Greenberger-Horne-Zeilinger nonlocality in phase space

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We show that the continuous-variable analogues to the multipartite entangled Greenberger-Horne-Zeilinger states of qubits violate Bell-type inequalities imposed by local realistic theories. Our results suggest that the degree of nonlocality of these nonmaximally entangled continuous-variable states, represented by the maximum violation, grows with increasing number of parties. This growth does not appear to be exponentially large as for the maximally entangled qubit states, but rather decreases for larger numbers of parties.

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Entanglement and nonlocality are the most outstanding features of quantum mechanics. In the rapidly advancing field of quantum communication and computation, entangled states are the key ingredients: they enable quantum teleportation [1], quantum cryptography [2], and many other potentially useful schemes. Bell showed that nonlocality can be revealed via the constraints that local realism imposes on the statistics of two physically separated systems [3]. These constraints, expressed in terms of the Bell inequalities, can be violated by quantum mechanics. Entanglement does not automatically imply nonlocality. The so-called Werner states are mixed states that are inseparable, but do not violate any Bell inequality [4]. Also pure entangled states can, if associated with a positive Wigner function, directly reveal a local hidden-variable description [3].

Towards possible applications in quantum communication, both theoretical and experimental investigations increasingly focus on quantum states with a continuous spectrum defined in an infinite-dimensional Hilbert space. These states can be relatively easily generated using squeezed light and beam splitters, as for instance the entangled two-mode squeezed vacuum state that has already proven its usefulness for quantum teleportation [5]. The two-mode squeezed vacuum state is an approximate version of the original Einstein-Podolsky-Rosen (EPR) state [6] where the quadrature amplitudes of the electromagnetic field play the roles of position and momentum of a particle. Its Wigner function is positive everywhere and hence it has a local hidden-variable description [3]. Thus, attempts to derive for this state violations of Bell inequalities based on homodyne measurements of the quadratures failed [7]. However, whether nonlocality is uncovered depends on the observables and the measurements considered in a specific Bell inequality and not only on the quantum state itself. It was shown by Banaszek and Wodkiewicz [8], that the two-mode squeezed vacuum state is nonlocal, as it violates a Clauser-Horne-Shimony-Holt (CHSH) inequality [9] when measurements of photon number parity are considered.

The nonlocality of the multipartite entangled qubit Greenberger-Horne-Zeilinger (GHZ) states can *in principle* be manifest in a single measurement and need not be statistical [10] as the violation of a Bell inequality that relies on mean values. But Mermin and others [11,12] also derived Bell-CHSH inequalities for N-particle systems. The aim of this paper is to apply those N-party inequalities to continuous-variable GHZ states [13] and thereby to prove their nonlocality. Since these states have a positive Wigner function, we shall follow the convenient strategy of Banaszek and Wodkiewicz [8] who exploited the fact that the Wigner function is connected to the quantum mean value of the photon number parity operator. Relying on this connection, we will demonstrate N-party nonlocality using mean-value inequalities [12], and we do not follow the original GHZ program utilizing a contradiction to local realism in a single measurement.

Let us identify the "position" and "momentum" of a particle with the quadrature amplitudes of a single electromagnetic mode (the real and imaginary part of the mode's annihilation operator). In Ref. [13], it has been shown that a sequence of beam splitter operations,

$$\hat{B}_{N-1\,N}(\pi/4)\hat{B}_{N-2\,N-1}(\cos^{-1}(1/\sqrt{3}))\\ \times\cdots\times\hat{B}_{12}(\cos^{-1}(1/\sqrt{N})),$$

applied to one momentum squeezed vacuum mode 1 and $N-1$ position squeezed vacuum modes 2 through N, yields an N-mode state with N-party entanglement between all modes. Here, an ideal (phase-free) beam splitter operation $\hat{B}_{ij}(\theta)$ acts on a pair of modes i and j with annihilation operators $\hat{a}_i$ and $\hat{a}_j$ like $\hat{a}_i\to\hat{a}_i\cos\theta+\hat{a}_j\sin\theta$, and $\hat{a}_j\to\hat{a}_i\sin\theta-\hat{a}_j\cos\theta$. The Wigner function of the pure entangled N-mode state is

$$W(\mathbf{x},\mathbf{p})=\left(\frac{2}{\pi}\right)^N\exp\Bigg\{-e^{-2r}\Bigg[\frac{2}{N}\left(\sum_{i=1}^{N}x_i\right)^2+\frac{1}{N}\sum_{i,j}^{N}(p_i-p_j)^2\Bigg]-e^{+2r}\Bigg[\frac{2}{N}\left(\sum_{i=1}^{N}p_i\right)^2+\frac{1}{N}\sum_{i,j}^{N}(x_i-x_j)^2\Bigg]\Bigg\},\quad(1)$$

where $\mathbf{x}=(x_1,x_2,\ldots,x_N)$ and $\mathbf{p}=(p_1,p_2,\ldots,p_N)$ are the positions and momenta of the N modes and r is the squeezing parameter (with equal squeezing in all initial modes). The state $W(\mathbf{x},\mathbf{p})$ is always positive, symmetric among the N modes, and becomes peaked at $x_i-x_j=0$ $(i,j=1,2,\ldots,N)$ and $p_1+p_2+\cdots+p_N=0$ for large squeezing r. For $N=2$, it equals the well-known EPR-state Wigner function that approaches $\delta(x_1-x_2)\delta(p_1+p_2)$ in the limit of infinite squeezing. Any nonzero squeezing yields N-partite entanglement and the position and momentum correlations can be exploited for quantum teleportation [14] between any two of N parties with the assistance of the remaining $N-2$ parties [13].

In order to prove the nonlocality exhibited by the state $W(\mathbf{x},\mathbf{p})$, we use the fact that the Wigner function is proportional to the quantum expectation value of a displaced parity operator [15]. We obtain the relation [8]

$$W(\boldsymbol{\alpha})=\left(\frac{2}{\pi}\right)^N\langle\hat{\Pi}(\boldsymbol{\alpha})\rangle=\left(\frac{2}{\pi}\right)^N\Pi(\boldsymbol{\alpha}), \tag{2}$$

where $\boldsymbol{\alpha}=\mathbf{x}+i\mathbf{p}=(\alpha_1,\alpha_2,\ldots,\alpha_N)$ and $\Pi(\boldsymbol{\alpha})$ is the quantum expectation value of the operator

$$\hat{\Pi}(\boldsymbol{\alpha})=\otimes_{i=1}^N\hat{\Pi}_i(\alpha_i)=\otimes_{i=1}^N\hat{D}_i(\alpha_i)(-1)^{\hat{n}_i}\hat{D}_i^\dagger(\alpha_i). \tag{3}$$

The operators $\hat{D}_i(\alpha_i)$ are phase-space displacement operators acting on mode i. Thus, $\hat{\Pi}(\boldsymbol{\alpha})$ is a product of displaced parity operators given by

$$\hat{\Pi}_i(\alpha_i)=\hat{\Pi}_i^{(+)}(\alpha_i)-\hat{\Pi}_i^{(-)}(\alpha_i), \tag{4}$$

with the projection operators

$$\hat{\Pi}_i^{(+)}(\alpha_i)=\hat{D}_i(\alpha_i)\sum_{k=0}^{\infty}|2k\rangle\langle 2k|\hat{D}_i^\dagger(\alpha_i), \tag{5}$$

$$\hat{\Pi}_i^{(-)}(\alpha_i)=\hat{D}_i(\alpha_i)\sum_{k=0}^{\infty}|2k+1\rangle\langle 2k+1|\hat{D}_i^\dagger(\alpha_i), \tag{6}$$

corresponding to the measurement of an even (parity $+1$) or an odd (parity -1) number of photons in mode i. This means that each mode is now characterized by a dichotomic variable similar to the single-particle spin or the single-photon polarization. Different spin or polarizer orientations are replaced by different displacements in phase space. These different settings of a measurement with two possible outcomes ± 1 for each possible setting is exactly what we need for the nonlocality test.

In the case of N-particle systems, such a nonlocality test is possible using the N-particle generalization of the two-particle Bell-CHSH inequality [12]. This inequality is based on the following recursively defined linear combination of joint measurement results

$$\begin{aligned}B_N\equiv&\tfrac{1}{2}[\sigma(a_N)+\sigma(a_N')]B_{N-1}\\&+\tfrac{1}{2}[\sigma(a_N)-\sigma(a_N')]B_{N-1}'=\pm 2,\end{aligned} \tag{7}$$

where $\sigma(a_N)=\pm 1$ and $\sigma(a_N')=\pm 1$ describe two possible outcomes for two possible measurement settings (denoted by a_N and a_N') of measurements on the Nth particle. Provided that $B_{N-1}=\pm 2$ and $B_{N-1}'=\pm 2$, Eq. (7) is true for a single run of the measurements, where $\sigma(a_N)$ becomes either $+1$ or -1 and so does $\sigma(a_N')$. Thus, induction proves Eq. (7) for any N with

$$B_2\equiv[\sigma(a_1)+\sigma(a_1')]\sigma(a_2)+[\sigma(a_1)-\sigma(a_1')]\sigma(a_2')=\pm 2, \tag{8}$$

which is trivially true (the expressions B_N' are equivalent to B_N but with all the a_i and a_i' swapped). Within the framework of local realistic theories with the hidden variables $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\ldots,\lambda_N)$ and the normalized probability distribution $P(\boldsymbol{\lambda})$, we obtain an inequality for the average value of $B_N\equiv B_N(\boldsymbol{\lambda})$,

$$\left|\int d\lambda_1 d\lambda_2\ldots d\lambda_N P(\boldsymbol{\lambda})B_N(\boldsymbol{\lambda})\right|\leqslant 2. \tag{9}$$

By the linearity of averaging, this is a sum of means of products of the $\sigma(a_i)$ and $\sigma(a_i')$. For example, if $N=2$, we obtain the CHSH inequality

$$|C(a_1,a_2)+C(a_1,a_2')+C(a_1',a_2)-C(a_1',a_2')|\leqslant 2, \tag{10}$$

with the correlation functions

$$C(a_1,a_2)=\int d\lambda_1 d\lambda_2 P(\lambda_1,\lambda_2)\sigma(a_1,\lambda_1)\sigma(a_2,\lambda_2). \tag{11}$$

Following Bell [3], an always positive Wigner function can serve as the hidden-variable probability distribution. In this sense, the EPR-state Wigner function could prevent the CHSH inequality being violated: $W(x_1,p_1,x_2,p_2)\equiv P(\lambda_1,\lambda_2)$. The same applies to the general Wigner function in Eq. (1): $W(\mathbf{x},\mathbf{p})\equiv P(\boldsymbol{\lambda})$ could be used to construct correlation functions

$$\begin{aligned}C(\mathbf{a})=&\int d\lambda_1 d\lambda_2\ldots d\lambda_N P(\boldsymbol{\lambda})\\&\times\sigma(a_1,\lambda_1)\sigma(a_2,\lambda_2)\cdots\sigma(a_N,\lambda_N),\end{aligned} \tag{12}$$

where $\mathbf{a}=(a_1,a_2,\ldots,a_N)$. However, for parity measurements on each mode with possible results ± 1 and different settings by different displacements, this would require unbounded δ functions for the local objective quantities $\sigma(a_i,\lambda_i)$ [8], as in this case the relation

$$C(\mathbf{a})\equiv\Pi(\boldsymbol{\alpha})=(\pi/2)^N W(\boldsymbol{\alpha}) \tag{13}$$

holds. This relation, which directly relates the correlation function to the Wigner function, is indeed crucial for the nonlocality proof of the continuous-variable states in Eq. (1). For the EPR state with $N=2$, we can now look at the combination [8]

$$\mathcal{B}_2=\Pi(0,0)+\Pi(0,\beta)+\Pi(\alpha,0)-\Pi(\alpha,\beta), \tag{14}$$

which according to Eq. (10) satisfies $|\mathcal{B}_2|\leqslant 2$ for local realistic theories. Here, we have chosen the displacement settings $\alpha_1=\alpha_2=0$ and $\alpha_1'=\alpha$, $\alpha_2'=\beta$.

Let us write the states in Eq. (1) as

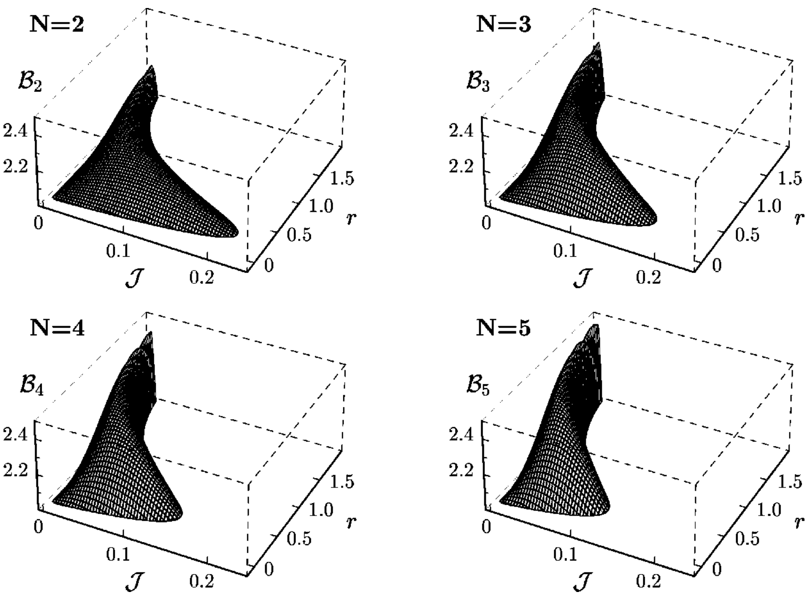

FIG. 1. Violations of the inequality $|\mathcal{B}_N|\leqslant 2$ imposed by local realistic theories with the entangled two-mode EPR ($N=2$, as in Ref. [8]), three-mode GHZ ($N=3$), four-mode GHZ ($N=4$), and five-mode GHZ ($N=5$) states.

$$\Pi(\boldsymbol{\alpha})=\exp\Bigg\{-2\cosh 2r\sum_{i=1}^{N}|\alpha_i|^2+\sinh 2r\Bigg[\frac{2}{N}\sum_{i,j}^{N}(\alpha_i\alpha_j+\alpha_i^*\alpha_j^*)-\sum_{i=1}^{N}(\alpha_i^2+\alpha_i^{*2})\Bigg]\Bigg\}. \quad (15)$$

For $N=2$ and $\alpha=\beta=i\sqrt{\mathcal{J}}$ in terms of the real displacement parameter $\mathcal{J}\geqslant 0$ [16], these states yield $\mathcal{B}_2=1+2\exp(-2\mathcal{J}\cosh 2r)-\exp(-4\mathcal{J}e^{+2r})$. In the limit of large $r(\cosh 2r\approx e^{+2r}/2)$ and small $\mathcal{J}$, this $\mathcal{B}_2$ is maximized for $\mathcal{J}e^{+2r}=(\ln 2)/3$: $\mathcal{B}_2^{\max}\approx 2.19$ [8], which is a clear violation of the inequality $|\mathcal{B}_2|\leqslant 2$. Smaller violations occur also for smaller squeezing and bigger $\mathcal{J}$. For any nonzero squeezing, some violation takes place (see Fig. 1).

Let us now examine the three-mode state and set $N=3$ in Eq. (15). According to the inequality of the correlation functions derived from Eqs. (7)–(9) with $N=3$,

$$|C(a_1,a_2,a_3')+C(a_1,a_2',a_3)+C(a_1',a_2,a_3)-C(a_1',a_2',a_3')|\leqslant 2, \quad (16)$$

for the possible combination

$$\mathcal{B}_3=\Pi(0,0,\gamma)+\Pi(0,\beta,0)+\Pi(\alpha,0,0)-\Pi(\alpha,\beta,\gamma), \quad (17)$$

a contradiction to local realism does not occur only if $|\mathcal{B}_3|\leqslant 2$. The corresponding settings here are $\alpha_1=\alpha_2=\alpha_3=0$ and $\alpha_1'=\alpha$, $\alpha_2'=\beta$, and $\alpha_3'=\gamma$. With the choice $\alpha=\sqrt{\mathcal{J}}e^{i\phi_1}$, $\beta=\sqrt{\mathcal{J}}e^{i\phi_2}$, and $\gamma=\sqrt{\mathcal{J}}e^{i\phi_3}$, we obtain

$$\mathcal{B}_3=\sum_{i=1}^{3}\exp(-2\mathcal{J}\cosh 2r-\tfrac{2}{3}\mathcal{J}\sinh 2r\cos 2\phi_i)-\exp\Bigg\{-6\mathcal{J}\cosh 2r-\tfrac{1}{3}\mathcal{J}\sinh 2r\times\sum_{i\neq j}^{3}[\cos 2\phi_i-4\cos(\phi_i+\phi_j)]\Bigg\}. \quad (18)$$

Apparently, because of the symmetry of the entangled three-mode state, equal phases ϕ_i should also be chosen in order to maximize $\mathcal{B}_3$. The best choice is $\phi_1=\phi_2=\phi_3=\pi/2$, which ensures that the positive terms in Eq. (18) become maximal and the contribution of the negative term minimal. Therefore we again use equal settings $\alpha=\beta=\gamma=i\sqrt{\mathcal{J}}$ and obtain

$$\mathcal{B}_3=3\exp(-2\mathcal{J}\cosh 2r+2\mathcal{J}\sinh 2r/3)-\exp(-6\mathcal{J}e^{+2r}). \quad (19)$$

The violations of $|\mathcal{B}_3|\leqslant 2$ that occur with this result are similar to the violations $|\mathcal{B}_2|\leqslant 2$ obtained for the EPR state, but the $N=3$ violations are even more significant than the $N=2$ violations (see Fig. 1). In the limit of large r (and small $\mathcal{J}$), we may use $\cosh 2r\approx\sinh 2r\approx e^{+2r}/2$ in Eq. (19). Then $\mathcal{B}_3$ is maximized for $\mathcal{J}e^{+2r}=3(\ln 3)/16$: $\mathcal{B}_3^{\max}\approx 2.32$. This requires even smaller displacements $\mathcal{J}$ than in the $N=2$ case for the same squeezing.

Let us now investigate the cases $N=4$ and $N=5$. From Eqs. (7)–(9) with $N=4$, the following inequality for the correlation functions can be derived:

$$\tfrac{1}{2}|C(a_1,a_2,a_3,a_4')+C(a_1,a_2,a_3',a_4)+C(a_1,a_2',a_3,a_4)+C(a_1',a_2,a_3,a_4)+C(a_1,a_2,a_3',a_4')+C(a_1,a_2',a_3,a_4')+C(a_1',a_2,a_3,a_4')+C(a_1,a_2',a_3',a_4)+C(a_1',a_2,a_3',a_4)$$

$$+C(a_1',a_2',a_3,a_4)-C(a_1',a_2',a_3',a_4)$$
$$-C(a_1',a_2',a_3,a_4')-C(a_1',a_2,a_3',a_4')$$
$$-C(a_1,a_2',a_3',a_4')-C(a_1,a_2,a_3,a_4)$$
$$-C(a_1',a_2',a_3',a_4')|\leqslant 2. \tag{20}$$

It is symmetric among all four parties as any inequality derived from Eqs. (7)–(9) is symmetric among all parties. For the settings $\alpha_1=\alpha_2=\alpha_3=\alpha_4=0$ and $\alpha_1'=\alpha$, $\alpha_2'=\beta$, $\alpha_3'=\gamma$, and $\alpha_4'=\delta$, complying with local realism implies $|\mathcal{B}_4|\leqslant 2$, where

$$\mathcal{B}_4=\tfrac{1}{2}[\Pi(0,0,0,\delta)+\Pi(0,0,\gamma,0)+\Pi(0,\beta,0,0)+\Pi(\alpha,0,0,0)$$
$$+\Pi(0,0,\gamma,\delta)+\Pi(0,\beta,0,\delta)+\Pi(\alpha,0,0,\delta)$$
$$+\Pi(0,\beta,\gamma,0)+\Pi(\alpha,0,\gamma,0)+\Pi(\alpha,\beta,0,0)$$
$$-\Pi(\alpha,\beta,\gamma,0)-\Pi(\alpha,\beta,0,\delta)-\Pi(\alpha,0,\gamma,\delta)$$
$$-\Pi(0,\beta,\gamma,\delta)-\Pi(0,0,0,0)-\Pi(\alpha,\beta,\gamma,\delta)]. \tag{21}$$

Similarly, for $N=5$ one finds

$$\mathcal{B}_5=\tfrac{1}{2}[\Pi(0,0,0,\delta,\epsilon)+\Pi(0,0,\gamma,0,\epsilon)+\Pi(0,\beta,0,0,\epsilon)$$
$$+\Pi(\alpha,0,0,0,\epsilon)+\Pi(0,0,\gamma,\delta,0)+\Pi(0,\beta,0,\delta,0)$$
$$+\Pi(\alpha,0,0,\delta,0)+\Pi(0,\beta,\gamma,0,0)+\Pi(\alpha,0,\gamma,0,0)$$
$$+\Pi(\alpha,\beta,0,0,0)-\Pi(\alpha,\beta,\gamma,\delta,0)-\Pi(\alpha,\beta,\gamma,0,\epsilon)$$
$$-\Pi(\alpha,\beta,0,\delta,\epsilon)-\Pi(\alpha,0,\gamma,\delta,\epsilon)-\Pi(0,\beta,\gamma,\delta,\epsilon)$$
$$-\Pi(0,0,0,0,0)], \tag{22}$$

which has to statisfy $|\mathcal{B}_5|\leqslant 2$ and contains the same settings as for $N=4$, but in addition $\alpha_5=0$ and $\alpha_5'=\epsilon$.

We can now use the entangled states in Eq. (15) with $N=4$ and $N=5$ and apply the inequalities to them. For the same reason as for $N=3$ (symmetry among all modes in the states and in the inequalities), the choice $\alpha=\beta=\gamma=\delta=\epsilon=i\sqrt{\mathcal{J}}$ appears to be optimal (maximizes positive terms and minimizes negative contributions).

With this choice, we obtain

$$\mathcal{B}_4=2\exp(-2\mathcal{J}\cosh 2r+\mathcal{J}\sinh 2r)-2\exp(-6\mathcal{J}\cosh 2r$$
$$-3\mathcal{J}\sinh 2r)+3\exp(-4\mathcal{J}\cosh 2r)$$
$$-\tfrac{1}{2}\exp(-8\mathcal{J}e^{+2r})-\tfrac{1}{2},$$

$$\mathcal{B}_5=5\exp(-4\mathcal{J}\cosh 2r+4\mathcal{J}\sinh 2r/5)$$
$$-\tfrac{5}{2}\exp(-8\mathcal{J}\cosh 2r-24\mathcal{J}\sinh 2r/5)-\tfrac{1}{2}. \tag{23}$$

As shown in Fig. 1, the maximum violation of $|\mathcal{B}_N|\leqslant 2$ (for our particular choice of settings) grows with increasing number of parties N. The asymptotic analysis (large r and small $\mathcal{J}$) yields for $N=5$: $\mathcal{B}_5^{\max}\approx 2.48$ with $\mathcal{J}e^{+2r}=5(\ln 2)/24$. At a certain amount of large squeezing, smaller displacements $\mathcal{J}$ than for $N\leqslant 4$ (at the same squeezing) are needed to approach this maximum violation. Another important observation is that in all four cases ($N=2,3,4,5$), violations occur for any nonzero squeezing. This requires the presence of N-partite entanglement for any nonzero squeezing, which is consistent with the results in Ref. [13]. Moreover, we see that not only for large squeezing but also for modest finite squeezing, the significance of the violations (at optimal displacements $\mathcal{J}$) grows with increasing N.

In the following, we will examine the general case of N parties. How does the maximum violation of the Bell-type inequalities derived with the continuous-variable GHZ states in general evolve with increasing number of parties, in particular, compared to the exponential growth for the qubit GHZ states [11,12]? At least for $N\leqslant 5$, the maximum violation grows, and this growth does not appear to be exponentially large, but rather seems to decrease. This conjecture has not been proven, since we did not consider all possible settings (all possible combinations of α_i and α_i'). However, there are strong hints that our choice of $\alpha_i=0$ and $\alpha_i'=i\sqrt{\mathcal{J}}$ is near optimal. In particular, that the nonlocality is always revealed for arbitrarily small squeezing (any nonzero squeezing) lets our choice appear more appropriate than other possible combinations. Having now much confidence in the choice of settings that we used for small numbers of parties, we will use the same settings for larger numbers of parties.

Considering odd numbers of parties N, we find the following expression for $\mathcal{B}_N$,

$$\text{if}\ \ N=3+8M:\quad \mathcal{B}_N=2^{(3-N)/2}\sum_{k=0}^{(N-1)/2}(-1)^k \times\binom{N}{2k+1}\Pi(\alpha_1',\alpha_2',\ldots,\alpha_{2k+1}', \alpha_{2k+2},\alpha_{2k+3},\ldots,\alpha_N), \tag{24}$$

where the first $2k+1$ arguments of Π are $\alpha_1'=\alpha_2'=\cdots=\alpha_{2k+1}'=i\sqrt{\mathcal{J}}$, and the remaining ones are $\alpha_{2k+2}=\alpha_{2k+3}=\cdots=\alpha_N=0$, and $M=0,1,2,3,\ldots$. Because of the symmetry of the states $\Pi(\boldsymbol{\alpha})$ in Eq. (15), all possible permutations of the $(2k+1)$ α_i''s with $\alpha_i'=i\sqrt{\mathcal{J}}$ and the $[N-(2k+1)]$ α_i's with $\alpha_i=0$ can be described by the same function $\Pi(\alpha_1',\alpha_2',\ldots,\alpha_{2k+1}',\alpha_{2k+2},\alpha_{2k+3},\ldots,\alpha_N)$.

Similarly, with the same settings $\alpha_i'=i\sqrt{\mathcal{J}}$ and $\alpha_i=0$, and again by exploiting symmetry, we obtain

$$\text{for}\ \ N=5+8M:\quad \mathcal{B}_N=2^{(3-N)/2}\sum_{k=0}^{(N-1)/2}(-1)^{k+1} \times\binom{N}{2k}\Pi(\alpha_1',\alpha_2',\ldots,\alpha_{2k}',\alpha_{2k+1}, \alpha_{2k+2},\ldots,\alpha_N), \tag{25}$$

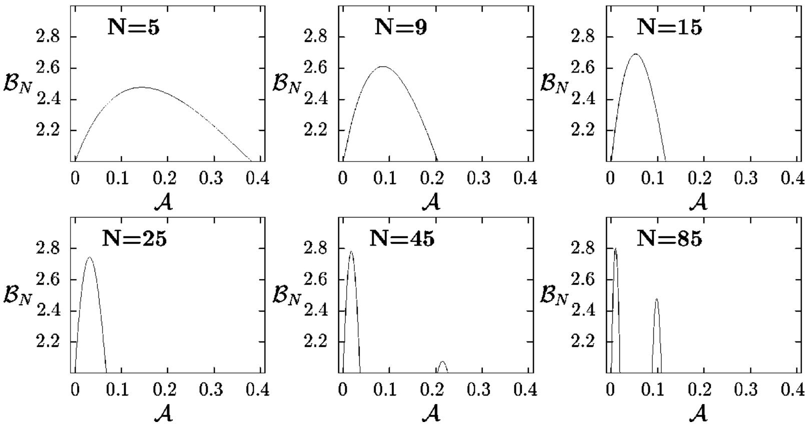

FIG. 2. Maximum violations of the inequality $|\mathcal{B}_N|\leqslant 2$ imposed by local realistic theories in the limit of large squeezing. $\mathcal{B}_N$ is plotted as a function of $\mathcal{A}\equiv\mathcal{J}e^{+2r}$ for different N.

$$\text{for } N=7+8M:\quad \mathcal{B}_N=2^{(3-N)/2}\sum_{k=0}^{(N-1)/2}(-1)^{k+1}\binom{N}{2k+1}\Pi(\alpha_1',\alpha_2',\ldots,\alpha_{2k+1}',\alpha_{2k+2},\alpha_{2k+3},\ldots,\alpha_N), \tag{26}$$

$$\text{for } N=9+8M:\quad \mathcal{B}_N=2^{(3-N)/2}\sum_{k=0}^{(N-1)/2}(-1)^{k}\binom{N}{2k}\Pi(\alpha_1',\alpha_2',\ldots,\alpha_{2k}',\alpha_{2k+1},\alpha_{2k+2},\ldots,\alpha_N). \tag{27}$$

The functions concerned in these formulas are explicitly given by [see Eq. (15)]

$$\Pi(\alpha_1',\alpha_2',\ldots,\alpha_{2k}',\alpha_{2k+1},\alpha_{2k+2},\ldots,\alpha_N)=\exp\left\{-2\mathcal{J}\cosh 2r(2k)+2\mathcal{J}\sinh 2r\times\left[2k-2\frac{(2k)^2}{N}\right]\right\}, \tag{28}$$

$$\Pi(\alpha_1',\alpha_2',\ldots,\alpha_{2k+1}',\alpha_{2k+2},\alpha_{2k+3},\ldots,\alpha_N)=\exp\left\{-2\mathcal{J}\cosh 2r(2k+1)+2\mathcal{J}\sinh 2r\times\left[2k+1-2\frac{(2k+1)^2}{N}\right]\right\}. \tag{29}$$

Let us first consider the case of zero squeezing, $r=0$. The sum from Eq. (24) becomes in this case

$$\mathcal{B}_N(r=0)=2^{(3-N)/2}(1+e^{-4\mathcal{J}})^{N/2}\sin[N\arctan(e^{-2\mathcal{J}})]. \tag{30}$$

As expected, without squeezing, no violations of the Bell-type inequalities are obtained for the unentangled, separable N-mode states: we find $\mathcal{B}_N(r=0)=2$ if $\mathcal{J}=0$ for any $N=3+8M$ and $|\mathcal{B}_N(r=0)|<2$ if $\mathcal{J}>0$. In the limit $N\to\infty$, we obtain $\mathcal{B}_N(r=0)\to 0$ for any $\mathcal{J}>0$. Similar expressions as in Eq. (30) can be found for $\mathcal{B}_N(r=0)$ in the other cases of odd N, $N=5+8M$, $N=7+8M$, and $N=9+8M$, and in fact, no violations occur. The inequality $|\mathcal{B}_N|\leqslant 2$ imposed by local realistic theories always remains satisfied for zero squeezing.

On the other hand, inferring from the results for $N\leqslant 5$ parties, the maximum violations of $|\mathcal{B}_N|\leqslant 2$ occur for large squeezing. Let us again consider the limit of large squeezing ($\cosh 2r\approx\sinh 2r\approx e^{+2r}/2$) and define $\mathcal{A}\equiv\mathcal{J}e^{+2r}$. Now we can write Eq. (28) and Eq. (29) as

$$\Pi(\alpha_1',\alpha_2',\ldots,\alpha_{2k}',\alpha_{2k+1},\alpha_{2k+2},\ldots,\alpha_N)=\exp[-2\mathcal{A}(2k)^2/N], \tag{31}$$

$$\Pi(\alpha_1',\alpha_2',\ldots,\alpha_{2k+1}',\alpha_{2k+2},\alpha_{2k+3},\ldots,\alpha_N)=\exp[-2\mathcal{A}(2k+1)^2/N]. \tag{32}$$

Figure 2 shows the maxima of the violations of $|\mathcal{B}_N|\leqslant 2$ (for our particular choice of settings), calculated with Eqs. (24)–(27) and the asymptotic results from Eqs. (31)–(32) for large squeezing. The maximum violation grows from $\mathcal{B}_5^{\max}\approx 2.48$ for $N=5$ to $\mathcal{B}_{85}^{\max}\approx 2.8$ for $N=85$. Within this range, a maximum violation near 2.8 is already attained with $N=45$ parties and there is only a very small increase from $N=45$ to $N=85$. On the other hand, between $N=5$ and $N=9$, the maximum violation goes up from 2.48 to about 2.6, which is still significantly less than the increase between $N=2$ ($\mathcal{B}_2^{\max}\approx 2.19$) and $N=5$. This confirms our conjecture based on the results for $N\leqslant 5$: apparently, the maximum violation indeed grows with increasing number of parties, but this growth seems to continuously decrease for larger numbers of parties. In fact, from $N=45$ to $N=85$, we see a second local maximum emerging rather than a significant further increase of the absolute maximum violation.

In Fig. 3, calculated with Eqs. (24)–(27) and Eqs. (28)–(29), violations of $|\mathcal{B}_N|\leqslant 2$ are compared between different numbers of parties at certain amounts of squeezing of the corresponding GHZ states. As stated earlier, the violations grow with N also for modest finite squeezing, but this increase is smaller than the increase of the maximum violations and becomes unrecognizable for small squeezing. An illustrating example is that a violation comparable to the

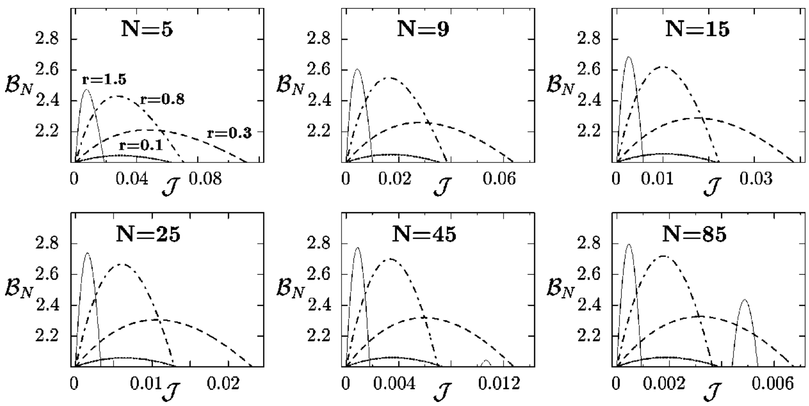

FIG. 3. Violations of the inequality $|\mathcal{B}_N|\leqslant 2$ imposed by local realistic theories for different N at certain amounts of squeezing of the N-mode GHZ states: $r=0.1$ (≈ 0.9 dB), $r=0.3$ (≈ 2.6 dB), $r=0.8$ (≈ 6.9 dB), and $r=1.5$ (≈ 13 dB). $\mathcal{B}_N$ is plotted as a function of $\mathcal{J}$. Note that the axes of the displacement parameter $\mathcal{J}$ vary in scale. The larger N becomes, the smaller become the displacements required.

maximum violation with the two-mode EPR state for large squeezing ($\mathcal{B}_2^{\max}\approx 2.19$) can be attained with a five-mode GHZ state built from five modestly squeezed states (about 2.6 dB each).

We conclude with a summary and an assessment of our results. We have considered pure multipartite entangled states described by continuous quantum variables and shown that they violate Bell-type inequalities imposed by local realism. An experimental nonlocality test based on these states and on our scheme is possible, but it would require detectors capable of resolving the number of absorbed photons [17]. Nevertheless, *the N-mode states, which we have unambiguously proven to exhibit nonlocality, can be relatively easily generated in practice, as opposed to the discrete-variable GHZ states on which all current multiparty nonlocality proofs rely*. Furthermore, entangled N-mode states similar to those considered here can even be produced using only one single-mode squeezed vacuum state and linear optics instead of N squeezed states [13]. Since it has been shown already that the entangled two-mode state created this way is nonlocal with respect to parity measurements [18], one can apply our analysis to the corresponding N-mode states and expect that they too are nonlocal.

The degree of nonlocality of the continuous-variable GHZ states, if represented by the maximum violation of the corresponding Bell-type inequalities, seems to grow with increasing number of parties. This growth, however, continuously decreases for larger numbers of parties. Thus, the evolution of the continuous-variable states' nonlocality with increasing number of parties and the corresponding evolution of nonlocality for the qubit GHZ states are qualitatively equal but quantitatively different (with an exponential increase for the qubits). The reason for this may be that the latter always relies on maximally entangled states, whereas the former depends on nonmaximally entangled states as long as the squeezing remains finite. In fact, an observation of the nonlocality of the continuous-variable states requires small but nonzero displacements $\mathcal{J}\propto e^{-2r}$, which is not achievable when the singular maximally entangled states for infinite squeezing are considered.

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