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FUNCTION VALUES AS BOUNDARY INTEGRALS

RICHARD ARENS AND I. M. SINGER

1. **Introduction.** For suitable families H of numerical-valued continuous functions on a topological space S (see 2.4 below) we show that there is a smallest closed subset B of S such that on B every function in the family attains its maximum absolute value. The construction is patterned after that of the "Šilov boundary" ([III] or, more conveniently, [I, p. 80]) where the family of functions has to be an algebra. Algebraically, our families have to be multiplicative semigroups.

We then proceed to show that for each $s \in S$ there is at least one regular Baire measure m_s on B such that if $g \in H$ and its reciprocal also belongs to H, then $\log |g(s)| = \int_B \log |g(b)| m_s(db)$. Our main effort is directed toward finding how this integral representation generalizes for those $g \in H$ whose reciprocal may not exist or at least be absent from H. In those cases where H is rich enough to ensure that the measures m_s are unique for each s, we obtain $\log |g(s)| \leqq \int_B \log |g(b)| m_s(db)$, which is to say that the geometric mean (relative to m_s) of $|g|$ on B is not less than its value at s. This generalizes a classical inequality due to Szegö [II] for regular functions on the disc S, where B is the usual boundary.

In the nonunique case we show (see 6.4) that there is for each g at least one measure for which the inequality holds, but there may be some for which it fails.

We intend to published elsewhere a more detailed discussion of these matters for a special type of Banach algebras generalizing more directly the classical case of functions regular on the disc (see §4). In this case, the boundary is a locally compact abelian group.

2. **Existence of the boundary.** Let H be a class of real or complex, bounded continuous functions on a topological space S. Consider a subset F of S with the properties:

2.1 *F is closed,*

2.2 *for each h in H, $|h(s)|$ attains on F the value* $\sup_{s \in S} |h(s)|$; and moreover

2.3 *If any subset F_1 of S has the properties of F in* 2.1 *and* 2.2, *then F_1 contains F.*

Such a set, if it exists, is evidently unique, and will be denoted by $\partial_H S$, and may be called the *Šilov boundary of S relative to H.*

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If the class H of functions is replaced by the class of their absolute values, $\partial_H S$, as well as its existence, is clearly unaffected. We therefore deal with non-negative real-valued functions only.

2.4 THEOREM. *Let S be a topological space and H a class of real-valued non-negative continuous functions on S such that*

2.41 *if $h, j \in H$ then $hj \in H$,*

2.42 *if $\epsilon > 0$ and $h \in H$, the set $\{s; h(s) \geqq \epsilon\}$ is compact,*

2.43 *each point s_0 in S has a basis of neighborhoods of the form*

$$\{s; h_1(s) < \epsilon, h_2(s) < \epsilon, \cdots, h_n(s) < \epsilon\},$$

where $h_1, \cdots, h_n \in H$ and $\epsilon > 0$.

Then $\partial_H S$ exists.

PROOF. Let K be the class of all subsets F of S having properties 2.1 and 2.2. Let K_0 be a maximal decreasing chain in K, regarded as ordered by inclusion. This K_0 exists by Zorn's lemma. Let F_0 be the intersection of all members of K_0. F_0 is closed, and also satisfies 2.2 since for $h \neq 0$, the set $\{s; h(s) = \max h\}$ (where max is always intended to refer to the maximum value of the function on S) intersects every F in K_0 and hence intersects F_0. It remains to prove that F_0 satisfies 2.3 (with $F = F_0$). Suppose B is closed in S and does not contain F_0. Then some point s_0 of F_0 has a neighborhood V defined as in 2.43 which does not meet B. Then $F_0 - V$ being closed does not satisfy 2.2. Hence max g is greater than sup $g(s)$ for s in $F_0 - V$. By taking h as a suitable power of g, we can obtain

$$2.44 \qquad \max_{s \in F_0 - V} h(s) < \frac{\epsilon \max h}{\max h_1 + \cdots + \max h_n + 1} \cdot$$

Let $s \in F_0$. If $s \in V$ then $h(s)\, h_i(s) < \epsilon \max h$. If $s \in F_0 - V$ then again $h(s) h_i(s) < \epsilon \max h$, by 2.44, and this is true for each i. Hence $\max hh_i < \epsilon \max h$. Now suppose $h(t) = \max h$. Then $h(t) h_i(t) = h_i(t) \max h < \epsilon \max h$ so that $t \in V$. Hence h does not attain its max on B at all. Thus 2.3 holds, and the theorem is proved.

2.5 COROLLARY. *Let the hypothesis of 2.4 hold. Then $s_0 \in \partial_H S$ if and only if for every neighborhood of s_0 there is an h in H which attains its max only in that neighborhood.*

The "only if" was demonstrated above. The "if" part follows from 2.3.

3. **Applications.** Let A be an algebra of continuous complex-valued functions vanishing at infinity on a locally compact space, with the

property that for $s \neq t$ there is an f in A such that $f(s) \neq f(t)$. The theorem of Šilov [III] says that then $\partial_A S$ exists. It may be of interest to prove that 2.4 really does generalize Šilov's theorem. To this end, compactify S with a point ∞ obtaining S^*. Let B be the class of all $\lambda+f$, $f \in A$ and λ complex. Let H be the class of all $|g|$, $g \in B$. Then 2.43 holds with S replaced by S^*, since B has a unit [I, 5G], and 2.41, 2.43 are obvious. Let $F=\partial_H S^* - \{\infty\}$. First of all, every element of A does attain its maximum modulus on F. Second, consider a closed subset F_1 of S which does not include F. Then $F_1 \cup \{\infty\}$ does not include $\partial_H S^*$, hence there is a g in B which does not attain its maximum modulus on the former set.

By scalar multiplication and suitable potentiation one can arrange that

$$\max_{s \in F_1 \cup \{\infty\}} |g(s)| < 1, \tag{3.1}$$

$$3 < \max_{s \in S^*} |g(s)|. \tag{3.2}$$

Let $g=\lambda+f$, $f \in A$. Then $|\lambda| < 1$ because f vanishes at ∞, whence $\max_{s \in F_1} |f(s)| < 1+|\lambda| < 2$, by 3.1. On the other hand, $|\lambda| < 1$ together with 3.2 shows that $\max |f| > 2$, whence f does not attain its maximum modulus on F_1. Thus F satisfies the definition of $\partial_A S$.

For later reference we sum this up briefly as follows.

3.3 *The Šilov boundary for A (with no unit) is obtained by deleting the point at infinity from the Šilov boundary for A-with-unit-adjoined.*

The fact that 2.4 does not require two algebraic operations makes possible another application. Let L be a convex topological linear space, and let S be a weakly compact subset of L. Let $\overline{L}$ be the class of linear continuous (possibly complex) functionals on L. Let H be the class of functions $h=|e^{\lambda+f}|$, $f \in \overline{L}$, and λ any complex constant. It is not hard to see that the sets of the form 2.43 with $\epsilon=1$ form a basis for the weak topology in L, and 2.41, 2.42 obviously hold. The set $\partial_H S$ is what has been called the *T-frontier* of S by Milman [IV] in the case of Banach spaces. It can also be seen that it is the weak closure of the set of extreme points of the closed convex hull of S (supposing that hull to be compact), by generalizing the argument presumably used by Milman.

4. **Examples.** Let T be a compact Hausdorff space. Let G be a group of homeomorphisms of T that acts transitively. Let A be a complete subalgebra, with unit, of $\mathfrak{C}(T, C)$ (C=complex numbers) with the obvious ring structure and norm, and satisfying:

4.01 *if $t_1 \neq t_2$ then $f(t_1) \neq f(t_2)$ for some f in A,* 4.02 *if* $f\in A$ *and* $x\in G$ *then* $fx\in A$ *where* $(fx)(t)=f(x(t))$.

Now 4.01 *shows that* T *can be imbedded in the space* S *of maximal ideals*[1] *of* A, *by sending* t *into* s *if* $f(s)=f(t)$ *for all* f *in* A. *We may therefore regard* A *as a set of functions on* S, *and regard* T *as a subset of* S.

4.1 THEOREM. *Under the preceding assumptions,* $T=\partial_A S$.

PROOF. Every f in A attains its maximum modulus on T, whence $\partial_A S$ is contained in T. From 4.02 we see that G induces automorphisms in A which induce homeomorphisms of S under which $\partial_A S$ is surely invariant. The induced homeomorphisms agree on T with the way G is supposed to act on T. Now G is transitive on T, and $\partial_A S$ is a nonvoid invariant subset of T. Hence $\partial_A S=T$, as asserted.

Many specific examples of this sort of algebra can be obtained as follows.

Let Γ be a discrete abelian group, and let π be such a subset of Γ that

4.21 $\alpha, \beta\in\pi$ *implies* $\alpha\beta\in\pi$ (*multiplicative notation*),

4.22 *the neutral element* 1 *of* Γ *belongs to* π,

4.23 Γ *is the smallest subgroup containing* π.

Now let $G=\{x, y, \cdots\}$ be the character group of Γ, and regard Γ as functions on G. Let A be closed subalgebra of $\mathfrak{C}(G, C)$ generated by Γ with the usual norm $\|f\|=\max |f(x)|$, $x\in G$. The group to act on G shall be G itself, acting as translations. Then 4.01, 4.02 are satisfied. Hence $G=\partial_A S$ where S is the space of maximal ideals of A.

The special case in which Γ is linearly ordered will be treated in detail in another paper.

The most important case of all this is when Γ is the integers and π is the set of non-negative integers. One arrives then at the classical theory of continuous functions on the unit disc which are regular in the interior. See Loomis [I, p. 81, Remark].

5. **Measures on $\partial_H S$ representing points of S.** We are about to generalize some of the following well-known facts.

Let $P(s, \theta)$ be Poisson's kernel, where s is complex and $|s|<1$, and θ is the angle of the variable boundary point of the circle $|z|=1$. Then $m_s(d\theta)=(1/2\pi)P(s, \theta)d\theta$ gives a measure on that circle such that if $f(z)$ is continuous for $|z|\leqq 1$ and regular for $|z|<1$ (or perhaps merely the real part of a regular function), then $f(s)=\int f(e^{i\theta})m_s(d\theta)$, integration being the over the boundary.

We shall construct, for each $s\in S$ (notation of 2.4), a regular measure m_s on, and supported by, $\partial_H S$, "representing" the point s. The

[1] Or rather, as we prefer to take it, the space of homomorphisms of A onto C, and we shall write, somewhat ambiguously, $f(s)$ for the value of s at f.

"representation" is however not such that $h(s)=\int h(x)m_s(dx)$ (integration being over $\partial_H S$) for $h\in H$, but rather this: if g and g^{-1} both belong to H then $\log g(s)=\int \log g(x)m_s(dx)$. In the special case of the regular functions, i.e., H the class of all $|f|$ where f is continuous on the disc and regular inside, the familiar result is easily recovered, as follows. Form $g=|e^f|$; then g and g^{-1} both belong to H so that by the result on log g, we have $Rf(s)=\int Rf(x)m_s(dx)$, from which the desired result can be concluded.

The hypothesis for this section will be now stated once for all.

5.01 *S is a topological space and H is a class of real-valued non-negative continuous functions on S such that*

5.02 *If $h, j\in H$ then $hj\in H$.*

5.03 *Every element of H attains its maximum on some fixed compact Hausdorff subset B of S.*

5.04 *There is an element k in H which is a positive constant less than 1, on B.*

It is clear that if 2.41–2.43 hold, then B exists. However, we have no use for 2.43 in this section.

We begin by introducing a linear space of continuous real functions on S.

Let E be the class of all real linear combinations of functions log g, where g and g^{-1} both belong to H.

5.1 LEMMA. *Let $h\in H$, $u\in E$, and suppose μ is a non-negative real number. Suppose $\mu \log h\leqq u$ on B. Then $\mu\log h\leqq u$ on S.*

PROOF. Let $v=-u$; then we have $\mu\log h+v\leqq 0$ on B. Let us assume that there is an s in S such that $\mu \log h(s)+v(s)>\epsilon>0$. (The ensuing contradiction will prove the theorem.) Select a positive integer p exceeding $-\epsilon^{-1}\log k$. Suppose $v=\sum\lambda_i \log g_i$ where the g_i have inverses in H. Then

$$5.12 \qquad \mu\log h+\sum\lambda_i\log g_i+\frac{1}{p}\log k<0 \qquad \textit{on } B,$$

and

$$5.13 \qquad \mu\log h(s)+\sum\lambda_i\log g_i(s)>\epsilon.$$

Now find rational approximations m/n, m_i/n for μ, λ_i respectively (with common positive denominator n) such that if the Greek letters in 5.12 and 5.13 are replaced by the approximations, then the inequalities still hold. It follows that

$$h^{pm}k^n(g_1^{m_1}g_2^{m_2}\cdots)^p<1 \qquad \text{on } B$$

whence the same is true on all of S and so

$$\frac{m}{n}\log h(s) + \sum \frac{m_i}{n}\log g_i(s) + \frac{1}{p}\log k < 0.$$

Considering the rational approximation of 5.13, we see that $-p^{-1}\log k > \epsilon$ which contradicts the choice of p.

5.2. THEOREM. *Let* 5.01–5.04 *hold. Let s in S be given. Then there is a regular Baire measure m_s defined on B such that*

$$\log g(s) = \int_B \log g(x) m_s(dx)$$

whenever both g and g^{-1} belong to H.

PROOF. For u in E let u' be the function u with domain restricted to B. This gives a linear mapping of E onto a subspace E' of $\mathfrak{C}(B, R)$. For s in S define $J(u') = u(s)$. Taking $h = k$ and $\mu = 0$ shows that J is non-negative, and it is clearly linear. Suppose now that $u \leqq 1$ on B. Let $\mu = -(\log k)^{-1}$, which is positive, and we have $\mu \log k \leqq -u$. Applying 5.1, we obtain $u(s) \leqq 1$. This shows that J has a bound not greater than 1. A well-known argument now gives an integral representation for $J(s)$ in terms of a measure m_s on B and this is the measure m_s on B such that $u(s) = \int_B u(x) m_s(dx)$. This is evidently the measure we promised to construct. It is not necessarily unique, as the following example, involving merely regular functions of 2 complex variables, will show.

Let Γ be J^2, where J is the group of integers, and let Π be the class of pairs of non-negative integers. This sets in motion the train of ideas presented in §4. The space S of maximal ideals of A is (homeomorphic to) the class of pairs (λ, μ) where λ, μ are complex and $|\lambda|$, $|\mu| \leqq 1$. By the maximum modulus principle, $\partial_A S$ is the class of pairs (λ, μ) where $|\lambda| = |\mu| = 1$ which is the character group of J^2. Reference to 4.1 produces the same result. For $f \in A$, $f(\lambda, \mu) = a_0 + a_1\lambda + b_1\mu + \cdots$, we surely have $f(0, 0) = \int_G f(\lambda, \mu)|d\lambda||d\mu|$ where $|\ |$ refers to the usual (Haar measure) of the circle group. Thus the Haar measure of the torus G can be used to represent the point $(0, 0)$ of S. But the 1-*dimensional* Haar measure of certain 1-*dimensional* closed subgroups will produce the same representation. In fact, let $\lambda = \nu^m$, $\mu = \nu^n$ (where $|\nu| = 1$ and m, n are positive integers) parametrize a subgroup G_1 of G. Then $\int f(\nu^m, \nu^n)|d\nu| = f(0, 0)$ (integration being over the circle group).

6. **Further properties of the measures.** From a theorem of Szegö

[II] one knows that for f regular in the disc and continuous on the boundary, the function $\log |f|$ is integrable (unless $f=0$) over the boundary of the disc, and in any case,

6.01 $$\log |f(0)| \leqq \int \log |f|,$$

integration being over the boundary, with the usual measure divided by 2π. For each interior point of the disc there is an analogous inequality, involving the measure representing the point in question. We shall generalize this proposition. It will actually be a consequence of findings on the totality of values of $I_s(\log h)$, where I_s runs through all possible extensions of J_s. In the classical case there is only one extension of J_s. If the reader were to deflate the following considerations to this special case, he would obtain a proof of 6.01 rather different from Szegö's, since it makes no use of the group structure of the boundary of the disc.

The hypothesis 5.01–5.04 will be assumed for this section just as for the last. The set E is that defined in the last section.

Now let l be any extended-real-valued (meaning that $\pm\infty$ are permitted as values) function defined on B. Then another extended-real-valued function, defined on all of S, can be constructed thus:

6.1 $$l^-(s) = \inf_{u \in E,\, u \geqq l \text{ on } B} u(s)$$

where we let the inf be $+\infty$ if there are no such u. A dual function l_- can be defined, using a sup, or we may simply define $l_- = -(-l)^-$.

Properties of this sort of prolonged upper envelope are listed in the following.

6.2 THEOREM. *The function l^- is upper semicontinuous, and*

6.21 $$l_- \leqq l \leqq l^- \text{ on } B,$$

6.22 $$l_1 \leqq l_2 \text{ on } B \text{ implies } l_1^- \leqq l_2^- \text{ on } S,$$

6.23 $$(l_1 + l_2)^- \leqq l_1^- + l_2^-;\ (cl)^- = cl^- \text{ for } c \geqq 0;$$

6.24 *for $l = \log h$, $h \in H$, the inequality $\log h \leqq (\log h)^-$ holds on all of S.*

We shall consider only the proof of 6.24. Suppose $u \geqq \log h$ on B. Then $u \geqq \log h$ on S by 5.1. Taking the inf of these u, we get 6.24.

In the definition (6.1) of $l^-(s)$ we consider the class of all $\sum \lambda_i \log g_i$ which dominate l on B. We note next that actually only sums of one term need to be considered.

6.25 THEOREM. *Suppose that $k^{-1} \in H$, so that $\log k \in E$. Then*

$$l^-(s) = \inf\left[\frac{1}{n}\log g(s)\right] < \infty$$

where (g, n) runs through all $g \in H$ such that $g^{-1} \in H$ and $g \geqq e^{nl}$, n being a positive integer.

PROOF. Let the right-hand side of 6.25 be called a. Then surely $a \geqq l^-(s)$. Now let $l^-(s) < b$. Then there is a $u \in E$ such that $u \geqq l$ on B and $u(s) < b$. Since $\log k$ may be assumed present in u, we can alter its coefficient a bit and get $u > l$ on B. With this elbow room we can replace all coefficients by rational numbers, and obtain a relation like $\sum n_i \log g_i > nl$, $\sum n_i \log g_i(s) < nb$ where the n_i and n are positive integers. Let $g = g_1^{n_1} g_2^{n_2} \cdots$. Then $(1/n)\log g(s) < b$, so that $a < b$. Thus $a \leqq l^-(s)$.

We want now to extend J_s to all of $\mathfrak{C}(B, R)$ and we want to see what are the possible values of the extension I_s for a fixed $l \in \mathfrak{C}(B, R)$.

6.3 THEOREM. *Let $l \in \mathfrak{C}(B, R)$; let $s \in S$; and suppose $l_-(s) \leqq j \leqq l^-(s)$. Then*

6.31 *$u_1 + \lambda l \leqq u_2 + \mu l$ implies $u_1(s) + \lambda_j \leqq u_2(s) + \mu j$ for every pair of elements u_1, u_2 of E;*

6.32 *by setting $J'(u + \lambda l) = u(s) + \lambda j$ one obtains a linear, non-negative extension to the linear space of the $u + \lambda l$ of the functional J_s, and its bound is still 1;*

6.33 *all linear non-negative extensions I_s of J_s from E to $\mathfrak{C}(B, R)$ can be obtained in this way.*

PROOF. We consider the case $\lambda < \mu$ in 6.31, leaving the others to the reader. Then $u = (\mu - \lambda)^{-1}(u_1 - u_2) \leqq l$ on B, so that $(\mu - \lambda)^{-1}(u_1(s) - u_2(s)) \leqq l_-(s) \leqq j$, from which the desired inequality follows.

Part 6.32 follows easily from 6.31. Part 6.33 follows from the fact that $l_-(s) \leqq I_s(l) \leqq l^-(s)$ for all such extensions of J_s.

6.34. COROLLARY. *If every J_s has only one non-negative linear extension to $C(B, R)$ or if (which is assuming at least as much) E is dense in $C(B, R)$, then there is a linear, order-preserving mapping $l \rightarrow l^*$ of $C(B, R)$ to $C(S, R)$ such that l^* is an extension of l, and such that if l is the restriction of $u \in E$ to B, then $l^* = u$.*

For the proof, let $l^* = l_-$. By 6.3, $l^* = l^-$. Finally, 6.2 insures that l^* has all the properties stated.

This theorem (6.34) constitutes a kind of general solution to Dirichlet's problem. We shall not pursue this topic, and return to the question of what values $I_s(\log h)$ can have, for h in H but having possible zeros on B so that 6.3 does not apply.

6.4. THEOREM. *Let $h \in H$, and $s \in S$, and suppose $k^{-1} \in H$ (see 5.04). Then the totality of values of $I_s(\log h)$, where I_s ranges over all extensions to $C(B, R)$ of J_s, is an interval possessing at least its upper end point, which is $(\log h)^-(s)$, plus possibly the point $-\infty$. In the latter event, the point $-\infty$ may or may not be the lower end point.*

PROOF. The set of measures $\{m_s\}$ representing the various I_s is clearly convex, and so therefore is the set of values $I_s(\log h)$ insofar as they are finite. Therefore, except for the detail about $-\infty$ which we shall settle by an example, the main thing to prove is the identity of the upper end point. It is of course clear as in 6.2 that $I_s(\log h) \leqq (\log h)^-(s)$. We shall now construct an I_s such that $I_s(\log h) \geqq (\log h)^-(s)$.

Let a be a real positive number. Let $j_a = (\log (h+a))^-(s) < \infty$ by 6.25. By 6.3 there exists an extension I_a of J_s such that $I_a(\log (h+a)) = j_a$. These I_a all belong to the unit ball of the conjugate space, $\mathfrak{C}(B, R)^-$, and hence cluster weakly at some I of the unit ball, as $a \rightarrow 0$.

Let $b \geqq a$. Then, since $\log h \leqq \log(h+a) \leqq \log(h+b)$, one has $(\log h)^-(s) \leqq j_a \leqq I_a(\log (h+b))$. For fixed b, the last term clusters at $I(\log (h+b))$ as $a \rightarrow 0$. Hence $(\log h)^-(s) \leqq I(\log (h+b))$. As $b \rightarrow 0$, $\log (h+b)$ tends monotonely to $\log h$ on B (in fact, on S), on which I is represented as an integral, so $(\log h)^-(s) \leqq I(\log h)$. This is the desired inequality. What we have proved means that $\log h$ is summable-I if and only if $(\log h)^-(s) \neq -\infty$.

From 6.24 and 6.4 we have the following:

6.41 COROLLARY. *If J_s has only one extension I_s, and m_s is the representing measure, then*

$$\log h(s) \leqq \int_B \log h(x) m_s(dx).$$

The following rather trivial example shows that our assumptions permit us to say nothing more about the values of $I_s(\log h)$ in general. Let $S = s_1, s_2, s_3$ be a set of three points, and let H be the class of all non-negative constant functions on S (to satisfy 5.04) plus the set of all non-negative functions which vanish for at least one point (to satisfy 2.43). Actually, all axioms made above are satisfied. Clearly $\partial_H S$ is S itself, and E is one-dimensional. Hence any measure on S with $m(S) = 1$ represents each of the three points. Let $h(s_1) = 0$, $h(s_2) = \lambda$, $h(s_3) = \mu$ with $0 \leqq \lambda \leqq \mu$. Then the set of values for $I(\log h)$ is the closed interval $[\log \lambda, \log \mu]$ plus (if $\lambda \neq 0$) the point $-\infty$. Notice that one may have $-\infty \neq I(\log h) < \log h(s)$.

The final remarks of this section are devoted to showing that the theorems proved above constitute a generalization of Szegö's inequality 6.01.

We return to the classical example already referred to at the end of §4, which is the Banach algebra A of functions continuous on the disc $S=\{|\lambda|\leqq 1\}$ and regular for $|\lambda|<1$. We let H be the class of $|f|$, with $f\in A$. Obviously $\partial_H S$ is the ordinary boundary $B=\{|\lambda|=1\}$. The set E (regarded as continuous functions on B) surely contains all real parts Rf for $f\in A$ since $\log|e^f|=Rf$ and $|e^f|$ has its inverse $|e^{-f}|$ in H. Since every trigonometric polynomial is the boundary value of some polynomial $f(z)$, the $\{Rf\}$ and thus E is dense in $C(B, R)$ so that each $s\in S$ has precisely one representing measure on S, which is now to be identified as that given by Poisson's integral. Let us consider merely $s=0$. By Cauchy's formula, if $f\in A$ then $f(0)=(2\pi)^{-1}\int_0^{2\pi}f(e^{i\theta})d\theta=\int_B f$ with a suitable normalization of the measure. Now let $l=\log|e^f|$. Then $l\in E$, so that Cauchy's formula gives $l(0)=\int_B l$. Thus the Haar measure on B does give the unique representing measure, since the class $\{Rf\}$ is large enough to define it. Now let $l=\log|f|$, $f\in A$. Then $\log|f(0)|\leqq(\log|f|)^-(0)$ by 6.24, and $(\log|f|)^-(0)\leqq\int_B\log|f|$, by 6.4. This gives 6.01. The example $f(\lambda)=\lambda$ exhibits a phenomenon that should not be overlooked (cf. Jensen's formula).

7. **Noncompact spaces of maximal ideals.** Suppose we have a commutative Banach algebra A and S is its space of maximal ideals, and H is the class of functions $|f|$, f in A, regarded as defined on S. Then 5.03, 5.04 will not be satisfied unless A has a unit. Hence we adjoin a unit, which adds a point at infinity to S, while 3.3 says that precisely the same thing happens to the Šilov boundary B. Now the representing measures will a priori require for their support the set $B^*=B\cup\{\infty\}$ in $S^*=S\cup\{\infty\}$. However, the point ∞ may be regarded as of measure 0 in every case, since all the f from A vanish there.

We shall formulate the result of applying these considerations to 5.2 and 6.4.

7.1 THEOREM. *Let A be a commutative Banach algebra. Let S be the space of maximal ideals. Let B be the Šilov boundary. We regard the elements of A as functions on S. Let s be any point of S. Then there is a regular Baire measure m_s defined on B such that*

7.11 $$m_s(B)\leqq 1,$$

7.12 $$\int_B f(x)m_s(dx)=f(s) \qquad \textit{for every } f \textit{ in } A;$$

and for each particular f the measure may be so chosen that (moreover)

7.13 $$\int_B \log |f(x)| m_s(dx) \geqq \log |f(s)|.$$

Naturally, if for some reason there is for some s only one measure satisfying 7.12, then 7.13 holds for that measure.

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FAMILIES OF CURVES

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Amasa Forrester in [1] proved the following theorem of a mixed Euclidean and topological character. If ϕ is a continuous map without fixed points on the Euclidean n-sphere such that ϕ^2 is the identity, then the chords $P\phi(P)$ for all points P of the sphere completely fill the interior of this sphere.

The object of this note is to generalize this theorem to a purely topological statement.

First we recall the definition of retract. If $B \subset A$ are two spaces, then B is a retract of A if there is r: $A \rightarrow B$ which leaves fixed all points of B. (If X and Y are spaces the symbol f: $X \rightarrow Y$ shall denote a continuous map from X to Y.)

Let I denote the unit interval. If F: $B \times I \rightarrow A$ and $t \in I$, define F_t: $B \rightarrow A$ by $F_t(b) = F(B, t)$ for all $b \in B$.

Observation. If F: $B \times I \rightarrow A$ and if B is a retract of A by the map r and if $p, q \in I$, then rF_p is homotopic to rF_q.

In fact such a homotopy is provided by G: $B \times I \rightarrow B$ defined by

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