Truth-table degrees and the Boone groups

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In [2, 3] Boone proved that for every recursively enumerable Turing degree of unsolvability D, there exists a finitely presented group G_D whose word problem is of degree D. While primarily concerned with Turing degrees, Boone remarks in two footnotes ([2, p. 523, note 8] and [3, p. 50, note 4]) that at all times the reducibility involved in his argument is truth-table reducibility and hence that the theorem remains true for truth-table degrees. Later, however, he withdrew this assertion and speculated that it might even be false (see J. C. Shepherdson's review [8]). We show here that Boone's claim is in fact correct, but that the verification requires argument additional to that given in [2, 3]. (See Boone's corrective note [4] for an indication of where his argument is lacking.) In this connection we should also mention the work of A. A. Fridman on the same theorem. In this original announcement [6], Fridman claimed that the theorem was true for truth-table degrees but did not repeat this claim in the complete version [7] of his work and indeed his argument breaks down in much the same way as does Boone's. However, as will be apparent, we shall make use of some of Fridman's ideas.

We shall assume throughout that the reader is familiar with [2, 3], in particular with the notation used there. We begin by considering the situation in [3] and then examine [2] later.

LEMMA 1. *$(?\,W)\, W =_G 1$ is truth-table reducible to $(?\,V$ k-free$)(\exists D)\, V =_{G_1} D$.*

Proof. Let W be a word of G. If W is $W_1 k^{-\varepsilon} V k^{\varepsilon} W_2$, where $\varepsilon = \pm 1$ and V is k-free, and $(\exists D)\, V =_{G_1} D$, then we call $W_1 V W_2$ a primitive k-reduction of W of standard type. Also we write $\pi_k[W]$ ($\delta_k[W]$) for the word obtained from W by deleting all symbols of W except k-symbols (by deleting all k-symbols). Then it follows that

(*) *$W =_G 1$ if and only if $\pi_k[W] = 1$ in $F(k)$, the free group on k, and there exists a sequence of primitive k-reductions of standard type which terminates in $\delta_k[W]$, and $\delta_k[W] =_{G_1} 1$.*

To show that the reduction is by truth-tables, we must show that there exists a recursive procedure to compute for each word W of G an m-tuple $(V_1, V_2, \cdots, V_m)$, where m depends on W, and a truth-table with m "question columns" and an "answer column" from which it may be determined whether or not $W =_G 1$ when the answers to the questions "does there exist D such that $V_i = D$ in G_1?" ($i = 1, 2, \cdots, m$) are given. We denote $(V_1, V_2, \cdots, V_m)$ by par(W) and the truth-table by $tt(W)$. It is convenient, at this point, to note that in prescribing how to obtain the truth-tables for words of G, we may give a case by case description provided that there is a recursive procedure to determine which case holds.

If either $\pi_k[W] \neq 1$ in $F(k)$ or $\delta_k[W] \neq_{G_1} 1$, then by (*) $W \neq_G 1$. In this case we may take par(W) to be arbitrary and $tt(W)$ to have "No" in every entry in the answer column.

Now suppose that $\pi_k[W] = 1$ in $F(k)$ and $\delta_k[W] =_{G_1} 1$. For brevity we shall consider a specific example and then give a general description rather than give a detailed proof by induction. So let W be

$$U_1 k^{-1} V_1 k V_2 k^{-1} V_3 k V_4 k^{-1} V_5 k U_2$$

and suppose that $U_1 V_1 V_2 V_3 V_4 V_5 U_2 =_{G_1} 1$. Then it follows from (*) that $W =_G 1$ if and only if

either (i) $(\exists D)\, V_1 =_{G_1} D$, $(\exists D)\, V_3 =_{G_1} D$, and $(\exists D)\, V_5 =_{G_1} D$

or (ii) $(\exists D)\, V_2 =_{G_1} D$, $(\exists D)\, V_1 V_2 V_3 =_{G_1} D$, and $(\exists D)\, V_5 =_{G_1} D$

or (iii) $(\exists D)\, V_4 =_{G_1} D$, $(\exists D)\, V_3 V_4 V_5 =_{G_1} D$, and $(\exists D)\, V_1 =_{G_1} D$

or (iv) $(\exists D)\, V_2 =_{G_1} D$, $(\exists D)\, V_4 =_{G_1} D$, and $(\exists D)\, V_1 V_2 V_3 V_4 V_5 =_{G_1} D$

or (v) $(\exists D)\, V_3 =_{G_1} D$, $(\exists D)\, V_2 V_3 V_4 =_{G_1} D$, and $(\exists D)\, V_1 V_2 V_3 V_4 V_5 =_{G_1} D$.

In this case par(W) is the 15-tuple

$$(V_1,\ V_3,\ V_5,\ V_2,\ V_1 V_2 V_3,\ V_5,\ \cdots,\ V_3,\ V_2 V_3 V_4,\ V_1 V_2 V_3 V_4 V_5)$$

which we think of as being composed of five triples or blocks where the decomposition is the obvious one relative to (i)–(v). Then a line of the truth-table contains "Yes" in the answer column if and only if there exists a block for which all three answers are "Yes".

It should be clear to the reader that an analogous procedure will give an appropriate m-tuple and truth-table for an arbitrary W. The essential point is that to each possible sequence of primitive k-reductions of standard type there must correspond a block of questions and that the entry in a line of the answer column is "Yes" if and only if there is a block of questions to which all the answers are "Yes".

We make use of the following lemma due originally to Fridman and later elegantly proved by L. A. Bokut' [1].

LEMMA. *(Fridman-Bokut')* $(?P)(\exists L)(\exists R)(\exists \Sigma) P =_{G_2} L\Sigma R$ *is recursively solvable.*

LEMMA 2. *$(?V$ k-free$)(\exists D)V =_{G_1} D$ is truth-table reducible to $(?\Sigma)\Sigma =_T q$.*

Proof. Either by using an enumeration of all provable equations in G_2 or else directly from the proof of the Fridman-Bokut' Lemma, we can recursively compute, for any P such that $(\exists L)(\exists R)(\exists\Sigma)P =_{G_2} L\Sigma R$, words L_P, R_P, and Σ_P such that $P =_{G_2} L_P\Sigma_P R_P$. This enables us to compute values for $\mathrm{par}(V)$ and $tt(V)$, for arbitrary V, as follows.

If V is t-free, then $(\exists D)V =_{G_1} D$ if and only if $(\exists R)V =_{G_2} R$. Since there is an algorithm to determine whether or not the latter holds (the existence of such an algorithm follows from Theorems V, XII, XIII and XVI and 0 of [3]), we can define $\mathrm{par}(V)$ to be arbitrary and $tt(V)$ to have all answers "Yes" or all answers "No" according as $(\exists R)V =_{G_2} R$ or not.

Now suppose that we have given a procedure to deal with all words V with fewer than n t-symbols. Let V be $Ut^\varepsilon P$ where $\varepsilon = \pm 1$ and U has $n-1$ t-symbols. Then $(\exists D)V =_{G_1} D$ if and only if (i) L_P, R_P, and Σ_P are defined, (ii) $(\exists D)UL_P\Sigma_P =_{G_1} D$, and (iii) $\Sigma_P =_T q$. Thus we define $\mathrm{par}(V)$ and $tt(V)$ as follows. Determine, recursively, if L_P, R_P, and Σ_P exist and compute them if they do. If they do not exist, then $\mathrm{par}(V)$ is arbitrary and $tt(V)$ has "No" in every entry in the answer column. Otherwise $\mathrm{par}(V) = (\Sigma_P, \mathrm{par}(UL_P\Sigma_P))$ and $tt(V)$ is derived from the diagram below.

Yes ⋮ Yes	$tt(UL_P\Sigma_P)$
No ⋮ No	$tt(UL_P\Sigma_P)$

This consists of two copies of $tt(UL_P\Sigma_P)$, one below the other, with an additional column to the left (for Σ_P) with "Yes" entries next to the upper copy of $tt(UL_P\Sigma_P)$ and "No" entries next to the lower. Then $tt(V)$ is obtained by making every entry in the answer column of the lower copy a "No". It follows from the equivalence involving (i), (ii), and (iii) that this is correct.

THEOREM. *$(?W)W =_G 1$ is truth-table equivalent to $(?\Sigma)\Sigma =_T q$.*

Proof. Since $\Sigma =_T q$ if and only if $k\Sigma^{-1}t\Sigma =_G \Sigma^{-1}t\Sigma k$, it is clear that $(?\Sigma)\Sigma =_T q$ is truth-table reducible to $(?W)W =_G 1$. The converse follows from Lemmas 1 and 2.

To complete the argument, we must verify that for any recursively enumerable truth-table degree D there exists a Thue system T, of the required form, such that $(?\Sigma)\Sigma =_T q$ has degree D. We therefore examine Boone's argument in [2]. In particular we look at §§ 8, 6, and 9 in that order.

By [2, Lemma 1], we may regard D as the degree of the problem $(?n)hs_1^{n+1}q_1h \vdash_1 hqh$. We require, firstly, the following truth-table version of Equivalence Theorem 1′ [2, p. 568].

LEMMA 3. *$(?\mathbf{W}, \mathbf{W}$ on $\mathfrak{Z}_3)\mathbf{W} \vdash_3 hqh$ is truth-table equivalent to $(?n)hs_1^{n+1}q_1h \vdash_1 hqh$.*

Proof. The argument given in § 8 of [2], taken exactly as it stands, shows that the above two problems are truth-table equivalent. This must be checked in detail but the verification is straightforward and we therefore leave it to the reader.

We also require the truth-table version of Theorem XIII [2, p. 566] but the procedure here is less clearcut.

LEMMA 4. $(?\mathbf{W}, \mathbf{W}$ on $\tilde{\mathfrak{Z}}_{[\mathbf{P}]})\mathbf{W} \vdash_{\tilde{\mathfrak{T}}_{[\mathbf{P}]}} 1$ is truth-table equivalent to $(?\mathbf{A}, \mathbf{A}$ on $\tilde{\mathfrak{Z}})\mathbf{A} \vdash_{\tilde{\mathfrak{T}}} \mathbf{P}$.

Proof. By [2, Lemma 39], $(?\mathbf{A}, \mathbf{A}$ on $\tilde{\mathfrak{Z}})\mathbf{A} \vdash_{\tilde{\mathfrak{T}}} \mathbf{P}$ is truth-table reducible to $(?\mathbf{W}, \mathbf{W}$ on $\tilde{\mathfrak{Z}}_{[\mathbf{P}]})\mathbf{W} \vdash_{\tilde{\mathfrak{T}}_{[\mathbf{P}]}} 1$.

The converse, however, is not so straightforward. Just as the p-reduction procedure of [3], which is derived from Britton's Lemma [5, Lemma 4], can cause the type of difficulty which we resolved in Lemma 1, so [2, Lemma 40] gives a reduction procedure which requires special care. If W is a word on $\tilde{\mathfrak{Z}}_{[\mathbf{P}]}$, let $\pi_{cd}[\mathbf{W}]$ be the word obtained from $\mathbf{W}$ by deleting all symbols except c and d. Also let $\mathfrak{T}_{cd} = (c, d; cd \leftrightarrow 1)$. Then it is not hard to show that if $\mathbf{W} =_{\tilde{\mathfrak{T}}_{[\mathbf{P}]}} 1$, then $\pi_{cd}[\mathbf{W}] =_{\mathfrak{T}_{cd}} 1$. There is an obvious recursive procedure to determine for any $\mathbf{W}$ whether or not $\pi_{cd}[\mathbf{W}] =_{\mathfrak{T}_{cd}} 1$ so it suffices to consider words $\mathbf{W}$ such that $\pi_{cd}[\mathbf{W}] =_{\mathfrak{T}_{cd}} 1$.

If $\mathbf{W}$ is c-free, then $\mathbf{W}$ must also be d-free and $\mathbf{W} \vdash_{\tilde{\mathfrak{T}}_{[\mathbf{P}]}} 1$ if and only if $\mathbf{W}$ is 1. A truth-table can certainly to be constructed for this case. So suppose that $\mathbf{W}$ contains c-symbols. From Lemma 40 of [2], $\mathbf{W} \vdash 1$ in $\tilde{\mathfrak{T}}[\mathbf{P}]$ if and only if there is a sequence

$$(*) \qquad \mathbf{W} \equiv \mathbf{A}_1c\mathbf{B}_1d\mathbf{C}_1 \rightarrow \mathbf{A}_1\mathbf{C}_1 \equiv \mathbf{A}_2c\mathbf{B}_2d\mathbf{C}_2 \rightarrow \cdots \rightarrow 1\ ,$$

where $\mathbf{B}_j \vdash \mathbf{P}$ in $\tilde{\mathbf{T}}$. We employ what is essentially the argument of Lemma 1. Given $\mathbf{W}$ we can, on the basis of the $\{c, d\}$-structure of $\mathbf{W}$, write down all such theoretically possible sequences $(*)$ — let these be $\mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_q$. Each

$\mathcal{R}_i$ will have the same number p of steps and $\mathcal{R}_i$ will be a genuine sequence if and only if $\mathbf{B}_{ij} \vdash \mathbf{P}$ in $\tilde{\mathfrak{T}}$, $j = 1, 2, \cdots, p$. Then

$$\mathrm{par}(\mathbf{W}) = (\mathbf{B}_{11}, \cdots, \mathbf{B}_{1p}, \cdots, \mathbf{B}_{q1}, \cdots, \mathbf{B}_{qp})$$

and the truth-table is constructed in the obvious way.

We can now establish that the equivalences in Boone's arguments can all be established by truth-tables. Given any r.e. truth-table degree D, we can, by Lemma 3, construct $\mathfrak{T}_3$ so that the word problem for $\mathfrak{T}_3$ relative to hqh has (truth-table) degree D and hence, by Lemma 4, $\mathfrak{T}''$ can be constructed such that its word problem relative to 1 has degree D. It follows easily from Theorem XIV of [2] that we can construct $\mathfrak{T}$ such that $(?\Sigma)\Sigma =_T q$ has degree D whence, by the Theorem, a finitely presented group can be constructed with word problem of degree D.

One final point should be noted. We have proved that the equivalences in Boone's arguments can all be established by truth-tables. However, our argument employs arbitrarily large truth-tables and it seems unlikely that it can be modified or replaced in such a way as to show that all these equivalences can be established by bounded truth-tables. There seems, therefore, to be no obvious answer to the question of whether or not the main theorem remains true for bounded truth-table degrees.

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REFERENCES

[1] L. A. BOKUT', *On a property of groups of Boone* (Russian), Algebra i Logika Sem. **5** (1966), No. 5, 5-23; **6** (1967), No. 1, 15-24.

[2] W. W. BOONE, *Word problems and recursively enumerable degrees of unsolvability. A first paper on Thue systems*, Ann. Math. **83** (1966), 520-571.

[3] ———, *Word problems and recursively enumerable degrees of unsolvability. A sequel on finitely presented groups*, Ann. Math. **84** (1966), 49-84.

[4] ———, *Word problems and recursively enumerable degrees of unsolvability. An emendation*, Ann. Math. **94** (1971),

[5] J. L. BRITTON, *The word problem*, Ann. Math. **77** (1963), 16-32.

[6] A. A. FRIDMAN, *Degrees of unsolvability of the problem of identity in finitely presentep groups*, Soviet Math. **3** (1962), Part 2, 1733-1737.

[7] ———, Degrees of unsolvability of the word problem for finitely defined groups (Russian), Izdatel'stvo "Nauka", Moscow 1967, 193 pp.

[8] J. C. SHEPHERDSON, *Review of* [2], [3] *and* [6], J. Symb. Logic **33** (1968), 296-297.

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