=\mathrm{Prob}\{\eta(t)>u\}$. We consider a static nonlinear system with threshold θ, which receives $s(t)$ and $\eta(t)$ as inputs and produces the output $y(t)=\Gamma[s(t)+\eta(t)-\theta]$, with the Heaviside function $\Gamma(u)=0$ for $u\leqslant 0$ and $\Gamma(u)=1$ otherwise.

We are first interested in computing a statistical autocorrelation function for the output signal $y(t)$. Since y assumes values 0 or 1 only, the expectation $E[y(t)y(t-\tau)]$ for fixed $\tau\neq 0$ and fixed t can be expressed as the probability

$$E[y(t)y(t-\tau)]=\mathrm{Prob}\{y(t)=1,\ y(t-\tau)=1\}, \tag{1}$$

which is also

$$E[y(t)y(t-\tau)]=\mathrm{Prob}\{s(t)+\eta(t)>\theta,\quad s(t-\tau)+\eta(t-\tau)>\theta\}. \tag{2}$$

Since s is a deterministic signal and η a *white* noise, one can write

$$E[y(t)y(t-\tau)]=x(t)x(t-\tau), \tag{3}$$

with

$$x(t)=E[y(t)]=\text{Prob}\{\eta(t)>\theta-s(t)\}=F_c[\theta-s(t)]. \tag{4}$$

For $\tau=0$, one has

$$E[y(t)y(t-\tau)]=\text{Prob}\{y(t)=1\}=x(t). \tag{5}$$

Both $x(t)$ and $x(t-\tau)$ are periodic in t and τ with period T_s. Because of the periodic coherent modulation introduced by $s(t)$, the stochastic signal $y(t)$ is nonstationary, yet it is cyclostationary with period T_s [7]. It is possible to construct a ''stationary'' autocorrelation function $R_{yy}(\tau)$ for $y(t)$ through a proper time averaging of $E[y(t)y(t-\tau)]$ over an interval T_s, when t or t modT_s uniformly covers $[0,T_s[$.

To avoid difficulties due to the idealized notion of a white noise and also to have the possibility of a direct numerical evaluation of every relevant quantity of the model, especially for the purpose of comparison with a simulation of the nonlinear system, we choose now to move to the context of discrete-time signals. The time scale is thus discretized with a step $\Delta t \ll T_s$ such that $T_s=N\Delta t$. Now, in practice, the white noise $\eta(t)$ only need be a noise with a correlation length shorter than Δt.

We define the stationary autocorrelation function as

$$R_{yy}(k\Delta t)=\frac{1}{N}\sum_{j=0}^{N-1} E[y(j\Delta t)y(j\Delta t-k\Delta t)], \tag{6}$$

which can also be written, according to Eqs. (3) and (5), as

$$R_{yy}(k\Delta t)=(\overline{x}-\overline{x^2})\,\hat{\delta}(k\Delta t)+\frac{1}{N}\sum_{j=0}^{N-1} x(j\Delta t)x[(j-k)\Delta t], \tag{7}$$

with $\hat{\delta}(k\Delta t)=1$ for $k=0$ and $\hat{\delta}(k\Delta t)=0$ otherwise and with the time average

$$\overline{x}=\frac{1}{N}\sum_{j=0}^{N-1} x(j\Delta t) \tag{8}$$

and a similar definition for the average $\overline{x^2}$. We note that since F_c varies between 1 and 0, for any nonidentically zero or one $x(t)$, one always has $\overline{x}-\overline{x^2}>0$.

In order to proceed into the frequency domain, the Fourier coefficients of the deterministic periodic signal $x(j\Delta t)$ are introduced as

$$X_n=\frac{1}{N}\sum_{j=0}^{N-1} x(j\Delta t)\exp\left(-i2\pi\frac{jn}{N}\right). \tag{9}$$

We define the discrete Fourier transform $\mathscr{F}$ of R_{yy}, over a time interval of an integer number $2M$ of periods T_s, as

$$\mathscr{F}[R_{yy}(k\Delta t)]=\sum_{k=-MN}^{MN-1} R_{yy}(k\Delta t)\exp\left(-i2\pi\frac{kl}{2MN}\right), \tag{10}$$

which affords a frequency resolution $\Delta\nu=1/(2MN\Delta t)$.

The autocorrelation function of Eq. (7) is formed by a pulse at the origin with magnitude $\overline{x}-\overline{x^2}$, superposed to a periodic component with period T_s [the second term on the right-hand side of Eq. (7)]. The Fourier transform of R_{yy} defines the output power spectral density P_{yy}, which will then be formed by a constant background with magnitude $\overline{x}-\overline{x^2}$, superposed to a series of spectral lines at integer multiples of $1/T_s$. Application of Eq. (10) leads to

$$P_{yy}\left(\frac{n}{T_s}\right)=\overline{x}-\overline{x^2}+2MNX_nX_n^*. \tag{11}$$

When the horizon $M\to+\infty$, the magnitude of the coherent spectral lines above the broadband noise background tends to infinity. This type of form of the power spectral density is typical for the output of a stochastically resonant system. We choose to define the signal-to-noise ratio, at frequency n/T_s on the output, as the ratio of the power contained in the spectral line alone to the power contained in the noise background in a frequency band of $1/T_s$ around n/T_s. The corresponding expression of the (SNR) signal-to-noise ratio then follows as

$$\mathscr{R}\left(\frac{n}{T_s}\right)=\frac{N|X_n|^2}{\overline{x}-\overline{x^2}}. \tag{12}$$

In addition to the SNR, another desirable characterization consists in the possibility of evaluating the phase shift between the output and the coherent input. This can be achieved through the computation of an input-output cross-correlation function. For fixed τ and t, we first consider the expectation

$$E[s(t)y(t-\tau)]=s(t)\text{Prob}\{y(t-\tau)=1\}=s(t)x(t-\tau). \tag{13}$$

$E[s(t)y(t-\tau)]$ is periodic in both t and τ, with period T_s. For the definition of a stationary cross-correlation function, a time average is taken when t or t modT_s uniformly covers $[0,T_s[$. This yields the cross-correlation function

$$R_{sy}(k\Delta t)=\frac{1}{N}\sum_{j=0}^{N-1} s(j\Delta t)x[(j-k)\Delta t], \tag{14}$$

which is interpretable as the cross-correlation function of $s(t)$ with the nonstationary mean output $x(t)=E[y(t)]$. R_{sy} of Eq. (14) is periodic with period T_s. Its frequency contents has only components at integer multiples of $1/T_s$. Through a Fourier transform of R_{sy} according to Eq. (10), one obtains a cross-power spectral density

$$P_{sy}\left(\frac{n}{T_s}\right)=2MNS_nX_n^*, \tag{15}$$

where S_n, defined according to Eq. (9), is the order n Fourier coefficient of $s(t)$.

The phase shift ϕ between the mean output $E[y(t)]$ and the coherent input $s(t)$, as it is also considered in [8], can here be evaluated, for a component with frequency n/T_s, from the argument of the complex number $P_{sy}(n/T_s)$ as

$$\phi\left(\frac{n}{T_s}\right)=\arg(S_nX_n^*). \tag{16}$$

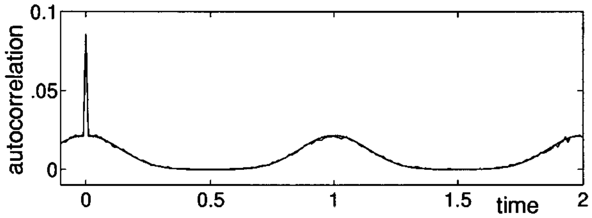

FIG. 1. Output autocorrelation function versus the time lag τ in units T_s : the smooth line is the theoretical expression of Eq. (7) and the noisy line (almost indistinguishable) is the experimental estimation, with $\Delta t = T_s/100$ at $\sigma_\eta = 0.5$.

With the present general and exact treatment, it is now easy to verify that the nonlinear system exhibits stochastic resonance for a large variety of conditions. For illustration we chose $\eta(t)$ a Gaussian white noise with zero mean and variance σ_η^2 and $s(t) = \cos(2\pi t/T_s)$. The threshold is $\theta = 1.2$ since for standard stochastic resonance the coherent input alone is unable to induce a transition of the output. We show in Fig. 1 the output autocorrelation function theoretically predicted by Eq. (7), when $\sigma_\eta = 0.5$ and $\Delta t = T_s/100$. Our first concern is to compare this theoretical autocorrelation function against a direct estimation of it, resulting from a numerical simulation of the nonlinear system. In the simulation, sample averages of terms of the form $y(t)y(t-\tau)$ were accumulated to provide an experimental estimation of the autocorrelation function, which is also presented in Fig. 1. As expected, because the model is exact, the theoretical and experimental autocorrelation functions are quite consistent and they would tend to perfectly superpose if the sample averages experimentally performed were estimated with a number of trials tending to infinity. From here, the rest of the comparison involves only ''mechanical'' Fourier transforms and thus cannot introduce discrepancies between theory and experiment that would be inherent to the system considered, since the autocorrelation functions agree.

Next, we show in Fig. 2 the output SNR, at frequencies $1/T_s$ and $2/T_s$, as a function of the input noise power density σ_η^2, as it results from Eq. (12). The nonmonotonic variation of the SNR that passes through a maximum for a specific input noise level is a clear signature of stochastic resonance. We observe, as visible with other stochastically resonant systems, that the resonance occurs at different noise levels for the first and second harmonics.

In the present case, Eq. (16) gives a phase shift $\phi(1/T_s) = 0$ between the components at frequency $1/T_s$ on the output and on the coherent input, for any value of σ_η^2 in the resonance region spanned in Fig. 2. This absence of input-output phase shift at the resonance is also observed in other stochastically resonant systems, although not always [8]. With the present model, we have the possibility to show that the behavior of the phase shift can be altered simply by changing some characteristics of the noise distribution or of the coherent input.

For instance, we change the periodic input to $s(t) = 0.5[\diagup(t/T_s) + \cos(2\pi t/T_s)]$, where $\diagup(t) = t$ mod1 is a sawtooth signal of amplitude 1 and period 1. The $1/T_s$ and $2/T_s$ harmonics of the output still resonate, as demonstrated by Fig. 2. The input-output phase shift given by Eq. (16), between the components at frequencies $1/T_s$ or $2/T_s$, displays, respectively, a nonmonotonic variation [Fig. 3(a)] and a monotonic decay [Fig. 3(b)] with the input noise. The shift in phase of an input component at frequency n/T_s can thus change with the overall frequency contents of the input, a typical nonlinear effect.

The present model allowed us to verify that many other conditions enable the system to resonate, for instance, uniformly or exponentially distributed noise. It also gives us the possibility to theoretically investigate various other issues of importance, for example, the issue of the optimal noise distribution to maximize resonance in the presence of a specified coherent signal $s(t)$ or conversely the optimal wave form for $s(t)$ in the presence of a specified noise distribution.

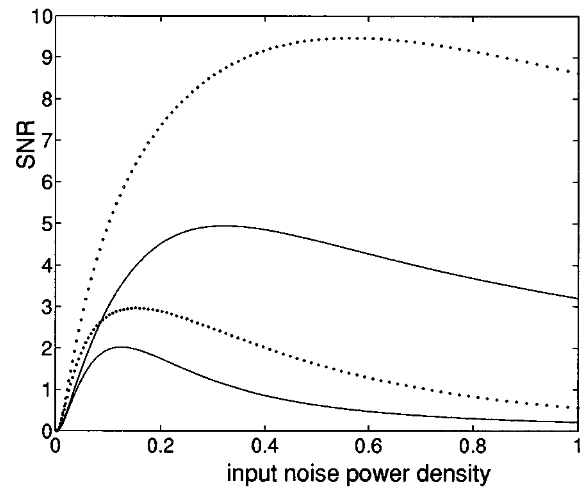

FIG. 2. Output SNR of Eq. (12) as a function of the input noise power density σ_η^2: the pair of dotted curves is with $s(t) = \cos(2\pi t/T_s)$ and the pair of solid curves with $s(t) = 0.5[\diagup(t/T_s) + \cos(2\pi t/T_s)]$. In each pair, the upper curve is for the first harmonic at $1/T_s$ and the lower curve for the second at $2/T_s$.

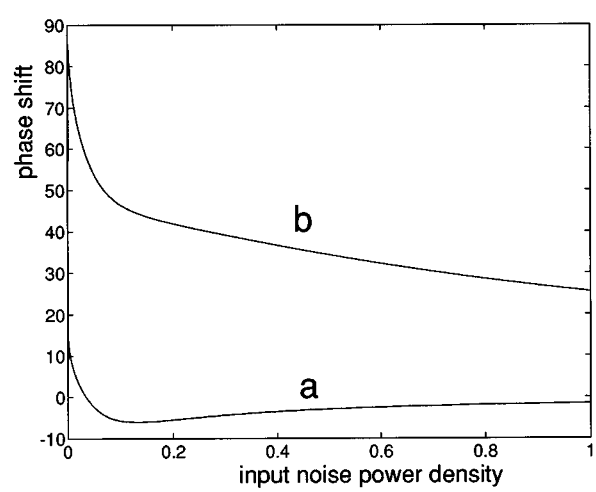

FIG. 3. Input-output phase shift (in degrees) of Eq. (16), as a function of the input noise power density σ_η^2, with $s(t) = 0.5[\diagup(t/T_s) + \cos(2\pi t/T_s)]$, at frequency (a) $1/T_s$ and (b) $2/T_s$.

As we mentioned, within the hypothesis of a white noise $\eta(t)$, the present theoretical treatment is exact. In practice, an actual physical noise will have a small, but not strictly zero, correlation length τ_c. When $\tau_c < \Delta t$, the physical output autocorrelation function will have a peak of width $\sim \tau_c$ around the origin, whose magnitude in $\tau = 0$ is correctly represented by Eq. (7) but whose exact shape will not be described by Eq. (7). The discrete-time treatment allows us to dispense with explicit assumptions concerning the exact shape of this narrow peak. The exact shape of this peak of duration $\sim \tau_c$ will start to manifest its influence on the output power spectral density in the frequency range of order $1/\tau_c$. Such high-frequency perturbations will generally leave unaffected the stochastic resonance effect that takes place in the much lower frequency range $1/T_s$ and will consequently be accurately described by the present theoretical treatment.

The study in [9] also considers a Heaviside nonlinearity, but preceded by a first- or second-order low-pass linear filter fed by a Gaussian white noise. In the presence of a sinusoidal coherent input, an exact analytical expression is derived for the cross-correlation coefficient (zero lag) of the output with the coherent input, from which a ratio is deduced that approaches the usual SNR for small coherent signals.

Our treatment deals with arbitrarily distributed white noise and arbitrary periodic input and provides exact expressions for the correlation functions, the power spectral densities, the input-output phase shift, and the output SNR. This represents an example of a stochastically resonant system that lends itself to an exact and general treatment. The success of such a complete theoretical analysis certainly relates to the static nature of the nonlinearity, which does not spread the correlation of the input noise. The correlation present in the random signal on the output essentially comes from the coherent input. These conditions greatly facilitate the calculation of the output autocorrelation function and as demonstrated here, they turn out to be sufficient to induce stochastic resonance. The present general and exact treatment of a conceptually very simple nonlinear system offers a unique theoretical framework to further investigate various aspects of stochastic resonance.

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