common—and how far we all are from solutions! But it is also pleasant simply to relax among those who share our view of the importance of mathematics. The social events are designed to give the maximum opportunity for informal, unrestricted contact.

Probably for many participants this may be the most vivid impression they carry away with them from the Congress—this sense of belonging to a community which transcends all barriers of nationhood, of politics, of race and religion—this sense that a common love of mathematics is a stronger force in uniting our profession than any other force could be which seeks to pull us apart. And we do not believe that there is any better way to create this wonderful sense of genuine comradeship than participation in an International Congress.

A CONJECTURED ANALOGUE OF ROLLE'S THEOREM FOR POLYNOMIALS WITH REAL OR COMPLEX COEFFICIENTS

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1. Introduction. We assume that $n \geqslant 2$ and let

$$P_n(z) = z^n + a_2 z^{n-2} + \cdots + a_n = \prod_1^n (z - z_j) \tag{1.1}$$

be a polynomial with $a_1 = 0$; hence its zeros z_j satisfy

$$z_1 + z_2 + \cdots + z_n = 0. \tag{1.2}$$

We also consider its derivative

$$P_n'(z) = nz^{n-1} + (n-2) a_2 z^{n-3} + \cdots + a_{n-1} = n \prod_1^{n-1} (z - w_k), \tag{1.3}$$

having the zeros w_k satisfying

$$w_1 + w_2 + \cdots + w_{n-1} = 0. \tag{1.4}$$

Evidently, the origin O of the complex plane $\mathbb{C}$ is the centroid of the set z_j and also the centroid of the w_k.

The set of $2n - 1$ points

$$R = \{ z_j ; w_k \} \tag{1.5}$$

we call *the complex Rolle set of* $P_n(z)$. If we turn the set R about the origin O by the angle θ, *the new set* $R(\theta)$ *is also a complex Rolle set.*

Indeed, by (1.1) and (1.3) we have

$$P_n(ze^{-i\theta}) = \prod_1^n \left(ze^{-i\theta} - z_j \right) = e^{-in\theta} \prod_1^n \left(z - z_j e^{i\theta} \right),$$

$$(d/dz) P_n(ze^{-i\theta}) = e^{-i\theta} P_n'(ze^{-i\theta}) = ne^{-i\theta} \prod_1^{n-1} \left(ze^{-i\theta} - w_k \right)$$

I. J. Schoenberg: I would very much like to know if our easy derivation of Theorem 2 from van den Berg's Theorem 1 generalizes so that Conjecture 1 can be derived from Theorem (4.2), on page 11, of Morris Marden's book [**2**, Theorem (4.2), page 11, for $p = n$ and $m_1 = m_2 = \cdots = m_n = 1$]. The w_k appear there as the foci of an algebraic curve of class $n - 1$ which touches all segments $z_r z_s$ $(r \neq s)$ in their midpoints. A short biography of mine appears on the back cover of the paper edition of my little book [**3**]. I may add to it that I played the violin and made chamber music most of my life, and that I have been a vegetarian since the age of ten.

$$= ne^{-in\theta} \prod_{1}^{n-1} \left(z - w_k e^{i\theta} \right).$$

These equations show that *the rotated set*

$$R(\theta) = \left\{ z_j e^{i\theta}, w_k e^{i\theta} \right\} \tag{1.6}$$

is the complex Rolle set of $P_n(ze^{-i\theta})$.

We say that the Rolle set R is *rectilinear*, provided that all its $2n - 1$ points z_j and w_k are on a straight line which must contain the centroid O of all these points. If all z_j are real, then by the usual Rolle theorem also all w_k are real. The invariance of the Rolle set by rotation implies that if all z_j are on a line L through O, then all w_k are on L, and so R is rectilinear.

From (1.2) and (1.4) we obtain the equations

$$\sum_{1}^{n} z_j^2 = -2 \sum_{j<j'} z_j z_{j'} = -2a_2, \tag{1.7}$$

$$w_k^2 = - \sum_{k<k'} w_k w_{k'} = -2\frac{n-2}{n} a_2, \tag{1.8}$$

whence, by eliminating a_2, we obtain the identity

$$\sum_{1}^{n-1} w_k^2 = \frac{n-2}{n} \sum_{1}^{n} z_j^2. \tag{1.9}$$

Thus the $2n - 1$ points of any complex Rolle set R satisfy the important identity (1.9).

The invariance of R by rotation around O has the following consequence:

THEOREM 0. *If the Rolle set* $R = \{z_j; w_k\}$ *is rectilinear, then we have the equation*

$$\sum_{1}^{n-1} |w_k|^2 = \frac{n-2}{n} \sum_{1}^{n} |z_j|^2. \tag{1.10}$$

Indeed, the rectilinearity of R implies that there is an angle ϕ such that

$$z_j = \pm |z_j| e^{i\phi} \text{ for all } j, \quad \text{and} \quad w_k = \pm |w_k| e^{i\phi} \text{ for all } k.$$

Substituting these into (1.9), and canceling the factor $e^{2i\phi}$, we obtain the equation (1.10). We may therefore state

COROLLARY 1. *The equation* (1.10) *is a necessary condition for the rectilinearity of the Rolle set* $R = \{z_j; w_k\}$.

These results suggest the following

CONJECTURE 1. *For any complex Rolle set* $R = \{z_j; w_k\}$ *we have the inequality*

$$|w_1|^2 + \cdots + |w_{n-1}|^2 \leqslant \frac{n-2}{n} \left(|z_1|^2 + \cdots + |z_n|^2 \right), \tag{1.11}$$

with the equality sign if and only if R is rectilinear.

In the sequel we first establish Conjecture 1 for the special case when $n = 3$; we actually settle this case in two different ways. For $n = 3$ the two points w_1 and w_2 are evidently on a line L through O. In Theorem 3 we prove Conjecture 1 for *any* complex Rolle set such that *all* w_k points are on a line L. In spite of the simple proof of Theorem 3 we have included the more complicated direct proof of Conjecture 1 for the case $n = 3$, because its approach might possibly generalize for arbitrary n (see the generalization for all n of our Theorem 1 to Theorem (2, 3) in Morris Marden's book [**2**, p. 11]. We also establish Conjecture 1 for the special case when $P_n(z)$ is a *binomial polynomial.*

Fred Sauer, of the MRC Computing Staff, has verified the inequality (1.11) for some 25 numerically given complex $P_n(z)$, for $n = 4$ and $n = 5$. For this help I am much obliged to Fred. I am also impressed by the speed and precision of the Jenkins-Traub algorithm used in solving the equations $P_n(z) = 0$ and $P_n'(z) = 0$.

2. A first proof of Conjecture 1 for $n = 3$. Let

$$T = (z_1, z_2, z_3) \tag{2.1}$$

be a non-degenerate triangle in the complex plane having the zeros of $P_3(z)$ as vertices. By (1.2) the centroid of T is in the origin O. We shall use the following theorem of van den Berg ([**1**], or [**3**, Chapter 7]).

THEOREM 1 (van den Berg). *Let E be the Steiner ellipse of the triangle. This is the ellipse which is inscribed in T such that E is tangent to the sides of T in their midpoints. Then the zeros of $P_3'(z)$ are identical with the focii w_1 and w_2 of the ellipse E* (see Fig. 1).

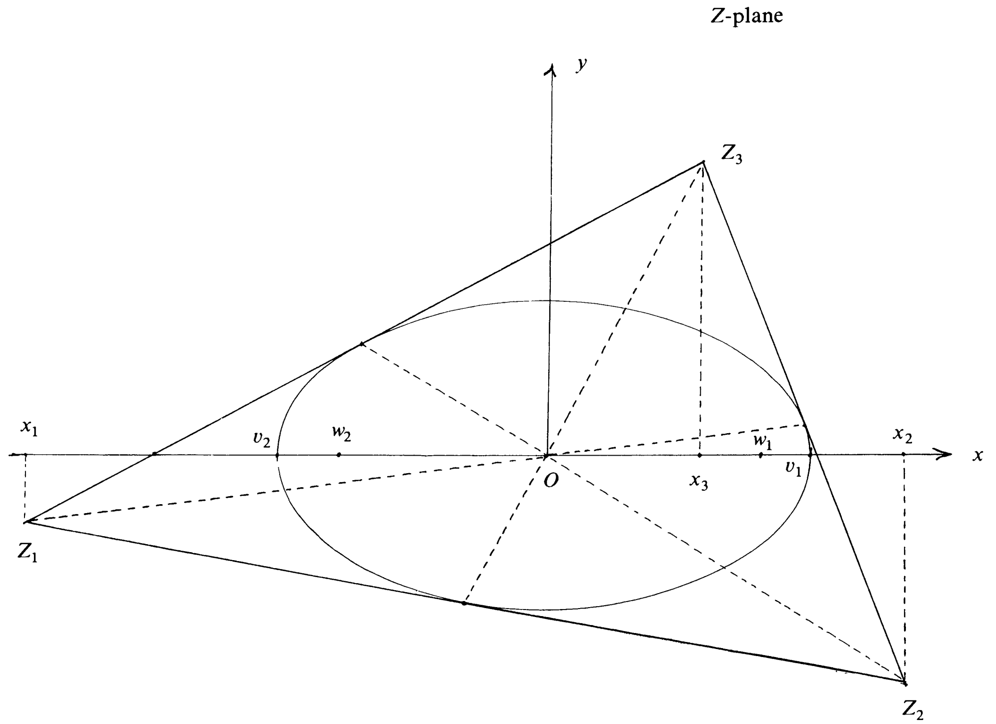

FIG. 1

With $z = x + iy$ and $z_j = x_j + iy_j$ we place $T \cup E$ so that the major axis v_1v_2 of E is on the real axis, its center being at the origin. Let a and b be the semi-axes of E and $a^2 - b^2 = c^2 = w_1^2 = w_2^2$. We now subject the plane to the affine transformation

$$A_t: \begin{array}{l} x(t) = x \\ y(t) = yt, \quad (0 \leqslant t \leqslant 1), \end{array}$$

which contracts E toward the x-axis. As the semi-axes of the new ellipse $E(t) = A_tE$ are a and bt, we find that the foci of $E(t)$, which we denote by $w_1(t)$ and $w_2(t)$, have the abscissae

$$w_1(t) = \sqrt{a^2 - b^2t^2} \quad \text{and} \quad w_2(t) = -\sqrt{a^2 - b^2t^2}\,.$$

As $t \to 0+$ we see

1. That the foci of $E(t)$ converge to the endpoints v_1 and v_2 of the major axis of E.
2. That $\lim z_j = x_j$ $(j = 1, 2, 3)$.

By continuity we conclude that $(x_1, x_2, x_3, v_1, v_2)$ is a complex Rolle set composed of five real points, so that by (1.9) we have $v_1^2 + v_2^2 = (1/3)(x_1^2 + x_2^2 + x_3^2)$ and therefore

$$\frac{1}{3}\sum_1^3 |z_j|^2 \geqslant \frac{1}{3}\sum_1^3 x_j^2 = v_1^2 + v_2^2 \geqslant |w_1|^2 + |w_2|^2. \tag{2.2}$$

A comparison of the extreme members proves the inequality (1.11).

Moreover, if the extreme members of (2.2) are equal, then we must have $\Sigma_1^3 |z_j|^2 = \Sigma_1^3 x_j^2$. This clearly implies that $|z_j|^2 = x_j^2$ for all j, and therefore that $z_j = x_j$ for all j. We have therefore established

THEOREM 2. *Conjecture* 1 *holds for* $n = 3$.

3. A generalization. We shall now generalize Theorem 2 in a certain direction and at the same time simplify its proof, as Theorem 1 will not be used. However, we add the new assumption that

(3.1) *All w_j are on a line L through O.*

We may as well assume that all w_j are real. This allows us to state

THEOREM 3. *If all w_j are real, then we have the inequality*

$$\sum_1^{n-1} w_k^2 \leqslant \frac{n-2}{n}\sum_1^n |z_j|^2, \tag{3.2}$$

with the equality sign if and only if all z_j are real.

Proof. By (1.9) we have

$$\sum_1^{n-1} w_k^2 = \frac{n-2}{n}\sum_1^n z_j^2,$$

and therefore

$$\sum_1^{n-1} w_k^2 = \frac{n-2}{n}\left|\sum_1^n z_j^2\right| \leqslant \frac{n-2}{n}\sum_1^n |z_j|^2. \tag{3.3}$$

This proves (3.2).

If we have the equality sign in (3.2), then it surely holds also in (3.3), and this implies that

$$\left|\sum_1^n z_j^2\right| = \sum_1^n |z_j|^2.$$

This last equation implies that all n complex numbers z_j^2 have the same argument. But this means that there is an angle θ so that $z_j^2 = |z_j|^2 e^{i\theta}$. This, however, implies that

$$z_j = \pm |z_j| e^{i\theta/2} \quad \text{for all } j,$$

and it follows that all points z_j are on a line L through O.

By the Gauss-Lucas theorem the convex hull of the z_j must contain the convex hull of the w_k. As the w_k are all real by assumption, it follows that we must have $\theta = 0$, hence all z_j are real. This completes our proof of Theorem 3.

4. Verifying Conjecture 1 for binomial polynomials. We say that our polynomial is *binomial* provided that it is of the form $P_n(z) = z^n + a_k z^{n-k}$, where we may as well assume that $a_k = 1$; hence

$$P_n(z) = z^n + z^{n-k} = z^{n-k}(z^k + 1), \quad (2 \leqslant k \leqslant n), \tag{4.1}$$

and

$$P_n'(z) = nz^{n-1} + (n-k)z^{n-k-1} = nz^{n-k-1}\left(z^k + \frac{n-k}{n}\right). \tag{4.2}$$

Denoting, as before, their zeros by z_j and w_k, we now find that

$$\sum_1^{n-1} |w_2|^2 = k\left(\frac{n-k}{n}\right)^{2/k} \quad \text{and} \quad \sum_1^n |z_j|^2 = k,$$

so that (1.11) amounts to the inequality

$$k\left(\frac{n-k}{n}\right)^{2/k} \leqslant \frac{n-2}{n}k.$$

This being evident if $k = 2$ or if $k = n$, there remains to prove

LEMMA 1. *We have the inequality*

$$(n-2)^k > n^{k-2}(n-k)^2 \quad \text{if} \quad 2 < k < n. \tag{4.3}$$

This we derive from the more general

LEMMA 2. *If the reals $x_1, \dots, x_k$, not all equal to each other, have the arithmetic mean*

$$a = \frac{1}{k}\sum_1^k x_j, \tag{4.4}$$

then

$$(x-a)^k > \prod_1^k (x - x_j) \quad \text{if} \quad x > x_j \ (j = 1, \dots, k). \tag{4.5}$$

Proof of Lemma 2. Taking logarithms, we see that (4.5) is equivalent to

$$\log(x-a) > \frac{1}{k}\sum_1^k \log(x - x_j) \quad \text{if} \quad x > \max x_j. \tag{4.6}$$

From (4.4) we have

$$x - a = \frac{1}{k}\sum_1^k (x - x_j)$$

and (4.6) amounts to

$$\log\left(\frac{1}{k}\sum_1^k (x - x_j)\right) > \frac{1}{k}\sum_1^k \log(x - x_j), \tag{4.7}$$

which follows from the *strict concavity* of $\log x$ in $0 < x < \infty$. Indeed, for any *strictly concave* function $f(x)$ in $(0, \infty)$, and for positive quantities $p_j = x - x_j$, not all equal to each other, we have the well-known inequality

$$f\left(\frac{1}{k}\sum_1^k p_j\right) > \frac{1}{k}\sum_1^k f(p_j).$$

For $f(x) = \log x$ this amounts to (4.7).

We now specialize Lemma 2 by choosing

$$x_1 = x_2 = \cdots = x_{k-2} = 0, \qquad x_{k-1} = x_k = k, \quad \text{and} \quad x = n.$$

For the mean value (4.4) we find

$$a = \frac{1}{k}\sum_1^k x_j = \frac{1}{k}\cdot 2k = 2,$$

while

$$\prod_{1}^{k-2}(x-x_j)=n^{k-2},\qquad (x-x_{k-1})(x-x_k)=(n-k)^2.$$

Now (4.5) goes over into the desired inequality (4.3).

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THE INTERVAL OF CONVERGENCE AND LIMITING FUNCTIONS OF A HYPERPOWER SEQUENCE

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1. Introduction. In this article we study the convergence of the infinitely iterated exponential

$$x=h(z)=z^{z^{z^{\cdot^{\cdot^{\cdot}}}}}\tag{1.1}$$

for real positive z. We prove that $h(z)$ exists for $z\in[e^{-e},e^{1/e}]$ by showing that $x=h(z)$ is the unique solution of the *first auxiliary equation*

$$f(z,x)\equiv x-z^x=0\tag{1.2}$$

for $(z,x)\in(1,e^{1/e}]\times[0,e]$, and also the unique solution of the *second auxiliary equation*

$$g(z,x)\equiv x-z^{z^x}=0\tag{1.3}$$

for $(z,x)\in[e^{-e},1)\times[0,\infty)$. In addition we prove that the sequence of hyperpowers implied by the right hand side of (1.1) diverges for $z\notin[e^{-e},e^{1/e}]$. For $(z,x)\in(0,e^{-e})\times[0,\infty)$ it turns out that the second auxiliary equation $g(z,x)=0$ has exactly three solutions $s_1(z)$, $s(z)$ and $s_2(z)$ satisfying

$$0<s_1(z)<s(z)<s_2(z)<1,$$

where $x=s(z)$ also solves $f(z,x)=0$, and we show here that the subsequences of odd and even hyperpowers converge to $s_1(z)$ and $s_2(z)$, respectively, hence the divergence of the original hyperpower sequence. For $z\in(e^{1/e},\infty)$ we prove that divergence is a direct consequence of the non-existence of solutions of the first auxiliary equation $f(z,x)=0$.

The same interval of convergence was established in an article by Knoebel [**2**], where the implicit function theorem was also employed, but where a proof by contradiction was used to prove the divergence of the hyperpower sequence for $z\in(0,e^{-e})$. Our more direct approach leads to estimates that are sharper than some of those obtained there, and yields in addition some interesting properties of the limiting functions of the hyperpower sequence (1.1).

J. M. de Villiers: I obtained a Ph.D. degree at Cambridge University in 1974 and was then appointed to the Department of Applied Mathematics at the University of Stellenbosch (about 50 kilometers from Cape Town), where I now hold the position of associate professor. The courses I teach, as well as my research interests, are mainly in the fields of applied functional analysis, differential equations and numerical analysis. The second author, P. N. Robinson, was a graduate student under my supervision at this department. For the last eight years I have been, in addition to my mathematical teaching duties, full-time conductor of the Stellenbosch University Choir. At the end of 1984 I laid down my post as choral conductor in order to devote more time to mathematical teaching and research.