We have seen that the sums of rearrangements of the alternating harmonic series depend only on the asymptotic density α. This behavior is in some sense specific to series like the harmonic series, as Theorem 2 indicates. Readers are invited to construct proofs for themselves or to consult Pringsheim's paper [3].

THEOREM 2 [3]. *Suppose* $\{a_n\}_{n=1}^{\infty}$ *is a sequence of real numbers such that* $|a_1| \geqslant |a_2| \geqslant |a_3| \geqslant \cdots$, $\lim_{n\to\infty} a_n = 0$, *and* $a_{2k-1} > 0 > a_{2k}$ *for* $k = 1,2,3,\ldots$.

(i) *If* $\lim_{n\to\infty} n|a_n| = \infty$, *and if S is a real number, there is a simple rearrangement of the series* $\sum_{k=1}^{\infty} a_k$ *with asymptotic density* $\frac{1}{2}$ *whose sum is S.*

(ii) *If* $\lim_{n\to\infty} na_n = 0$, *if* $\sum_{k=1}^{\infty} b_k$ *is a simple rearrangement of the series* $\sum_{k=1}^{\infty} a_k$ *for which the asymptotic density* α *exists, and if* $0 < \alpha < 1$, *then* $\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} a_k$.

The authors gratefully acknowledge support from NSF (Cowen and Kaufman) and NSERC (Davidson).

References

1. T. J. Ia. Bromwich, An Introduction to the Theory of Infinite Series, 2nd ed., Macmillan, London, 1947.
2. H. P. Manning, Irrational Numbers, Wiley, New York, 1906.
3. A. Pringsheim, Über die Werthveränderungen bedingt convergierten Reihe und Producte, Mathematische Annalen, 22 (1883) 455–503.
4. W. Rudin, Principles of Mathematical Analysis, 2nd, ed., McGraw-Hill, New York, 1964.
5. E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, Amer. ed., Cambridge University Press, 1943.

A STRONG CONVERSE TO GAUSS'S MEAN-VALUE THEOREM

R. B. BURCKEL

Department of Mathematics, Kansas State University, Manhattan, KS 66506

The theorem of Gauss in the title affirms that

$$h(a) = \frac{1}{2\pi}\int_0^{2\pi} h(a + re^{i\theta})\,d\theta \tag{1}$$

holds for all a in a region Ω, all $r > 0$ such that the closure of the disc $D(a,r) = \{z \in \mathbb{C}: |z-a| < r\}$ lies in Ω, and all functions h that are harmonic throughout Ω. Most books on function theory or potential theory prove this elementary result as well as the following converse due to Koebe [4]: *If h is continuous in the region* Ω *and* (1) *holds for all a and r such that* $\overline{D}(a,r) \subset \Omega$, *then h is harmonic in* Ω. In fact, the somewhat stronger version in which the equality is required to hold only at each a for some sequence $r_n(a) \to 0$ is often proved. What does not seem to be well known is that, when h is continuous on $\overline{\Omega}$, one radius suffices. This strong converse of Gauss's theorem is due to Kellogg [3] and is not trivial. However, for Dirichlet regions this strong converse is as easy to prove as Koebe's theorem and should be presented in elementary texts. The theorem for Dirichlet regions is due to Volterra [7] (with a supplemental hypothesis) and to Vitali [6] (where the supplemental hypothesis is removed). Here is their proof in modern dress, presented in dimension two, though the reader will see that it is valid in any dimension.

LEMMA. *Let U be a bounded open subset of the complex plane* $\mathbb{C}$ *and let* $f: \overline{U} \to \mathbb{R}$ *be continuous and for each* $a \in U$ *have the following* restricted mean-value property:

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta})\,d\theta \text{ for some } r = r(a) > 0 \text{ such that } \overline{D}(a,r) \subset U. \tag{2}$$

Then $\max f(\overline{U}) = \max f(\partial U)$.

Proof. (Cf. Cimmino [1]) Let $M = \max f(\overline{U})$. It suffices to see that the closed subset $f^{-1}(M)$ of

$\bar{U}$ meets ∂U. If this is not the case, then by compactness there is a point $a \in f^{-1}(M) \subset U$ that is nearest to ∂U. From

$$M = f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + r(a)e^{i\theta})\,d\theta$$

and the maximality of M we see that $f(a + r(a)e^{i\theta}) = M$ for each $\theta \in [0, 2\pi]$. That is, the circle $\{z \in \mathbb{C}: |z - a| = r(a)\}$ lies wholly in $f^{-1}(M)$. Since, obviously, some point of this circle is closer to ∂U than a, we have a contradiction to the choice of a.

THEOREM. *Let U be a bounded open subset of $\mathbb{C}$ for which the Dirichlet problem is solvable. Then any continuous real-valued function on $\bar{U}$ that has the restricted mean-value property is harmonic in U.*

Proof. Let g be such a function. Since the Dirichlet problem is solvable for U, there exists a continuous function h on $\bar{U}$ that is harmonic in U and coincides with g on ∂U. Since h has the (unrestricted) mean-value property (by Gauss's theorem), the functions $f_1 = g - h$, $f_2 = h - g$ each satisfy the hypotheses of the lemma. Since $f_1 = f_2 = 0$ on ∂U, we infer from the lemma that $f_1 \leqslant 0$, $f_2 \leqslant 0$ throughout U. That is, $g \equiv h$.

FINAL REMARKS. A version of this proof occurs in Courant and Hilbert [**2**, pp. 279–281]. There the reader will also find elementary examples showing that both the boundedness of U and the continuity of h on $\bar{U}$ are essential to the validity of the theorem. There are, however, some fascinating versions of the theorem for the cases $U = \mathbb{C}$ or h only continuous on U. For discussions of these results and extensive references to the literature see Netuka [**5**] and Zalcman [**8**], [**9**]. The latter paper is especially readable.

References

1. G. Cimmino, Formole di maggiorazione nel problema di Dirichlet per le funzioni armoniche, Rend. Seminario Mat. Padova, 3 (1932) 46–66; Jbuch 58, p. 508; Zbl 5, p. 16.

2. R. Courant and D. Hilbert, Methods of Mathematical Physics, vol. 2, Partial Differential Equations, Wiley-Interscience, New York, 1962; MR 31 #4968; Zbl 121, p. 78; Jbuch 63, p. 449.

3. O. D. Kellogg, Converses of Gauss' theorem on the arithmetic mean, Trans. Amer. Math. Soc., 36 (1934) 227–242; Jbuch 60, p. 430; Zbl 9, p. 112.

4. P. Koebe, Herleitung der partiellen Differentialgleichung der Potentialfunktion aus deren Integraleigenschaft, Sitzungsber. Berlin Math. Gesell., 5 (1906) 39–42; Jbuch 37, p. 384.

5. I. Netuka, Harmonické funkce a věty o průměru, Časopis pro pěstování matematiky, 100 (1975) 391–409; MR 57 #3411; Zbl 314 #31007.

6. G. Vitali, Sopra una proprietà caratteristica delle funzioni armoniche, Rend. Accad. d. Lincei Roma (5), 21, part 2 (1912) 315–320; Jbuch 43, p. 492.

7. V. Volterra, Alcune osservazioni sopra propiertà atte ad individuare una funzione, Rend. Accad. d. Lincei Roma (5), 18, part 1 (1909) 263–266; Jbuch 40, p. 453.

8. L. Zalcman, Analyticity and the Pompeiu problem, Arch. Rational Mech. and Anal., 47 (1972) 237–254; MR 50 #582; Zbl 251 #30047.

9. ______, Offbeat integral geometry, this MONTHLY, 87 (1980) 161–175.

10. S. Zaremba, Contributions à la théorie d'une équation fonctionnelle de la physique, Rend. Circ. Mat. Palermo, 19 (1905) 140–150; Jbuch 36, p. 822.