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Interaction corrections at intermediate temperatures: Longitudinal conductivity and kinetic equation

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It is well known that electron-electron interaction in two-dimensional disordered systems leads to logarithmically divergent Altshuler-Aronov corrections to conductivity at low temperatures ($T\tau \ll 1$; τ is the elastic mean-free time). This paper is devoted to the fate of such corrections at intermediate temperatures $T\tau \gtrsim 1$. We show that in this (ballistic) regime the temperature dependence of conductivity is still governed by the same physical processes as the Altshuler-Aronov corrections—electron scattering by Friedel oscillations. However, in this regime the correction is linear in temperature; the value and even the *sign* of the slope depends on the strength of electron-electron interaction. (This sign change may be relevant for the ''metal-insulator'' transition observed recently.) We show that the slope is directly related to the renormalization of the spin susceptibility and grows as the system approaches the ferromagnetic Stoner instability. Also, we obtain the temperature dependence of the conductivity in the cross-over region between the diffusive and ballistic regimes. Finally, we derive the quantum kinetic equation, which describes electron transport for arbitrary value of $T\tau$.

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I. INTRODUCTION

Temperature-dependent corrections to conductivity due to electron-electron interactions has been a subject of theoretical[1–7] and experimental[8–10] studies for more than two decades. Recently the interest in the matter was renewed with appearance of new data[11,12] showing a sign change in the temperature dependence of conductivity in two dimensions (2D). Theoretical discussions[12] that followed emphasized the question of whether that data indicated a non-Fermi-liquid behavior. However, the experiments were performed in a regime where the temperature T was of the same order of magnitude as the inverse scattering time τ^{-1} (obtained from the Drude conductivity), while preexisting calculations were focused on the two limiting cases: the diffusive regime[1–3] $T\tau \ll 1$, and the ballistic regime[6,7] $T\tau \gg 1$.

In the diffusive limit one finds[1–3] for the logarithmically divergent correction to the diagonal conductivity $\delta\sigma$:

$$\delta\sigma = -\frac{e^2}{2\pi^2\hbar}\ln\left(\frac{\hbar}{T\tau}\right)\left[1+3\left(1-\frac{\ln(1+F_0^\sigma)}{F_0^\sigma}\right)\right], \quad (1.1)$$

where F_0^σ is the interaction constant in the triplet channel which depends on the interaction strength. It is clear, that the sign of this logarithmically divergent correction may be positive (metallic) or negative (insulating), depending on the value of F_0^σ.[13]

The result[6,7] for the ballistic region frequently cited in literature reads

$$\delta\sigma = -\frac{e^2}{\pi\hbar}\left(\frac{T\tau}{\hbar}\right) f(r_s), \quad (1.2)$$

where $f(r_s)$ is a positive function of the gas parameter of the system r_s. In a sharp contrast to Eq. (1.1), Eq. (1.2) predicts always metallic sign of the interaction correction.

The absence of a rigorous calculation at intermediate temperatures $T\tau/\hbar \simeq 1$ may have contributed to the notion that those two limits are governed by different physical processes. In this paper we prove that notion erroneous: the results (1.1) and (1.2) are due to the same physical process, namely, elastic scattering of electrons by the self-consistent potential created by all the other electrons. Therefore, these two different expressions are in fact the two limits of a single interaction correction. We calculate the correction within assumptions of the Fermi-liquid theory (other limitations of our approach we discuss below) and present the cross-over function between the diffusive and ballistic limits.

Moreover, we show that the existing theory for the ballistic limit (1.2) is incomplete. First, the results of Refs. 4–7 account only for Hartree-like interaction terms missing the exchange or Fock terms. Second, this theory essentially employs a perturbative expansion in terms of the interaction strength, which breaks down for stronger coupling. Both issues lead to the change in the theoretical prediction even on a qualitative level.

The consequence of the first point is that the correction to conductivity (1.2) is always negative unlike the correction in the diffusive limit that changes sign depending on the value of F_0^σ. This sign change is due to competition between the universal (and positive) Fock correction and the coupling-specific (and negative) Hartree contribution. If the Fock potential (or, to be more precise, singlet channel) is properly taken into account, then the sign of the correction in the ballistic limit is also not universal (being positive for weak interaction in contrast to Refs. 4–7).

As follows from the second point, for the stronger interaction the Hartree correction should be modified to include higher order processes. For this case we show (see Sec. III) that in fact it should be replaced by the triplet channel correction, which is characterized by the Fermi liquid constant F_0^σ. This constant measures the strength of the spin-exchange interaction. If $F_0^\sigma < 0$, the spin-exchange interaction tends to align electron spins and (if it is strong enough) leads to the ferromagnetic Stoner instability.[13] Even though this constant is unknown, it can be found experimentally by means of independent measurement of the spin susceptibility of the system. As a function of temperature the interaction correction to conductivity is almost always monotonous, except for a narrow region of parameters (where it is so small that it can hardly be observed).

The remainder of the paper is organized as follows. The following section is devoted to qualitative discussion of the physics involved. In the same section we summarize our results. Then we present two alternative approaches to the microscopic calculation. In Sec. III we use the traditional perturbation theory[1] to derive the results presented in Sec. II, while in Sec. IV the same results are obtained using the quantum kinetic equation that we derive. The advantage of the kinetic equation approach is that it can be readily used to discuss the temperature behavior of quantities other than conductivity. These results are advertised in Conclusions and will be published elsewhere.[14]

II. QUALITATIVE DISCUSSION AND RESULTS

In this section we describe the scattering processes contributing to the temperature dependence of conductivity. We show that unlike the standard Fermi-liquid T^2 corrections, the leading correction to conductivity is accumulated at large distances, of the order $v_F/\min(T,\sqrt{T/\tau})$. In the ballistic limit such correction is linear in temperature and we derive this result here using a text-book quantum mechanical approach. The diffusive limit is discussed in detail in Ref. 1. The resulting correction $\delta\sigma \sim \ln T$ seems to be rather different from the linear one, but we show that both corrections arise due to the same physics—coherent scattering by Friedel oscillations. Throughout the paper we keep the units such that $\hbar = 1$, except for the final answers.

A. Scattering by Friedel oscillations

We start with the simplest case of a weak short-range interaction $V_0(\vec{r}_1-\vec{r}_2)$ and show how one can obtain the correction to conductivity in the ballistic limit, i.e., due to a single scatterer. This discussion is similar to that of Ref. 15, where the correction to the one-particle density of states (DOS) was discussed, and also of Ref. 16, which describes the correction to the conductivity in the diffusive limit.

Consider a single impurity localized at some point, taken as the origin. The impurity potential $U(\vec{r})$ induces a modulation of electron density close to the impurity. The oscillating part of the modulation is known as the Friedel oscillation, which in 2D can be written as

$$\delta\rho(\vec{r}) = -\frac{\nu\lambda}{2\pi r^2}\sin(2k_F r). \tag{2.1}$$

Here r denotes the distance to the impurity and its potential is treated in the Born approximation $\lambda = \int U(\vec{r})d\vec{r}$. In 2D the free electron DOS is given by $\nu = m/\pi\hbar^2$ and m is the electron mass, k_F is the Fermi momentum.

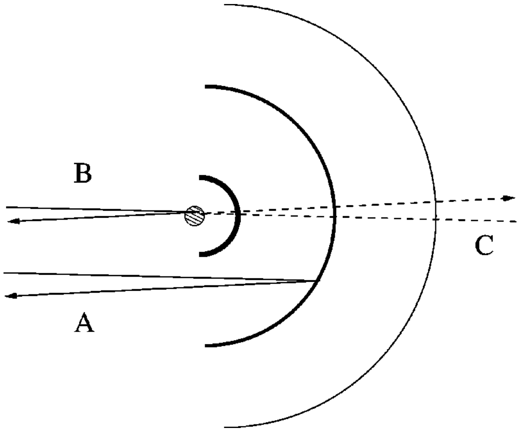

FIG. 1. Scattering by the Friedel oscillation. Interference between the two paths A and B contributes mostly to backscattering. The Friedel oscillation is created due to backscattering on the impurity, path C.

Taking into account electron-electron interaction $V_0(\vec{r}_1-\vec{r}_2)$ one finds additional scattering potential due to the Friedel oscillation Eq. (2.1). This potential can be presented as a sum of the direct (Hartree) and exchange (Fock) terms[17]

$$\delta V(\vec{r}_1,\vec{r}_2) = V_H(\vec{r}_1)\,\delta(\vec{r}_1-\vec{r}_2) - V_F(\vec{r}_1,\vec{r}_2), \tag{2.2a}$$

$$V_H(\vec{r}_1) = \int d\vec{r}_3 V_0(\vec{r}_1-\vec{r}_3)\,\delta\rho(\vec{r}_3), \tag{2.2b}$$

$$V_F(\vec{r}_1,\vec{r}_2) = \frac{1}{2}V_0(\vec{r}_1-\vec{r}_2)\,\delta n(\vec{r}_1,\vec{r}_2), \tag{2.2c}$$

where by $\rho(\vec{r})$ we denote diagonal elements of the one electron density matrix n,

$$n(\vec{r}_1,\vec{r}_2) = \sum_k \Psi_k^*(\vec{r}_1)\Psi_k(\vec{r}_2). \tag{2.3}$$

The factor 1/2 indicates that only electrons with the same spin participate in exchange interaction. As a function of distance from the impurity the Hartree-Fock energy δV oscillates similarly to Eq. (2.1).

The leading correction to conductivity is a result of interference between two semi-classical paths depicted on Fig. 1. If an electron follows path "A," it scatters off the Friedel oscillation created by the impurity and path "B" corresponds to scattering by the impurity itself. Interference is most important for scattering angles close to π (or for backscattering), since the extra phase factor accumulated by the electron on path "A" (e^{i2kR} with R being the length of the extra path interval relative to "B" and $2k$ being the difference between initial and final momenta for that extra path interval) is canceled by the phase of the Friedel oscillation $e^{-i2k_F R}$ so that the amplitudes corresponding to the two paths are coherent. As a result, the probability of backscattering is greater than the classical expectation (taken into account in the Drude conductivity). Therefore, taking into account interference effects leads to a correction to conductivity. We note that the interference persists to large distances, limited only by temperature $R \approx 1/|k-k_F| \leqslant v_F/T$. Thus there is a possibility for the correction to have a nontrivial temperature dependence. The sign of the correction depends on the sign of the coupling constant that describes electron-electron interaction.

To put the above argument onto more rigorous footing and to find the temperature dependence of the correction consider now a scattering problem in the potential Eq. (2.2). Following the textbook approach,[18] we write a particle's wave function as a sum of the incoming plane wave and the out-coming spherical wave (in 2D it is given by a Bessel function, which we replace by its asymptotic form)

$$\Psi = e^{i\vec{k}\cdot\vec{r}} + i f(\theta)\sqrt{\frac{2\pi}{kr}}e^{ikr}. \tag{2.4}$$

Here $f(\theta)$ is the scattering amplitude, which we will discuss in the Born approximation. For the impurity potential itself the amplitude $f(\theta)$ weakly depends on the angle. At zero temperature it determines the Drude conductivity σ_D, while the leading temperature correction is proportional to T^2, as is usual for Fermi systems. We now show that this is not the case for the potential Eq. (2.2). In fact, taking into account Eq. (2.2) leads to enhanced backscattering and thus to the conductivity correction that is linear in temperature.

First, we discuss the Hartree potential Eq. (2.2b). Far from the scatterer the wave function of a particle can be found in the first order of the perturbation theory as $\Psi = e^{i\vec{k}\cdot\vec{r}} + \delta\Psi(\vec{r})$, where the correction is given by[18]

$$\delta\Psi(\vec{r}) = i\int d\vec{r}_1 V_H(\vec{r}_1) e^{i\vec{k}\cdot\vec{r}_1}\sqrt{\frac{2\pi}{k|\vec{r}-\vec{r}_1|}}e^{ik|\vec{r}-\vec{r}_1|}. \tag{2.5}$$

Here $|\vec{r}-\vec{r}_1| \approx r - \vec{r}\cdot\vec{r}_1/r$, since we are looking at large distances. Substituting the form of the potential Eq. (2.2b) and introducing the Fourier transfer of the electron-electron interaction V_0 we can rewrite Eq. (2.5) as

$$\delta\Psi(\vec{r}) = -i\frac{\nu\lambda}{\sqrt{2\pi}}V_0(q)\frac{e^{ikr}}{\sqrt{kr}}\int\frac{d\vec{r}_1}{r_1^2}\sin(2k_F r_1)e^{i\vec{q}\cdot\vec{r}_1}, \tag{2.6}$$

where

$$\vec{q} = \vec{k} - k\vec{r}/r, \qquad |q| = 2k\sin(\theta/2),$$

with θ being the angle of scattering. Comparing to Eq. (2.4) we find the scattering amplitude as a function of θ (it also depends on the electron's energy $\epsilon = k^2/2m$)

$$f(\theta) = -\frac{\nu\lambda}{2\pi}V_0(q)\int\frac{d\vec{r}}{r^2}\sin(2k_F r)e^{i\vec{q}\cdot\vec{r}}. \tag{2.7}$$

The integral can be evaluated exactly[19] and the result is given by

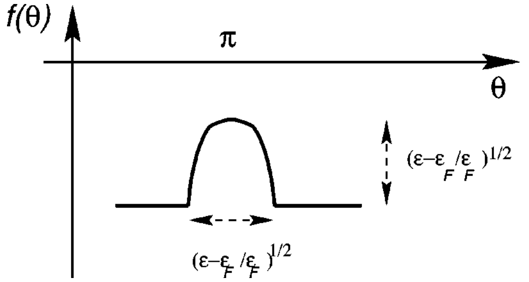

FIG. 2. Scattering amplitude. The singularity for backscattering is due to interference of paths A and B on Fig. 1.

$$f(\theta) = -\frac{\nu\lambda}{2\pi}V_0(q)\begin{cases}\dfrac{\pi}{2}, & |q| < 2k_F, \\ \arcsin\left(\dfrac{2k_F}{q}\right), & |q| > 2k_F.\end{cases} \tag{2.8}$$

Let us examine this expression more closely. Since $|q| \leqslant 2k$, the scattering amplitude Eq. (2.8) for small k weakly depends on the angle through the Fourier component of the interaction $V_0(q)$, see background value of $f(\theta)$ on Fig. 2. However, we are dealing with electronic excitations close to the Fermi surface, so in fact k is close to k_F, $|k-k_F|/k_F \ll 1$. If $k > k_F$, then the scattering amplitude Eq. (2.8) has a nontrivial angular dependence around $\theta = \pi$ shown on Fig. 2.

According to Eq. (2.8) such dependence is only possible in the region $|q| > 2k_F$. This translates into the condition $|\theta - \pi| < [2(k-k_F)/k_F]^{1/2}$, which determines the singular dependence of the width of the feature in the scattering amplitude on the energy of the scattered electron. Finally, using the fact that $\arcsin(1-x) = \pi/2 - \sqrt{2x}$, we find that the dependence of the height of the feature in the scattering amplitude is also singular: $\delta f(\theta) \simeq [(k-k_F)/k_F]^{1/2}$.

The transport scattering rate τ^{-1} is determined by the scattering cross section and can be found with the help of the amplitude Eq. (2.8), as well as the constant amplitude f_0 of the scattering by the impurity itself

$$\tau^{-1}(\epsilon) = \int\frac{d\theta}{2\pi}(1-\cos\theta)|f_0 + f(\theta)|^2. \tag{2.9}$$

The leading energy dependence of τ^{-1} comes from the interference term, which is proportional to $f(\theta)$. Then integration around $\theta = \pi$ is dominated by the feature of $f(\theta)$ resulting in a term of order $(\epsilon - \epsilon_F)/\epsilon_F$. It is this term that gives rise to the linear temperature dependence we are after. Since we are interested in this leading correction only, in all other terms we can set $k \approx k_F$ and write the scattering rate as

$$\tau^{-1}(\epsilon) = \tau_0^{-1} + \frac{\nu\lambda}{2}V_0(2k_F)\frac{\epsilon - \epsilon_F}{\epsilon_F}\,\eta(\epsilon - \epsilon_F)f_0. \tag{2.10}$$

Here $\eta(x)$ is the Heaviside step function and τ_0^{-1} is the zero-temperature rate that determines the Drude conductivity (indeed, the $\theta = \pi$ feature in $f(\theta)$ only exists for $k > k_F$ and at $T = 0$ there are no electrons with $k > k_F$).

To obtain the scattering time we have to integrate the energy-dependent rate Eq. (2.10) with the derivative of the Fermi distribution function $n_F(\epsilon)$

$$\tau=\int d\epsilon\,\tau(\epsilon)\frac{\partial}{\partial\epsilon}n_F(\epsilon).$$

Then the second term in Eq. (2.10) leads to a linear correction to the Drude conductivity, small as T/ϵ_F. However this is not the only contribution to the temperature dependence. At finite temperatures we also have to modify the Friedel oscillation Eq. (2.1) as follows:

$$\delta\rho(\vec{r})=-\frac{\nu\lambda T^2}{2\pi v_F^2\sinh^2\left(\frac{rT}{v_F}\right)}\sin(2k_Fr).$$

Consequently, the scattering amplitude Eq. (2.7) becomes temperature dependent

$$f(\theta)=-\frac{\nu\lambda}{2\pi}V_0(q)\int\frac{d\vec{r}_2}{r_2^2}e^{-2r_2(T/v_F)}\sin(2k_Fr_2)e^{i\vec{q}\cdot\vec{r}_2}$$

$$=-\frac{\nu\lambda}{2\pi}V_0(q)\arcsin\left(\frac{4k_F}{p}\right), \tag{2.11}$$

$$p=\sqrt{\left(\frac{2T}{v_F}\right)^2+(q+2k_F)^2}+\sqrt{\left(\frac{2T}{v_F}\right)^2+(q-2k_F)^2}.$$

Neglecting the small temperature dependent term in the denominator in Eq. (2.11) brings us back to Eq. (2.8). Keeping this term leads to the same feature in $f(\theta)$ as the one on Fig. 2, only now its width and magnitude are proportional to $\sqrt{T}$. The resulting correction to the conductivity is therefore similar to the one discussed above. Up to a numerical coefficient

$$\frac{\delta\sigma}{\sigma_D}=-2\nu V_0(2k_F)\frac{T}{\epsilon_F}. \tag{2.12}$$

The conductivity Eq. (2.12) is the same correction as the one calculated in Ref. 6, see also Eq. (1.2), up to a numerical factor. It is also clear that Eq. (2.12) is not the full story. We have forgotten about the Fock part of the potential Eq. (2.2) Substituting Eq. (2.4) into Eq. (2.3), we find the perturbation of the density matrix [which appears in the Fock potential Eq. (2.2c)] $\delta n(\vec{r}_1,\vec{r}_2)\approx\delta\rho[(\vec{r}_1+\vec{r}_2)/2]$. Then the argument can be repeated. The only difference is that the leading temperature correction comes from the Fourier component at $q=0$, rather than $q=2k_F$. What is most important, the Fock potential enters with the opposite sign. Therefore the expression for the conductivity Eq. (2.12) has to be corrected

$$\sigma=\sigma_D\left[1-\nu[2V_0(2k_F)-V_0(0)]\frac{T}{\epsilon_F}\right]. \tag{2.13}$$

The sign of the correction is thus not universal and depends on the details of electron-electron scattering. If the weak interaction is reasonably long ranged, then $V_0(0)\gg V_0(2k_F)$, so that the correction in Eq. (2.13) has the sign opposite to that in Ref. 6.

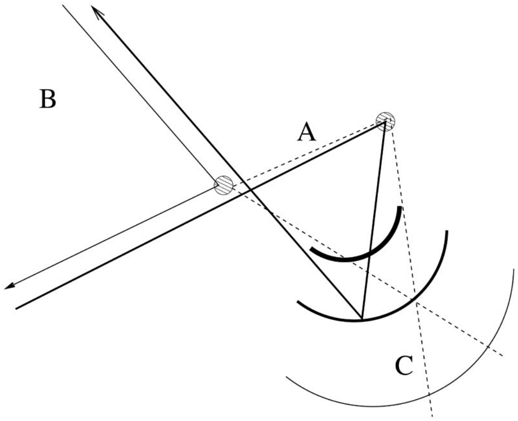

FIG. 3. Scattering process with two impurities and the Friedel oscillation. Scattering to all angles is affected by interference. The relevant Friedel oscillation is created by the self-intersecting path C.

So far we have considered the effect of a single impurity. The extension of the above arguments to the case of many impurities is straightforward. In particular, one can consider a scattering process, which involves two impurities and the Friedel oscillation shown on Fig. 3. It is clear that this process contributes to the scattering amplitude at any angle, and not just for backscattering as the single impurity process on Fig. 1 (which is typical for the diffusive motion of electrons). Such processes were discussed in detail (although using a slightly different language) in Ref. 16. Scattering by Friedel oscillations created by multiple impurities results in a conductivity correction (1.1) that is logarithmic in temperature and is typical for 2D diffusive systems.[1]

Comparing the scattering processes on Figs. 1 and 3, one can clearly see that conductivity corrections, which arise from these processes are governed by the same physics: coherent scattering by the Friedel oscillation, which means that the ballistic and diffusive regions should be analyzed on the same footing. In the next subsection we present the results of such anlysis, postponing the actual calculations until Secs. III and IV.

B. Results

Let us first consider the case of a weak, short range interaction potential. Then the interaction can be treated in the lowest order of perturbation theory, so that the resulting correction is proportional to the interaction constant:

$$\delta\sigma_w=\frac{e^2}{\hbar}\left[\gamma_1\frac{T\tau}{\pi\hbar}\left[1-\frac{3}{8}w(T\tau)\right]-\frac{\gamma_2}{4\pi^2}\ln\left(\frac{E_F}{T}\right)\right]. \tag{2.14}$$

Similarly to Eq. (2.13), it can be written as a sum of Hartree and Fock contributions (similar expression for γ_1 in one-dimensional systems was obtained in Ref. 20):

$$\gamma_1=\nu[V_0(0)-2V_0(2k_F)],\quad \gamma_2=\nu[V_0(0)-2\langle V_0(k)\rangle_{\mathrm{FS}}], \tag{2.15}$$

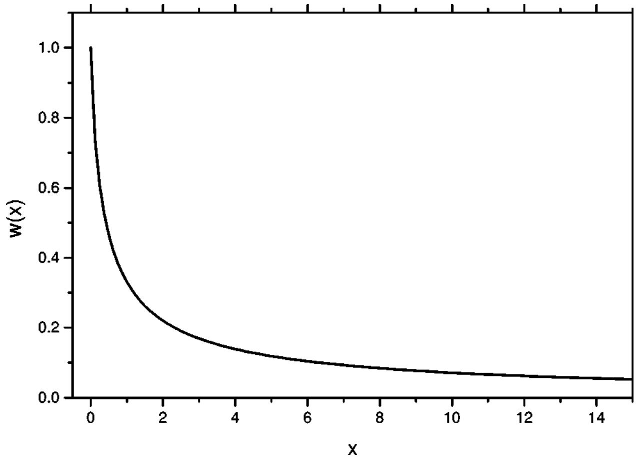

FIG. 4. Dimensionless function $w(x)$, which is defined so that $w(0)=1$.

where $\langle\cdots\rangle_{\mathrm{FS}}$ stands for the average over the Fermi surface. Here we kept the notation for the electron-electron interaction adopted in the previous section. Then the Hartree correction is proportional to the Fourier component of $V_0(q)$ at $q=2k_F$, while for the Fock correction $q=0$. The two corrections have different sign as we discussed above. The extra factor of 2 in the Hartree correction is due to electron spin degeneracy.

Note, that Eq. (2.14) is defined only up to a temperature-independent constant which is determined by the ultraviolet contribution. We have chosen the argument of the logarithm to be E_F/T instead of the usual $1/T\tau$ to emphasize that contrary to the naive expectations the logarithmic term persists up to temperatures much larger than $1/\tau$, see also Ref. 15.

The different expressions for the Hartree terms in γ_1 and γ_2 are related to the fact that the single impurity scattering, see Fig. 1, and multiple impurity case, see Fig. 3, allow for different possible scattering angles. The dimensionless function $w(T\tau)$ describes the crossover between ballistic and diffusive regimes. In the ballistic limit $T\tau\gg 1$ it vanishes as

$$w(x\gg 1)\approx\frac{8}{3\pi x}\left[\ln(2x)-\frac{1}{4}(\ln x-1)(6\ln 2-1)\right].$$

In the opposite limit $T\tau\ll 1$ it approaches a constant value [$\mathcal{C}\approx 0.577\cdots$ is the Euler's constant and $\zeta'(x)$ is a derivative of the Riemann zeta function]

$$w(x\ll 1)\approx 1+\frac{2\pi x}{9}\left(\ln x-\ln 2-\mathcal{C}+\frac{3}{4}+6\zeta'(2)\right),$$

so that the linear correction does not completely vanish in the diffusive limit, but competes with the logarithmic term and in semiconductor structures with low Fermi energy it might be important except for the lowest temperatures. The full function $w(x)$ is plotted on Fig 4.

If the Coulomb interaction is considered, then the lowest order in interaction is not sufficient since for the long range interaction $\nu V_0(q\approx 0)\gg 1$. Although the interaction itself is still independent of the electron spin, summation of the perturbation theory depends on the spin state of the two electrons involved. In the first order correction Eq. (2.14) all spin channels gave identical contributions. The total number of channels is 4 and they can be classified by the total spin of the two electrons: one state with the total spin zero ("singlet") and three states with the total spin 1 ("triplet," the three states differ by the value of the *z* component of the total spin). For long range interaction the perturbation theory for the Hartree correction singlet and triplet channels is different. It is known[1,2] (see also Sec. III), that the singlet channel contribution should be combined with the Fock correction as a renormalization of the coupling constant. However, the final result is universal due to dynamical screening: the singlet channel modification of the coupling does not affect the result. What remains of the Hartree term is the triplet channel contribution, which now depends on the corresponding Fermi-liquid constant F_0^σ. Thus, the total correction to the conductivity can be written as a sum of the "charge" (which combines Fock and singlet part of Hartree) and triplet contributions

$$\sigma=\sigma_D+\delta\sigma_T+\delta\sigma_C, \tag{2.16a}$$

where the charge channel correction is given by

$$\delta\sigma_C=\frac{e^2}{\pi\hbar}\frac{T\tau}{\hbar}\left[1-\frac{3}{8}f(T\tau)\right]-\frac{e^2}{2\pi^2\hbar}\ln\left(\frac{E_F}{T}\right), \tag{2.16b}$$

and the triplet channel correction is

$$\delta\sigma_T=\frac{3F_0^\sigma}{(1+F_0^\sigma)}\frac{e^2}{\pi\hbar}\frac{T\tau}{\hbar}\left[1-\frac{3}{8}t(T\tau;F_0^\sigma)\right] -3\left(1-\frac{1}{F_0^\sigma}\ln(1+F_0^\sigma)\right)\frac{e^2}{2\pi^2\hbar}\ln\left(\frac{E_F}{T}\right). \tag{2.16c}$$

Here the factor of 3 in the triplet channel correction Eq. (2.16c) is due to the fact that all three components of the triplet state contribute equally. We reiterate that the corrections Eqs. (2.16) are defined only up to a temperature independent (however not necessarily Fermi-liquid constant independent) term, see also discussion after Eq. (2.14).

We should warn the reader here, that we describe the interaction in the triplet channel by one coupling constant F_0^σ. For the weak coupling limit, it corresponds to the approximation $V_0(2k_F)\simeq\langle V_0(k)\rangle_{\mathrm{FS}}$. This approximation overestimates the triplet channel contribution to the ballistic case for $r_s=\sqrt{2}e^2/(\kappa\hbar v_F)\ll 1$. However, in this limit contribution itself is much smaller than the singlet one. For better estimates in this regime one should use

$$F_0^\sigma\to-\frac{1}{2}\frac{r_s}{r_s+\sqrt{2}}$$

in the first line of Eq. (2.16c) and

$$F_0^\sigma\to-\frac{1}{2\pi}\frac{r_s}{\sqrt{2-r_s^2}}\ln\left(\frac{\sqrt{2}+\sqrt{2-r_s^2}}{\sqrt{2}-\sqrt{2-r_s^2}}\right),\quad r_s^2<2,$$

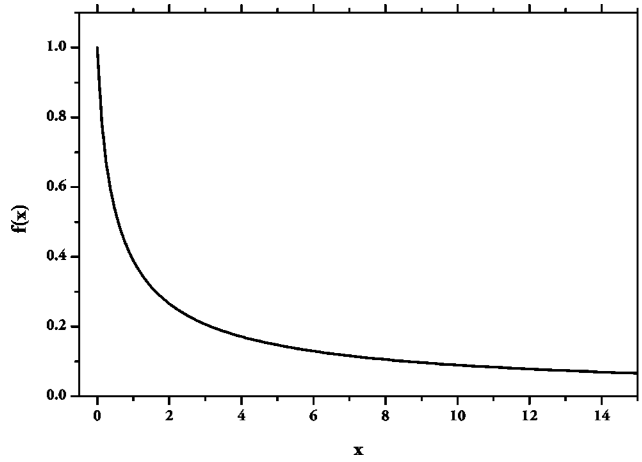

FIG. 5. Dimensionless function $f(x)$, defined so that $f(0)=1$.

$$F_0^\sigma \to -\frac{1}{\pi}\frac{r_s}{\sqrt{r_s^2-2}}\arctan\sqrt{\tfrac{1}{2}r_s^2-1}, \qquad r_s^2>2$$

in the second line. For $r_s \gtrsim 1$ our replacament is well justified even within weak coupling scheme.

Similar to Eq. (2.14) the dimensionless functions $f(x)$ and $t(x;F_0^\sigma)$ describe the cross-over between ballistic and diffusive limits. They are plotted on Figs. 5 and 6 and full expressions are given by Eqs. (3.36). The universal function $f(x)$ has the following limits

$$f(x \gg 1) \approx -\frac{1}{3\pi x}\left(2(\ln x - 1)\ln 2 - \frac{7}{2}\ln(2x)\right), \tag{2.17a}$$

$$f(x \ll 1) \approx 1 - \gamma_1 x + \frac{\pi}{6} x \ln x, \tag{2.17b}$$

$$\gamma_1 = -\frac{\zeta'(2)}{\pi} + \frac{\pi}{6}\left(\mathcal{C} + \frac{1}{3}\ln 2\right) \approx 0.7216.$$

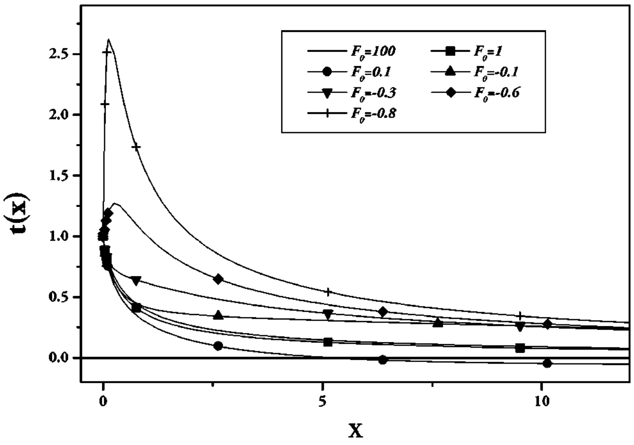

FIG. 6. Dimensionless function $t(x,F_0^\sigma)$ defined so that $t(0,F_0^\sigma)=1$.

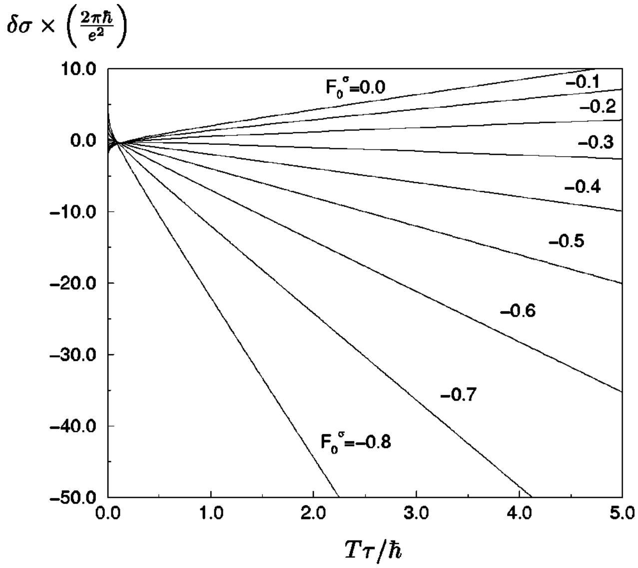

FIG. 7. Total interaction correction to conductivity. The divergence at $T\tau/\hbar \to 0$ is due to the usual logarithmic correction (Ref. 1). Curve $F_0^\sigma=0$ corresponds to the universal behavior of completely spin polarized electron gas. The correction is defined up to a temperature independent part, see Eq. (3.33) and discussion after Eq. (2.14).

The function $t(x;F_0^\sigma)$ depends on the coupling constant and therefore its asymptotic form also depends on F_0^σ. For very small $x \ll 1+F_0^\sigma$ the asymptotic form is

$$t(x \ll 1+F_0^\sigma) \approx 1 - \gamma_2 x + \frac{\pi}{18} x \ln x\left(3 + \frac{1}{1+F_0^\sigma}\right),$$

$$\gamma_2 = -\frac{\zeta'(2)}{3\pi}\left(3+\frac{1}{1+F_0^\sigma}\right) - \frac{\pi\gamma_3}{9(1+F_0^\sigma)} + \frac{\pi}{18}\left[\mathcal{C}\left(3+\frac{1}{1+F_0^\sigma}\right) + \ln 2\left(1+\frac{3}{1+F_0^\sigma}\right)\right], \tag{2.18}$$

$$\gamma_3 = 1 - \frac{5F_0^\sigma - 3}{1+F_0^\sigma} - \left(\frac{5}{2} - 2F_0^\sigma\right)\frac{\ln(1+F_0^\sigma)}{F_0^\sigma}.$$

Notice that at $T\tau \to 0$, Eqs. (2.16) reproduce the known result (1.1). Let us point out that for numerical reasons contributions of scaling functions w,f,t change the result only by few percents and they can be neglected for all the practical purposes.

Notice that while the charge channel correction Eq. (2.16b) is universal, the triplet channel correction Eq. (2.16c) is proportional to F_0^σ, which might be negative. That leads to the conclusion, that the overall sign of the total correction Eqs. (2.16) depends on value of F_0^σ: it can be either positive or negative, see Fig. 7.

Combining together all of the above results we plot the total correction to the conductivity on Fig. 7 for different

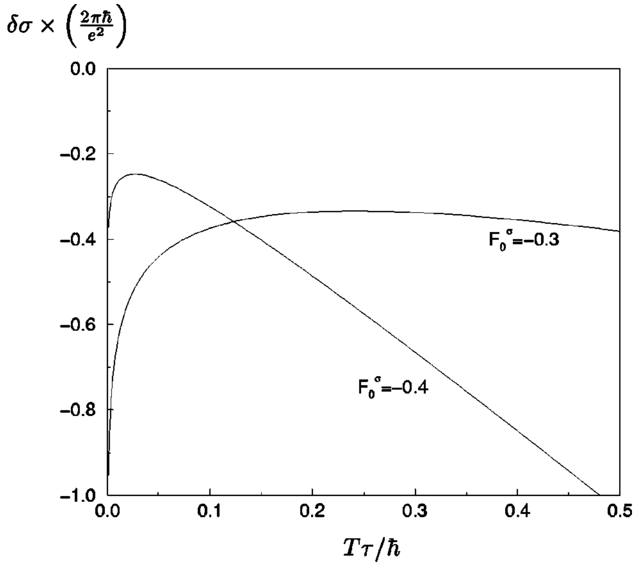

FIG. 8. The nonmonotonous correction to conductivity. Note the difference in the overall scale relative to the previous figure.

values of F_0^σ. The divergence at low temperature is due to the usual logarithmic correction.[1] Although the exact value of F_0^σ cannot be calculated theoretically (in particular, its relation to the conventional measure of the interaction strength, r_s, is unknown for $r_s > 1$), in principle it can be found from a measurement of the Pauli spin susceptibility

$$\chi = \frac{\nu}{1+F_0^\sigma}, \tag{2.19}$$

where the density of states ν should be obtained from a measurement of the specific heat (at $\tau^{-1} \ll T \ll E_F$). The constant F_0^σ is the only parameter in our theory which describes all the data, including the Hall coefficient and the magnetoresistance in the parallel field. The theory for interaction corrections in the magnetic field will be addressed in the forthcoming paper.[14]

The correction in Fig. 7 is almost always monotonous, except for a narrow region $-0.45 < F_0^\sigma < -0.25$. A typical curve in this region is shown in Fig. 8. Note, however, that the overall magnitude of the correction in the range of $T\tau$ in Fig. 7 is so small that it can hardly be observed.

When the interaction becomes so strong that the system approaches the Stoner instability, F_0^σ ceases to be a constant and becomes momentum dependent. Thus the result Eq. (2.16) is no longer valid. Although the simple condition $\delta\sigma_T < \sigma_D$ suggests that this happens at $T \approx (1+F_0^\sigma)E_F$, the more detailed analysis (see Sec. III E) shows that it happens much earlier. In fact, the approximation of the constant F_0^σ is valid in the parameter region defined by the inequality

$$\frac{T}{E_F} < (1+F_0^\sigma)^2, \tag{2.20}$$

see Sec. III E for the origin of this inequality. We were not able to make a reliable calculation of $\delta\sigma(T)$ at higher temperatures.

III. PERTURBATION THEORY

In this section we show how the announced results Eq. (2.16) can be obtained with the help of the traditional perturbation theory. We try to explain the most important points of the calculation in detail. The comprehensive review of the diagrammatic technique for disordered systems can be found in Ref. 1. We start by a brief discussion of the case of a weak, short-range interaction potential. Although this case is artificial and is unrelated to any experiment, it is governed by the same physics as the general problem, and it is simple enough to allow a transparent presentation. To generalize to stronger coupling, we need to recall the basic ideas of the Landau Fermi-liquid theory and to identify the soft modes in the system. Then we present the calculation leading to Eq. (2.16). Finally, to establish the relation of our results to existing literature, we briefly discuss scattering on a single impurity (this discussion is completely analogous to the one in Sec. II but uses the language of diagrams).

A. Hartree-Fock considerations

The static conductivity of a system of electrons is given by the Kubo formula

$$\sigma_{\alpha\beta} = -\lim_{\omega\to 0} \mathrm{Re}\left[\frac{1}{\Omega_n}\int_0^{1/T} d\tau \langle \mathbf{T}_\tau \hat{j}_\alpha(\tau)\hat{j}_\beta(0)\rangle e^{i\Omega_n\tau}\right]_{i\Omega_n\to\omega}, \tag{3.1}$$

where $\hat{j}_\alpha(\tau)$ is the operator of the electric current at imaginary time τ and the analytic continuation of the function defined at Matzubara frequencies $\Omega_n = 2\pi T n$ to function analytic at $\mathrm{Im}\omega > 0$ is performed.

Assuming that electrons interact by means of a weak, short-range interaction (range shorter than $v_F \min(\tau, 1/T)$, $V(r)$) it is sufficient to consider the lowest order of the perturbation theory. The perturbation theory can be conveniently expressed in terms of Feinman diagrams. The lowest order diagrams for the interaction correction to the conductivity are shown on Fig. 9. The Hartree term corresponds to the diagrams "a," while the Fock contribution corresponds to diagrams "b." Evaluation of the correction consists of two main steps: (i) analytic continuation to real time, and (ii) disorder averaging. While these two steps can be performed in any order without affecting the result, it is more convenient (for technical reasons) to start with step (i).

Although analytic continuation in Eq. (3.1) is now a textbook task, we include a brief discussion of the standard procedure in the Appendix to make the paper self-contained. After the continuation any physical quantity is expressed in terms of exact (i.e., not averaged over disorder) retarded and advanced Green's functions of the electronic system, which are defined as

$$G_{12}^{R(A)}(\epsilon) = \sum_j \frac{\Psi_j^*(\vec{r}_1)\Psi_j(\vec{r}_2)}{\epsilon - \epsilon_j \pm \imath 0}, \tag{3.2}$$

where j labels the exact eigenstates of the system and ϵ_j are the exact eigenvalues, counted from the Fermi energy

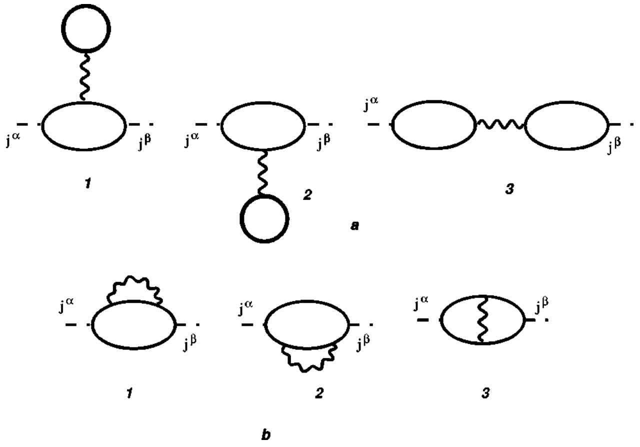

FIG. 9. Interaction correction to conductivity in the lowest order of perturbation theory. Here solid lines correspond to Matsubara Green's functions $-G(i\epsilon_n;\vec r_1,\vec r_2)$ and the wavy line represents the interaction potential $-V(\vec r_1-\vec r_2)$.

$$\left(\frac{-\nabla^2}{2m}+U(\vec r)\right)\Psi_j(\vec r)=(\epsilon_j+\epsilon_F)\Psi_j(\vec r).$$

Here $U(\vec r)$ is the disorder potential.

The resulting expression for the correction to the symmetric part of the conductivity (the Hall conductivity will be discussed in a separate publication[14]) can be written as[16]

$$\delta\sigma_{\alpha\beta}=\int_{-\infty}^{\infty}\frac{d\Omega}{8\pi^2}\left[\frac{\partial}{\partial\Omega}\left(\Omega\coth\frac{\Omega}{2T}\right)\right]\int d^2r_3d^2r_4 \times\mathrm{Im}\{V(\vec r_3-\vec r_4)(B_F^{\alpha\beta}(\Omega;\vec r_3,\vec r_4) -2B_H^{\alpha\beta}(\Omega;\vec r_3,\vec r_4)+\{\alpha\leftrightarrow\beta\})\}, \tag{3.3}$$

where the extra factor of 2 in the Hartree term is due to the summation over electron spin. Here we denoted products of four Green's functions as $B_{F(H)}$. For the Fock term we have

$$B_F^{\alpha\beta}(\Omega;\vec r_3,\vec r_4) =\int\frac{d^2r_1d^2r_5}{\mathcal{V}}\{\hat J_1^\alpha G_{15}^R(\epsilon)\hat J_5^\beta G_{53}^A(\epsilon)G_{34}^R(\epsilon-\Omega)G_{41}^A(\epsilon) \tag{3.4a}$$

$$+\hat J_1^\alpha G_{15}^A(\epsilon)\hat J_5^\beta G_{53}^R(\epsilon)G_{34}^R(\epsilon-\Omega)G_{41}^R(\epsilon) \tag{3.4b}$$

$$+2\hat J_1^\alpha G_{13}^R(\epsilon)G_{35}^R(\epsilon-\Omega)\hat J_5^\beta G_{54}^R(\epsilon-\Omega)G_{41}^A(\epsilon) \tag{3.4c}$$

$$-\hat J_1^\alpha G_{15}^A(\epsilon)\hat J_5^\beta G_{53}^A(\epsilon)G_{34}^R(\epsilon-\Omega)G_{41}^A(\epsilon) \tag{3.4d}$$

$$-\hat J_1^\alpha G_{13}^A(\epsilon)G_{35}^R(\epsilon-\Omega)\hat J_5^\beta G_{54}^R(\epsilon-\Omega)G_{41}^A(\epsilon)\}, \tag{3.4e}$$

where $\mathcal{V}$ is the area of the system. Equations (3.4c) and (3.4e) come from the diagram "b3" on Fig. 9 and the rest of Eq. (3.4) correspond to diagrams "b1" and "b2." For the Hartree term the expression is similar,

$$B_H^{\alpha\beta}(\Omega;\vec r_3,\vec r_4) =\int\frac{d^2r_1d^2r_5}{\mathcal{V}} \times\{\hat J_1^\alpha G_{15}^R(\epsilon)\hat J_5^\beta G_{53}^A(\epsilon)G_{44}^R(\epsilon-\Omega)G_{31}^A(\epsilon) \tag{3.5a}$$

$$+\hat J_1^\alpha G_{15}^A(\epsilon)\hat J_5^\beta G_{53}^R(\epsilon)G_{44}^R(\epsilon-\Omega)G_{31}^R(\epsilon) \tag{3.5b}$$

$$+2\hat J_1^\alpha G_{13}^R(\epsilon)G_{45}^R(\epsilon-\Omega)\hat J_5^\beta G_{54}^R(\epsilon-\Omega)G_{31}^A(\epsilon) \tag{3.5c}$$

$$-\hat J_1^\alpha G_{15}^A(\epsilon)\hat J_5^\beta G_{53}^A(\epsilon)G_{44}^R(\epsilon-\Omega)G_{31}^A(\epsilon) \tag{3.5d}$$

$$-\hat J_1^\alpha G_{13}^A(\epsilon)G_{45}^R(\epsilon-\Omega)\hat J_5^\beta G_{54}^R(\epsilon-\Omega)G_{31}^A(\epsilon)\}. \tag{3.5e}$$

Again, Eqs. (3.5c) and (3.5e) correspond to the diagram "a3" in Fig. 9. The current operator is defined as

$$f_1(\vec r)\hat{\vec J}f_2(\vec r)=\frac{ie}{2m}[(\vec\nabla f_1)f_2-(f_1\vec\nabla f_2)]-\frac{e\vec A(\vec r)}{m}f_1(\vec r)f_2(\vec r). \tag{3.6}$$

In the above expressions terms corresponding to diagrams "b3" and "a3" on Fig. 9 allow for at least one of the spatial integrations to be performed with the help of the identity

$$\int d\vec r_5G_{35}^R(\epsilon)\hat J_5^\beta G_{54}^R(\epsilon)=-ie(\vec r_3-\vec r_4)^\beta G_{34}^R(\epsilon). \tag{3.7}$$

Now it is clear that Hartree terms Eqs. (3.5c) and (3.5e) vanish identically, since there the identity (3.7) should be applied with coordinates $\vec r_3$ and $\vec r_4$ being equal to each other. In the Fock terms Eqs. (3.4e) and (3.4c) one needs to further multiply the result of Eq. (3.7) by the interaction potential $V(\vec r_3-\vec r_4)$. In the case of the short range interaction potential this also gives vanishing contribution. Thus we conclude, that the diagram "a3" on Fig. 9 does not contribute for any form of the interaction, while the diagram "b3" vanishes for the short-range interaction.

The same identity can also be applied to terms Eqs. (3.4d) and (3.5d), which also vanish by the same reason. Thus the task of averaging over disorder is now simplified because we only need to average two Fock terms Eqs. (3.4a) and (3.4b) and two Hartree terms Eqs. (3.5a) and (3.5b). These expressions contain only Green's functions of noninteracting electrons and can be averaged using the standard diagrammatic technique of the theory of disordered systems (see Ref. 1 for review). The diagrams for averaged quantities can be constructed using the four "building blocks" (we use the momentum representation since translational invariance is restored after averaging).

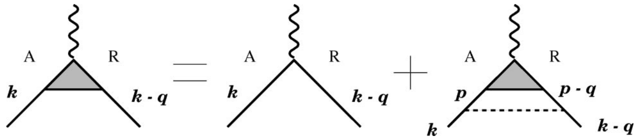

FIG. 10. Dressed interaction vertex.

(1) The average electronic Green's function (denoted as a solid line; there should be no confusion with the previous use of the solid line for exact Green's functions before averaging), which in momentum space can be written as

$$\langle G^{R(A)}\rangle(k,\epsilon)=\frac{1}{\epsilon-\xi_k\pm\dfrac{i}{2\tau}}. \tag{3.8}$$

(2) The disorder potential, which is assumed to be Gaussian with the correlator

$$\langle U(\vec r_1)U(\vec r_2)\rangle=\frac{1}{2\pi\nu\tau}\delta(\vec r_1-\vec r_2).$$

In the diagrams this correlator is represented by the dotted line.

(3) The dressed interaction vertex Γ (q and Ω are momentum and frequency of the interaction propagator), which represents a geometric series in disorder potential shown on Fig. 10.

$$\Gamma(\vec q;\Omega)=1+\frac{1/\tau}{S-\dfrac{1}{\tau}}, \tag{3.9a}$$

where we denote

$$S=\sqrt{\left(i\Omega+\frac{1}{\tau}\right)^2+v_F^2q^2}. \tag{3.9b}$$

(4) The averaged product of a retarded and an advanced Green's functions (sometimes referred to as the diffuson), where we have summed up a geometric series shown on Fig. 11.

Using these building blocks we can average the products of Green's functions as shown on Fig. 12. It is convenient to write the averaged $B_{F(H)}$ in the momentum representation. The product B_F, which appears in the Fock term, can be viewed as a function of coordinates of the two interaction vertices and can be transformed to the momentum space as

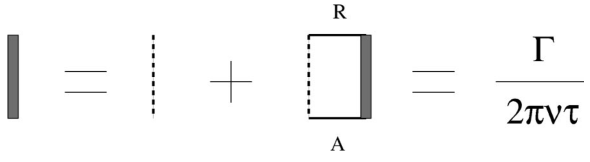

FIG. 11. Diffuson-geometric series of impurity lines.

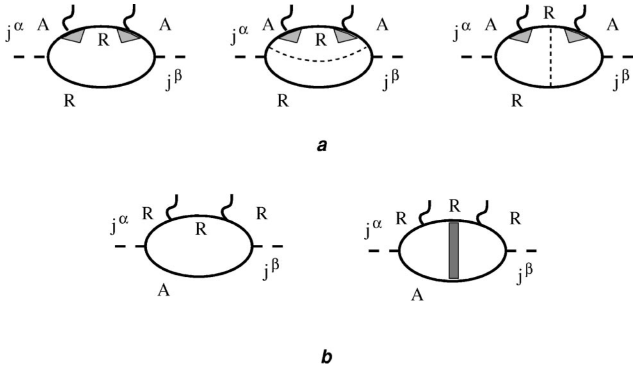

FIG. 12. Averaged product of four Green's functions. The wavy lines indicate $\delta(\vec p_{\rm in}-\vec p_{\rm out}+\vec q)$ for the Fock contribution $B_F(q)$ and $2\pi\delta(\vec p_{\rm in}-\vec p_{\rm out}+\vec q)\delta[\widehat{\vec n\vec p_{\rm in}}]$ for the Hartree contribution $B_H(q,n_1,n_2)$.

$$\langle B_F(\Omega;\vec r_1,\vec r_2)\rangle=\int\frac{d^2q}{(2\pi)^2}e^{i\vec q(\vec r_3-\vec r_4)}\langle B_F(q,\Omega)\rangle. \tag{3.10}$$

Using the explicit expressions Eqs. (3.8) and (3.9) we can write the analytic form of the averaged B_F

$$\frac{B_F(q,\Omega)}{\sigma_D}=\frac{(\Gamma^2-1)\tau}{S}+\frac{\Gamma(\Gamma+1)}{v_F^2q^2}\left(\frac{i\Omega+\dfrac{1}{\tau}}{S}-1\right)^2. \tag{3.11}$$

In the absence of magnetic field, $B_{F(H)}^{\alpha\beta}=\delta^{\alpha\beta}B_{F(H)}$, which is why we did not include the Greek indices in Eq. (3.11).

The Hartree contribution is considered analogously. One can write

$$\langle B_H(\Omega;\vec r_1,\vec r_2)\rangle=\int\frac{d\theta_1}{(2\pi)}\frac{d\theta_2}{(2\pi)}\frac{d^2q}{(2\pi)^2}e^{ik_F(\vec n_1-\vec n_2)(\vec r_1-\vec r_2)}\times B_H(\Omega;\vec n_1,\vec n_2,\vec q), \tag{3.12}$$

where $\vec n_i=(\cos\theta_i,\sin\theta_i)$ indictates the direction of the momentum. Then, disorder averaging of $B_H(\Omega;\vec n_1,\vec n_2,\vec q)$ is performed with the help of the same diagrams (see Fig. 12) but the expression for the vertices changed as indicated in the figure caption.

Accordingly, the expression for the dressed vertex (3.9a), see also Fig. 10, is changed to

$$\Gamma_H(\vec n,\vec n_k;\vec q;\Omega)=2\pi\delta(\widehat{\vec n\vec n_k})+\frac{1}{S_H}\frac{S}{S\tau-1},$$

$$S_H(\vec n,\vec q;\Omega)=i\Omega-iv_F\vec q\vec n+1/\tau, \tag{3.13}$$

where $\vec n_k$ corresponds to the direction of the momentum k on Fig. 10. The final expression for B_H is similar to Eq. (3.11)

$$\frac{B_H(\Omega;\vec{n}_1,\vec{n}_2,\vec{q})}{\sigma_D} = -\frac{-2\pi\delta(\widehat{\vec{n}_1\vec{n}_2})\tau}{S}+\frac{\tau}{S}\int\frac{d\theta_3}{2\pi}\Gamma_H(\vec{n}_1,\vec{n}_3)\Gamma_H(\vec{n}_2,n_3)$$
$$+\int\frac{d\theta_3}{2\pi}\int\frac{d\theta_4}{2\pi}(\vec{n}_3\vec{n}_4)\frac{\Gamma_H(\vec{n}_1,n_3)\Gamma_H(\vec{n}_2,n_4)}{S_H(\vec{n}_3)S_H(\vec{n}_4)}$$
$$+(\vec{n}_1\vec{n}_2)\frac{\Gamma}{S_H(\vec{n}_1)S_H(\vec{n}_2)}, \qquad (3.14)$$

and we suppressed the arguments q,Ω in the right-hand-side of the equation. We note in passing, that by construction of Eqs. (3.4a), (3.4b), (3.5a), and (3.5b) that

$$B_F(\Omega;\vec{r},\vec{r})=B_H(\Omega;\vec{r},\vec{r}),$$

and, therefore, according to Eqs. (3.10) and (3.12) the relation

$$B_F(\Omega;\vec{q})=\int\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}B_H(\Omega;\vec{n}_1,\vec{n}_2,\vec{q})$$

must hold [this can be easily verified using explicit expresions (3.11) and (3.14)].

We are now prepared to calculate the temperature dependence of the conductivity from Eq. (3.3). We substitute Eqs. (3.11) and (3.14) into Eq. (3.3). As we will see, the main contribution to the temperature dependence is provided by wave vectors $q_T \simeq \max[T,(T/\tau)^{1/2}]/v_F$. On the other hand the potential $V(\vec{r})$ has a range much shorter than $1/q_T$. This enables us to use the following approximations:

$$\int\frac{d^2r_3d^2r_4}{\mathcal{V}}V(\vec{r}_3-\vec{r}_4)e^{i\vec{q}(\vec{r}_3-\vec{r}_4)}\approx V(0),$$

$$\int\frac{d^2r_3d^2r_4}{\mathcal{V}}V(\vec{r}_3-\vec{r}_4)e^{ik_F(\vec{n}_1-\vec{n}_2)(\vec{r}_3-\vec{r}_4)}$$
$$\approx V\left(2k_F\sin\frac{\widehat{\vec{n}_1\vec{n}_2}}{2}\right),$$

where $V(k)$ in the right-hand-side of the above equations denotes the Fourier transform of the interaction potential.

Altogether, we now write the conductivity correction as

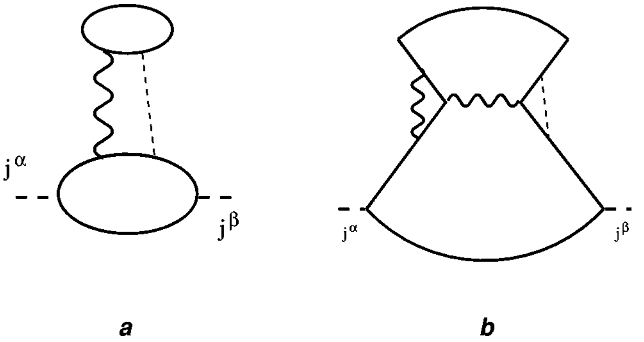

FIG. 13. (a) Single impurity contribution to the Hartree term, see Sec. III F for a detailed discussion. (b) Second order correction to the Hartree term (a).

$$\delta\sigma=\int_{-\infty}^{\infty}\frac{d\Omega}{8\pi^2}\frac{\partial}{\partial\Omega}\left(\Omega\coth\frac{\Omega}{2T}\right)\int\frac{d^2q}{(2\pi)^2}$$
$$\times\mathrm{Im}\left\{V_0(0)B_F(q,\Omega)-2\int\frac{d\theta_1}{2\pi}\int\frac{d\theta_2}{2\pi}\right.$$
$$\left.\times V_0\left(2k_F\sin\frac{\widehat{\vec{n}_1\vec{n}_2}}{2}\right)B_H(\Omega;\vec{n}_1,\vec{n}_2,\vec{q})\right\}. \qquad (3.15)$$

Evaluating this integral (where we only keep the temperature dependent part, see Sec. III F for details) one arrives to the same result Eq. (2.14), but with the coefficient in the form Eq. (2.15), in agreement with the discussion of Sec. II.

Let us now turn to the case of the Coulomb potential, where the scheme of the calculation (as described so far) breaks down. In the Fock term we have $V(0)$, which diverges for the Coulomb interaction $[V(q)\sim 1/q]$. To obtain meaningful results one needs to take into account the effect of dynamical screening. The Hartree term seems to work better since using just the static screening makes the result finite. However, this is wrong also, since in this case diagrams with extra interaction lines do not contain any smallness [see, e.g., Fig. 13; there the correction is $\sim V(2k_F)V(0)$]. Thus one can not justify the perturbation theory in the interaction potential. The way out of this problem is the standard theory of Landau Fermi liquid, which we briefly discuss in the following subsection.

B. Soft modes

As we already discussed, the main contribution to the temperature dependence of physical quantities comes from the processes characterized by spatial scales much larger than the Fermi wavelength λ_F. Therefore, there is a scale separation in the problem; all the Fermi liquid parameters[21] F_i are established at small distances of the order of λ_F, and are not affected by disorder if the relation $\epsilon_F\tau\gtrsim 1$ holds. On the other hand, all the temperature and disorder dependence is determined by infrared behavior of the system where F_i can be considered as fixed.

Therefore, our first step is to identify the terms in the interaction Hamiltonian, which may produce the biggest contributions at temperatures much smaller than the Fermi energy. This procedure contains nothing new in comparison with the standard identification of singlet, triplet, and Cooper channels, see Ref. 22, and we present here the main steps to make the paper self-contained.

The original interaction Hamiltonian has the form

$$\hat{H}_{\text{int}}=\sum_{\vec{q},\vec{p}_i}\frac{V(q)}{2}\psi^{\dagger}_{\sigma_1}(\vec{p}_1)\psi^{\dagger}_{\sigma_2}(\vec{p}_2)\psi_{\sigma_2}(\vec{p}_2+\vec{q})\psi_{\sigma_1}(\vec{p}_1-\vec{q}),$$

and we imply summation over repeated spin indices. Soft modes of the system correspond to the situation when two of the fermionic operators have momenta close to each other. The difference of the momenta $q*$ defines the scale $1/q*\gg\lambda_F$, which is the smallest lengthscale allowed in the theory. Therefore, we explicitly separate the Hamiltonian into a part that contains all the soft modes (first three terms) and a correction $\delta\hat{H}$, which does not contain such pairs of fermionic operators

$$\hat{H}_{\text{int}}=\hat{H}_{\rho}+\hat{H}_{\sigma}+\hat{H}_{pp}+\delta\hat{H}. \tag{3.16}$$

The explicit expressions for the entries of the Hamiltonian (3.16) are the following. The interaction in the singlet channel (charge dynamics) is described by

$$\hat{H}_{\rho}=\frac{1}{2}\sum_{|\vec{q}|<q*,\vec{p}_i}\left[V(q)+\frac{F^{\rho}(\widehat{\vec{n}_1\vec{n}_2})}{\nu}\right]\times[\psi^{\dagger}_{\sigma_1}(\vec{p}_1)\psi_{\sigma_1}(\vec{p}_1-\vec{q})][\psi^{\dagger}_{\sigma_2}(\vec{p}_2)\psi_{\sigma_2}(\vec{p}_2+\vec{q})], \tag{3.17}$$

where $\vec{n}_i=\vec{p}_i/|p_i|$, the dimensionless parameter $F^{\rho}(\widehat{\vec{n}_1\vec{n}_2})$ is related to the original interaction potential $V(q)$ by

$$F^{\rho}(\theta)=-\frac{\nu}{2}V\left(2k_F\sin\frac{\theta}{2}\right), \tag{3.18}$$

and ν is the thermodynamic density of states of noninteracting electrons (introduced here to make F^{ρ} dimensionless).

Interaction in the triplet channel (spin density dynamics) is governed by

$$\hat{H}_{\sigma}=\frac{1}{2}\sum_{\substack{\vec{p}_i\\|\vec{q}|<q*}}\sum_{j=x,y,z}\frac{F^{\sigma}(\widehat{\vec{n}_1\vec{n}_2})}{\nu}\times[\psi^{\dagger}_{\sigma_1}(\vec{p}_1)\hat{\sigma}^{j}_{\sigma_1\sigma_2}\psi_{\sigma_2}(\vec{p}_1-\vec{q})]\times[\psi^{\dagger}_{\sigma_3}(\vec{p}_2)\hat{\sigma}^{j}_{\sigma_3\sigma_4}\psi_{\sigma_4}(\vec{p}_2+\vec{q})], \tag{3.19}$$

where parameters $F^{\sigma}(\widehat{\vec{n}_1\vec{n}_2})$ are

$$F^{\sigma}(\theta)=-\frac{\nu}{2}V\left(2k_F\sin\frac{\theta}{2}\right). \tag{3.20}$$

Finally, the Hamiltonian

$$\begin{aligned}\hat{H}_{pp}=\sum_{|\vec{q}|<q*,\vec{p}_i}\Bigg\{&\frac{F^{e}(\widehat{\vec{n}_1\vec{n}_2})}{\nu}[\psi^{\dagger}_{\sigma_1}(\vec{p}_1)\hat{\sigma}^{y}_{\sigma_1\sigma_2}\psi^{\dagger}_{\sigma_2}(\vec{q}-\vec{p}_1)]\\&\times[\psi_{\sigma_3}(\vec{p}_2)\hat{\sigma}^{y}_{\sigma_3\sigma_4}\psi_{\sigma_4}(\vec{q}-\vec{p}_2)]\\&+\sum_{j=x,y,z}\frac{F^{o}(\widehat{\vec{n}_1\vec{n}_2})}{\nu}[\psi^{\dagger}_{\sigma_1}(\vec{p}_1)\tilde{\sigma}^{j}_{\sigma_1\sigma_2}\psi^{\dagger}_{\sigma_2}(\vec{q}-\vec{p}_1)]\\&\times[\psi_{\sigma_3}(\vec{p}_2)(\tilde{\sigma}^{j})^{\dagger}_{\sigma_4\sigma_3}\psi_{\sigma_4}(\vec{q}-\vec{p}_2)]\Bigg\}\end{aligned} \tag{3.21}$$

describes singlet F^{e} and triplet F^{o} pairing fluctuations. The parameters in this Hamiltonian are

$$F^{e,o}(\theta)=\frac{\nu}{4}\left[V\left(2k_F\sin\frac{\theta}{2}\right)\pm V\left(2k_F\cos\frac{\theta}{2}\right)\right], \tag{3.22}$$

where plus and minus signs correspond to even (e) and odd (o) pairing, respectively. Here $\hat{\sigma}^{j}_{\sigma_1\sigma_2}$ are the elements of the Pauli matrices in spin space

$$\hat{\sigma}^{x}=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad\hat{\sigma}^{y}=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\qquad\hat{\sigma}^{z}=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

and $\tilde{\sigma}^{j}=\hat{\sigma}^{y}\hat{\sigma}^{j}$.

Deriving Eqs. (3.17)–(3.22), we used the condition, $q*\ll k_F$. This condition allowed us to make the following approximation:

$$(\vec{p}_1-\vec{p}_2)^2\approx4k_F^2\sin^2\left(\frac{\widehat{\vec{n}_1\vec{n}_1}}{2}\right).$$

We also used the identity

$$\begin{aligned}2\delta_{\sigma_1\sigma_2}\delta_{\sigma_3\sigma_4}&=\delta_{\sigma_1\sigma_3}\delta_{\sigma_2\sigma_4}+\hat{\sigma}^{j}_{\sigma_1\sigma_3}\hat{\sigma}^{j}_{\sigma_4\sigma_2}\\&=\hat{\sigma}^{y}_{\sigma_1\sigma_3}\hat{\sigma}^{y}_{\sigma_2\sigma_4}+\tilde{\sigma}^{j}_{\sigma_1\sigma_3}(\tilde{\sigma}^{j})^{\dagger}_{\sigma_4\sigma_2}.\end{aligned}$$

So far, the representation (3.16) of original interaction is exact. The only advantage of this representation is that it explicitly separates the term δH which does not contain coupling to the low energy excitations of the fermionic system. Therefore, the contribution of δH to physical quantities is regular and not infrared divergent [such as $(T/v_Fq*)^2$]. Therefore, for the electron system with weak short range interaction, δH can be disregarded at all.

Moreover, even if the interaction is not weak or long range, δH can be treated in all the orders of perturbation theory without generating a soft mode. If this term does not break the translational symmetry at short distances, its only effect is to renormalize the interaction parameters F's in Eqs. (3.17), (3.19), and (3.21) and the Fermi velocity in the noninteracting part of the Hamiltonian. For instance, one obtains for the two-dimensional electron gas with the Coulomb interaction $V(q)=2\pi e^2/(\kappa|q|)$

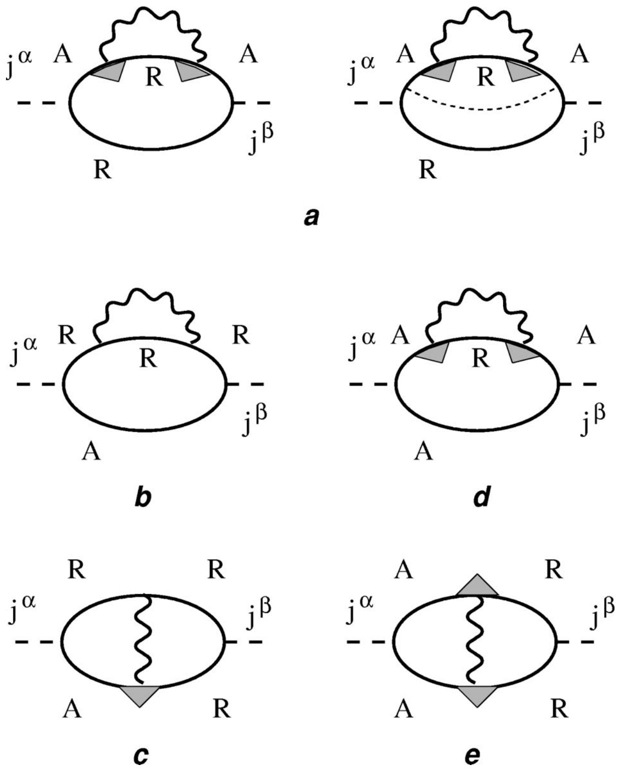

FIG. 14. Conductivity diagrams, group I. Diagrams (a), (b), and (d) were evaluated for the short range interaction in Sec. III A. In the diffusive regime (Ref. 1) only diagrams (a),(d),(e), were considered at $\omega,qv_F\ll 1/\tau$.

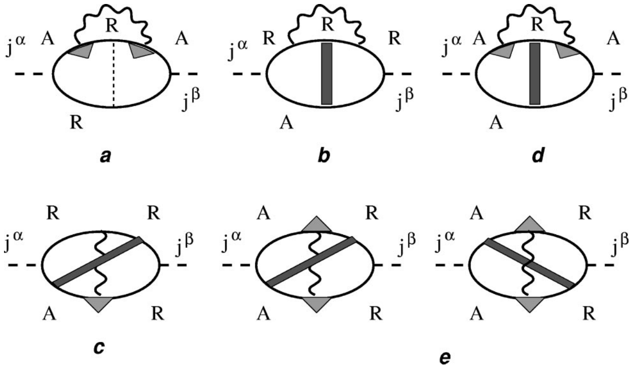

FIG. 15. Conductivity diagrams, group II. Diagrams (a) and (b) were evaluated for the short range interaction in Sec. III A. In the diffusive regime (Ref. 1) only diagrams (a),(d),(e), were considered at $\omega,qv_F\ll 1/\tau$.

$$F^{\rho}(\theta)=F^{\sigma}(\theta)=-\frac{1}{2}\,\frac{r_s}{\sqrt{2}\left|\sin\frac{\theta}{2}\right|+r_s},\tag{3.23}$$

where

$$r_s\equiv\frac{\sqrt{2}e^2}{\kappa\hbar v_F}\tag{3.24}$$

is the conventional parameter characterizing interaction strength and κ is the low frequency dielectric constant of the host material. Expression (3.23) is applicable only for $r_s\ll 1$, however, keeping it in denominator is legitimate for small angle scattering.

For stronger interaction $r_s\gtrsim 1$, but still far from the Wigner crystal instability[23] $r_s\lesssim 37$ exact calculation of the parameters F from the first principles (as well as their explicit expressions in terms of r_s) is not possible. Nevertheless, to study the behavior of the system at distances much larger than λ_F, one can still disregard the term δH in Eq. (3.16). Then parameters F are no longer bound by Eqs. (3.18), (3.20), and (3.22) [or by Eq. (3.23) for the Coulomb interaction] but rather should be treated as starting parameters for the low-energy theory. The form of Eqs. (3.17), (3.19), and (3.21) is guarded by symmetries of the system: Eq. (3.17) is guarded by translational symmetry and charge conservation; Eq. (3.19) is guarded by translational symmetry and symmetry with respect to spin rotations; and Eq. (3.21) is guarded by all above symmetries and the electron-hole symmetry, which holds approximately at low energies.

All the consideration above essentially repeats the basics of the Landau Fermi-liquid theory.[21] We reiterate, that this theory does not imply that the interaction is weak; the only assumption here is that no symmetry is broken at small distances.

C. Disorder averaging

To study the interaction correction to conductivity due to charge and triplet channel interactions introduced in the previous subsection, we follow the same route as in the case of the short-range interaction. In particular, the charge channel correction is a direct generalization of the Fock term. We start, however, with the discussion of disorder averaging.

The correction to conductivity Eq. (3.3) represents the first order perturbation theory in the original potential $V(q)$, valid when the potential is weak. For stronger coupling we make use of the effective Hamiltonian Eq. (3.16). Although the diagrams for conductivity look similar to the Fock term "b" on Fig. 9, their content is now quite different. First, the wavy line now represents the propagator for one of the soft modes in Eq. (3.16). Therefore the expression for the conductivity Eq. (3.3) should be rewritten as

$$\delta\sigma_{\alpha\beta}=-\int_{-\infty}^{\infty}\frac{d\Omega}{8\pi^2}\left[\frac{\partial}{\partial\Omega}\left(\Omega\coth\frac{\Omega}{2T}\right)\right]\times\mathrm{Im}\int\frac{d^2r_3d^2r_4}{\mathcal{V}}\{[\mathcal{D}^A(\Omega,\vec{r}_3,\vec{r}_4)+\mathrm{Tr}\hat{\mathcal{D}}^A_T(\Omega,\vec{r}_3,\vec{r}_4)]\times(B_F^{\alpha\beta}(\Omega,\vec{r}_3,\vec{r}_4)+\{\alpha\leftrightarrow\beta\})\},\tag{3.25}$$

where $\mathcal{D}^A$ and $\hat{\mathcal{D}}^A_T$ are advanced propagators for charge and triplet channels [$\hat{\mathcal{D}}_T$ is a 3×3 matrix as follows from Eq. (3.19), see also Sec. III E] and B_F is the product of electronic Green's functions given by Eq. (3.4), the same as in the Fock term. Deriving Eq. (3.25) we assumed that the spin rotational symmetry is preserved, i.e., no Zeeman splitting or the spin-orbit interaction is present. We also neglected the dependence of the interaction propagators on the direction of the electron momenta. Lifting of those two assumptions is straightforward but it will not be done in the present paper. To the leading order in $1/k_F l$ we can average the propagators independently of B_F (see, e.g., Ref. 1). Here we proceed with averaging B_F and the discussion of the propagators follows.

We have already averaged the product B_F of four Green's functions for the case of the short-range potential. There the three terms Eqs. (3.4e), (3.4c), and (3.4d) vanished due to the particular form of the potential. Now we have to take these terms into account and consider the full set of diagrams shown on Figs. 14 and 15. These diagrams can be evaluated in exactly the same way as those in Sec. III A (where we considered a subset of these diagrams).

As a result, the averaged B_F has a form similar to Eq. (3.11) and can again be expressed in terms of the dressed vertex Γ [see Eq. (3.9)]. We are still interested in the longitudinal conductivity and thus disregard the Hall contribution. Thus, after averaging the correction Eq. (3.25) takes the form

$$\delta\sigma = -e^2 v_F^2 \pi\nu \int_{-\infty}^{\infty} \frac{d\Omega}{4\pi^2} \frac{\partial}{\partial\Omega}\left(\Omega \coth\frac{\Omega}{2T}\right) \int \frac{d^2q}{(2\pi)^2} \times \mathrm{Im}\{[\mathcal{D}^A(\Omega,q) + \mathbf{Tr}\hat{\mathcal{D}}_T^A(\Omega,q)]\tilde{B}_F(\Omega,q)\},$$

$$\tilde{B}_F(\Omega,q) = -\frac{2\tau\left(i\Omega+\frac{1}{\tau}\right)\Gamma}{S^3} + \frac{(\Gamma^2-1)\tau^2}{S} + \frac{v_F^2q^2 - 2\left(i\Omega+\frac{1}{\tau}\right)^2}{S^5}\frac{\Gamma^2}{2} + \frac{\tau\Gamma(\Gamma+1)}{v_F^2q^2} \times \left(\frac{i\Omega+\frac{1}{\tau}}{S} - 1\right)^2 - \frac{2\Gamma^2}{S^3}\left(\frac{i\Omega+\frac{1}{\tau}}{S} - 1\right) + \frac{\Gamma^3 v_F^2 q^2}{\tau S^6}, \tag{3.26}$$

where quantities Γ and S are defined in Eq. (3.9).

It is important to emphasize that

$$\tilde{B}_F(\Omega, q=0) = 0 \tag{3.27}$$

[to see this one should use explicit expressions (3.9) in Eq. (3.26)]. This property is not accidental—it is guarded by the gauge invariance of the system: no interaction with zero momentum transfer can affect the value of the closed loop.

To proceed further with the actual calculation of the correction (3.26) we need to specify the interaction propagator. It will be done in the following two subsections.

D. Charge channel

In this section we discuss the charge channel correction, described by the Hamiltonian (3.17). Because the effective interaction is characterized by the momentum transfer much smaller than the Fermi wave vector, the random phase approximation (RPA), see Fig. 16, is applicable. To simplify further considerations, we approximate the Fermi liquid parameter F^ρ by its zero angular harmonic

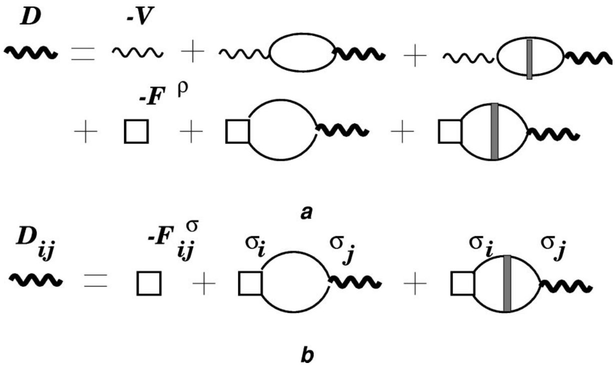

FIG. 16. Interaction propagator in the (a) singlet and (b) triplet channel.

$$F^\rho(\theta) \approx F_0^\rho, \tag{3.28}$$

this approximation does not affect the final result because of the long range nature of the Coulomb potential $V(q\to 0)\to\infty$.

Consequently, we write the charge channel propagator in the form

$$\mathcal{D}^A(\Omega,q) = -\frac{\nu V(q) + F_0^\rho}{\nu + (\nu V(q) + F_0^\rho)\Pi^A}, \tag{3.29a}$$

where the polarization operator is given by

$$\Pi^A(\Omega,q) = \nu\left[1 - \frac{i\Omega}{S - \frac{1}{\tau}}\right], \tag{3.29b}$$

using the notation (3.9b). The polarization operator (3.29b) differs from the more standard one (used, for instance, in Ref. 16) since the diffusion approximation has not been made yet. Indeed, expanding the polarization operator in small Ω and q we can recover the usual diffusive form. In terms of the scattering time it corresponds to the limit $T\tau \ll 1$. We do not do that here since we want to calculate the conductivity for all values of $T\tau$.

The form of the propagator (3.29) and expression for the conductivity correction (3.26) suggests that there could be two contributions. First, the propagator Eq. (3.29) has a pole which corresponds to the 2D plasmon. However, the plasmon dispersion relation is

$$(v_F q_{\mathrm{pl}})^2 \nu V(q_{\mathrm{pl}}) = 2\Omega\left(\Omega + \frac{i}{\tau}\right),$$

i.e., $(v_F q_{pl})^2 \ll |\Omega(\Omega + i/\tau)|$ at all distances larger than the screening radius. According to the gauge invariance condition (3.27) this contribution is strongly suppressed (by a factor of the order of $\max[T,(T/\tau)^{1/2}]d_{\mathrm{sc}}/v_F$, with d_{sc} being the screening radius $\nu V(1/d_{\mathrm{sc}}) = 1$) and we will not take it into account.

Second, at frequencies smaller than the plasmon frequency we can neglect the unity in the denominator in Eq. (3.29), which corresponds to the unitary limit, i.e.,

$$\mathcal{D}^A = -\frac{1}{\Pi^A} = -\frac{1}{\nu}\,\frac{S-\dfrac{1}{\tau}}{S-\dfrac{1}{\tau}-i\Omega}. \tag{3.30}$$

Thus the original coupling $V(q)$ as well as the renormalization of the coupling by the Fermi liquid parameter Eq. (3.17) does not affect the resulting propagator. In other words the propagator becomes universal.

It is important to emphasize that Eq. (3.30) gives the upper bound for the strength of the repulsive interaction. This is guaranteed by stability of the electron system with respect to the Wigner crystallization, i.e., by the condition $\nu V(q)+F_0^\rho>0$ at $q<q^*$. Therefore, we always have

$$\frac{\nu V+F_0^\rho}{\nu+(\nu V+F_0^\rho)\Pi}<\frac{1}{\Pi},$$

so that Eq. (3.30) is indeed the upper bound for the propagator Eq. (3.29a). Note, that the above condition is satisfied regardless of the sign of F_0^ρ. In particular, it is possible to have $F_0^\rho<-1$ so that the so-called compressibility of the system $\nu/(1+F_0^\rho)$ is negative. This fact, however, has nothing to do with stability of the Fermi liquid and does not affect transport phenomena.[24]

Using the propagator Eq. (3.30) in the expression for the correction Eq. (3.26) we obtain after momentum integration

$$\begin{aligned}\delta\sigma_C = &-e^2\tau\int_0^\infty \frac{d\Omega}{2\pi}\,\frac{\partial}{\partial\Omega}\left(\Omega\coth\frac{\Omega}{2T}\right)\\ &\times\left\{\frac{2}{\pi}\arctan\Omega\tau+\frac{1}{\pi\Omega\tau}+\frac{\Omega\tau}{2\pi}H(\Omega\tau)\ln 2\right.\\ &+\frac{1}{\pi}[1+H(\Omega\tau)]\arctan\frac{1}{\Omega\tau}+\frac{\Omega\tau}{4\pi}\\ &\left.\times\left[\frac{1}{2}+H(\Omega\tau)\right]\ln\left(1+\frac{1}{\Omega^2\tau^2}\right)\right\},\end{aligned} \tag{3.31}$$

where the dimensionless function $H(x)$ is defined as

$$H(x)=\frac{1}{4+x^2}.$$

In the frequency integral Eq. (3.31) we single out the first two terms as being dominant in the ballistic and diffusive limits respectively with the rest being the crossover function. The diffusive limit is given by

$$\begin{aligned}\delta\sigma_C(T\tau\ll 1) &= -e^2\tau\int_0^\infty\frac{d\Omega}{2\pi}\,\frac{\partial}{\partial\Omega}\left(\Omega\coth\frac{\Omega}{2T}\right)\frac{1}{\pi\Omega\tau}\\ &= -\frac{e^2}{2\pi^2}\ln\left(\frac{E_F}{T}\right).\end{aligned} \tag{3.32}$$

In the opposite limit we can replace $\arctan\Omega\tau$ by $\pi/2$. Then the integral is divergent in the ultraviolet, but that large constant can be incorporated in the definition of τ. This is done as follows:

$$\int_0^\infty d\Omega\,\frac{\partial}{\partial\Omega}\left(\Omega\coth\frac{\Omega}{2T}\right)\to -2T+E_F\coth\frac{E_F}{2T}, \tag{3.33}$$

where E_F is put for the upper limit of the integral. This is consistent with the approximations in momentum integration, where one typically relies on fast convergence in order to set the integration limit (otherwise determined by the Fermi energy) to infinity and to set all momenta in the numerator to the Fermi momentum in magnitude. Since we are interested in temperatures $T\ll E_F$, the second term is essentially a temperature independent (although infinite) constant. The temperature dependent correction to the conductivity is determined by the first term. As a result

$$\delta\sigma_C(T\tau\gg 1)=-e^2\tau\int_0^\infty\frac{d\Omega}{2\pi}\,\frac{\partial}{\partial\Omega}\left(\Omega\coth\frac{\Omega}{2T}\right)=e^2\frac{T\tau}{\pi}. \tag{3.34}$$

Integrating the full expression Eq. (3.31) we find the correction valid at all values of $T\tau$,

$$\delta\sigma_C=-\frac{e^2}{2\pi^2}\ln\left(\frac{E_F}{T}\right)+e^2\frac{T\tau}{\pi}\left[1-\frac{3}{8}f(T\tau)\right], \tag{3.35}$$

where the dimensionless function $f(x)$ is defined as a dimensionless integral

$$\begin{aligned}f(x)=\frac{8}{3}\int_0^\infty dz&\left[\frac{\partial}{\partial z}(z\coth z)-1\right]\\ &\times\left\{\frac{xz}{\pi}H(2xz)\ln 2+\frac{1}{\pi}[1+H(2xz)]\arctan\frac{1}{2xz}\right.\\ &+\frac{xz}{2\pi}\left[\frac{1}{2}+H(2xz)\right]\ln\left(1+\frac{1}{(2xz)^2}\right)\\ &\left.+\frac{2}{\pi}\arctan\Omega\tau-1\right\}.\end{aligned} \tag{3.36}$$

The factor 3/8 is introduced for convenience, so that $f(0)=1$. The integral can be evaluated analytical in the two limiting cases and the result is given by Eq. (2.17). In the intermediate regime the integral can be evaluated numerically and the result is plotted on Fig. 5.

E. Triplet channel

In this section we discuss the correction in the triplet channel. Similar to the case of the charge channel, we need to derive the interaction propagator in the triplet channel and then use Eq. (3.26). As follows from the Hamiltonian Eq. (3.19), the triplet channel propagator is now a 3×3 matrix. Apart from this minor complication, the propagator can be found using the same RPA approximation as the one used in Sec. III D, see Fig. 16.

Similarly to the charge channel, we take the Fermi liquid coupling $\hat{F}^\sigma$ to be independent of electron momenta

$$F^\sigma(\theta)\approx F_0^\sigma. \tag{3.37}$$

Unlike the case of the charge channel, this approximation slightly affects final results [see discussion after Eqs. (2.16) for the drawbacks of this approximation as well as for its remedies]. Then the matrix equation for the triplet propagator has the form

$$[\mathcal{D}_T]_{ij}=-\delta_{ij}\frac{F_0^\sigma}{\nu}-\frac{F_0^\sigma}{\nu}\sum_{k=x,y,z}\hat{\Pi}_{ik}[\mathcal{D}_T]_{kj}, \tag{3.38}$$

where $i,j=x,y,z$.

In the absence of the magnetic field and spin-orbit scattering each electronic Green's function is a diagonal matrix in the spin space, and therefore

$$\Pi_{ik}^A(q,\Omega)=\delta_{ik}\Pi^A(q,\Omega), \tag{3.39}$$

where $\Pi^A(q,\Omega)$ is the polarization operator given by Eq. (3.29b). Altogether, using Eq. (3.39) in the equation (3.38), we find the triplet channel propagator as

$$[\mathcal{D}_T^A(\Omega,q)]_{ij}=-\delta_{ij}\frac{F_0^\sigma}{\nu+F_0^\sigma\Pi^A(\Omega,q)}. \tag{3.40}$$

Before we continue, let us discuss the validity of the approximation Eq. (3.37). Consider the situation close to the Stoner instability $F_0^\sigma\to-1$. In this case the pole of the propagator Eq. (3.40) describes a magnetic excitation in the system. In the ballistic case $(q>1/l)$ it is a slow, overdamped spin wave

$$-i\omega\approx(1+F_0^\sigma)v_F|q|.$$

The main contribution to the temperature dependent conductivity correction comes from this pole at $\omega\sim T$. The corresponding typical momenta are $k^*\sim T/[(1+F_0^\sigma)v_F]$. Although we are using the momentum independent F_0^σ, it is known[25] that fluctuations in the triplet channel produce a nonanalytic correction to the spin susceptibility, so up to a numerical coefficient $F^\sigma\approx F_0^\sigma(1-|q|/k_F)$. Such momentum dependence can only be neglected if $k^*\ll k_F(1+F_0^\sigma)$. This translates into a limitation for the temperature range where the results listed in Sec. II B are valid:[26]

$$T\ll T^*\approx(1+F_0^\sigma)^2E_F. \tag{3.41}$$

At higher temperatures $T>T^*$ our theory is not applicable.

Having discussed the validity of the approach, we proceed with the straightforward calculation: one has to substitute the propagator Eq. (3.40) into the expression for the correction Eq. (3.26) and evaluate the integral. The result of the momentum integration is given by

$$\begin{aligned}\delta\sigma_T=&-3e^2\tau\int_{-\infty}^{\infty}\frac{d\Omega}{4\pi}\frac{\partial}{\partial\Omega}\left(\Omega\coth\frac{\Omega}{2T}\right)\Bigg\{\left(1-\frac{1}{F_0^\sigma}\ln(1+F_0^\sigma)\right)\\&\times\frac{1}{\pi\Omega\tau}+\frac{F_0^\sigma}{(1+F_0^\sigma)}\Bigg[\frac{2}{\pi}\arctan\Omega\tau+\frac{\Omega\tau}{2\pi}\{[H(\Omega\tau)\\&+h_1(\Omega\tau)]\ln2+h_4(\Omega\tau)\}+\frac{1}{\pi}[1+H(\Omega\tau)\\&+(\Omega\tau)^2h_3(\Omega\tau)]\arctan\frac{1}{\Omega\tau}+\frac{\Omega\tau}{4\pi}\\&\times\left[\frac{1}{2}+H(\Omega\tau)+h_2(\Omega\tau)\right]\ln\left(1+\frac{1}{\Omega^2\tau^2}\right)\Bigg]\Bigg\},\end{aligned} \tag{3.42a}$$

where we introduce notations

$$h_1(x)=\tilde{H}(x;1+2F_0^\sigma)[5+6F_0^\sigma-4(2+3F_0^\sigma)H(x)], \tag{3.42b}$$

$$\begin{aligned}h_2(x)=&h_1(x)+\tilde{H}(x;F_0^\sigma)\\&\times\left[-\frac{1}{2}(1+F_0^\sigma)+F_0^\sigma x^2\left(\frac{1}{2}-(1+F_0^\sigma)\tilde{H}(x;F_0^\sigma)\right)\right],\end{aligned} \tag{3.42c}$$

$$\begin{aligned}h_3(x)=&\tilde{H}(x;1+2F_0^\sigma)[-1-2F_0^\sigma+(2+3F_0^\sigma)H(x)]\\&+\frac{F_0^\sigma}{2}\tilde{H}(x;F_0^\sigma)[1+F_0^\sigma x^2\tilde{H}(x;F_0^\sigma)],\end{aligned} \tag{3.42d}$$

$$\begin{aligned}h_4(x)=&\tilde{H}(x;F_0^\sigma)\left[\frac{5F_0^\sigma-3}{2}+\frac{1-(F_0^\sigma)^2}{F_0^\sigma}\ln(1+F_0^\sigma)\right]\\&+h_5(x)\frac{1+F_0^\sigma}{F_0^\sigma}\ln(1+F_0^\sigma),\end{aligned} \tag{3.42e}$$

$$\begin{aligned}h_5(x)=&(2F_0^\sigma-1)\tilde{H}(x;1+2F_0^\sigma)+\tilde{H}^2(x;F_0^\sigma)\\&\times\left[\left(\frac{1}{2}-2F_0^\sigma\right)(1+F_0^\sigma)^2-(F_0^\sigma)^2x^2\left(\frac{1}{2}+2F_0^\sigma\right)\right].\end{aligned} \tag{3.42f}$$

Here we introduce a dimensionless function $\tilde{H}(x;y)$

$$\tilde{H}(x;y)=\frac{1}{(1+y)^2+(xy)^2},$$

which is related to the function $H(x)$ introduced in Sec. III D simply by $H(x)=\tilde{H}(x;1)$.

The expression in brackets turns into its counterpart in the charge channel in the unitary limit ($F_0^\sigma \to \infty$). Its first term describes the diffusive limit described in Ref. 1 (the formal difference in the coefficient stems from the difference in the definition of the coupling constant). The frequency integral is evaluated in the same way as in Eq. (3.32). Similar to our discussion of the charge channel correction [see, e.g., Eq. (3.34)], we identify the second term in Eq. (3.42a) with the ballistic limit (which we discuss in more detail in the next section). The intermediate temperature regime is described by the expression [which appeared previously in Sec. II B, Eq. (2.16c)]:

$$\delta\sigma_T = -3\left(1-\frac{1}{F_0^\sigma}\ln(1+F_0^\sigma)\right)\frac{e^2}{2\pi^2}\ln\left(\frac{E_F}{T}\right) + \frac{3F_0^\sigma}{(1+F_0^\sigma)}e^2\frac{T\tau}{\pi}\left[1-\frac{3}{8}t(T\tau;F_0^\sigma)\right], \qquad (3.43)$$

where the dimensionless function $t(x;F_0^\sigma)$ is defined as

$$t(x;F_0^\sigma) = \frac{8}{3}\int_0^\infty dz\left[\frac{\partial}{\partial z}(z\coth z)-1\right] \times\left\{\frac{xz}{\pi}\{[H(2xz)+h_1(2xz)]\ln 2 + h_4(2xz)\} + \frac{1}{\pi}[1+H(2xz)+4x^2z^2h_3(2xz)]\arctan\frac{1}{2xz} + \frac{xz}{2\pi}\left[\frac{1}{2}+H(2xz)+h_2(2xz)\right]\ln\left(1+\frac{1}{(2xz)^2}\right) + \left[\frac{2}{\pi}\arctan(2xz)-1\right]\right\}. \qquad (3.44)$$

Except for the limiting cases [see Eq. (2.18)] the integral in Eq. (3.44) has to be evaluated numerically. We plot the result for several values of F_0^σ in Fig. 6.

F. Single impurity limit

In the previous sections we obtained the expression for the correction to conductivity averaged over disorder. To complete the calculation we needed to separately average the interaction propagator and use the result to evaluate the integral in Eq. (3.26). In doing this we assumed that the dimensionless conductance of the system is large or in terms of the scattering time $\tau E_F \gg 1$. We have not, however, assumed anything about the relative value of the scattering rate and temperature. In other words, the correction Eq. (3.26) is valid in both the diffusive $T\tau \ll 1$ and ballistic $T\tau \gg 1$ limits. It also describes the cross-over behavior at intermediate temperatures.

The temperature behavior of the interaction correction in the limiting cases can of course be obtained from the general result Eq. (2.16). As we pointed out in Sec. II B, in the diffusive limit our results coincide with the standard theory, Ref. 1. On the other hand the correction in the ballistic limit is subject to conflicting claims in literature.[6,27] Unfortunately, neither result is completely correct. Therefore we discuss the ballistic limit in some detail, starting with diagrams before averaging (i.e., diagrams on Figs. 9). This way we are able to point out exactly which diagram produces the dominant result and which diagrams were missed in existing theories.

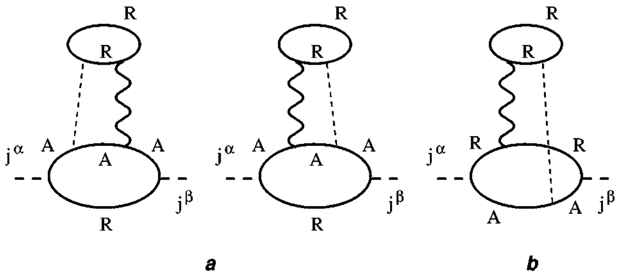

FIG. 17. Single impurity diagrams for Hartree channel.

We begin by discussing the Hartree term. This contribution was considered in Ref. 6 in the framework of the temperature dependent dielectric function. The physical idea was that electrons tend to screen the charged impurities and thus modify the scattering rate. In what follows we show which diagrams describe this process and how to calculate the resulting correction, which appears to be the same (up to a numerical factor miscalculated in Ref. 6; see below for detailed explanation). The important difference between the two approaches is that the impurity screening picture described only the direct (Hartree) interaction, while missing on the exchange part. The latter was later considered in Ref. 27. We think that this consideration is erroneous, and we discuss the Fock term in Sec. III F 2.

1. Single impurity limit for Hartree term

The goal of this discussion is to show which diagrams correspond to the ballistic limit of the Hartree term (as discussed in Sec. II) and how it relates to other interaction corrections we discuss in this paper. The Hartree term corresponds to averaging the two diagrams on Fig. 9, where the wavy line represents a weak interaction potential. In this case the diagram "a3" of Fig. 9 is equal to zero even before the averaging (as a total derivative) and we only need to average the diagrams "a1" and "a2." The rigorous procedure would involve dressing the interaction vertices according to Fig. 10

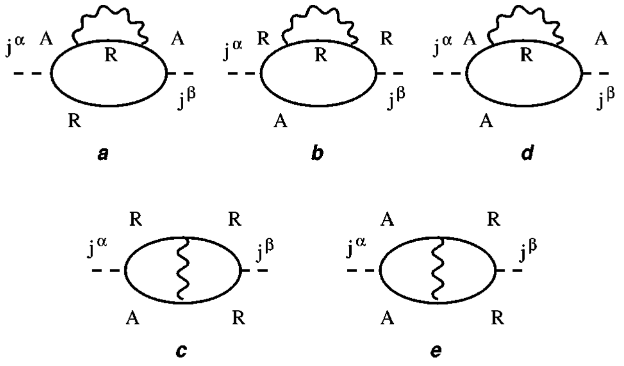

FIG. 18. Fock channel diagrams without impurity lines.

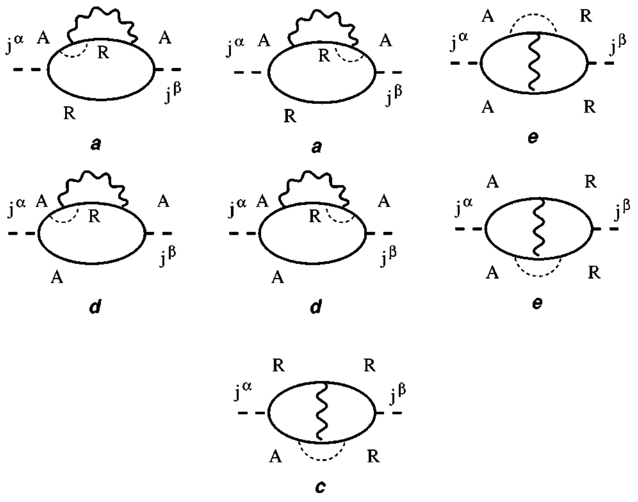

FIG. 19. Single impurity diagrams for Fock channel with the impurity line dressing one interaction vertex.

and adding diffusons Fig. 11 as it was done in Sec. III A (see Fig. 12), evaluating the resulting expression, and finally taking the limit $T\tau\rightarrow\infty$. However, the same result can be obtained by making the expansion by noticing that impurity line brings smallness $1/T\tau$. Therefore, high temperature limit may be studied by considering diagrams on Fig. 17 directly. Such approach is completely equivalent to that of Ref. 6. The result [which can also be obtained from the general expression Eq. (2.16c)] is similar to the one obtained in Ref. 6 (the difference is the extra factor of ln2 found in Ref. 6 due to an error in this reference, which consists in putting the energy of the scattered electron on the Fermi shell rather than integrating over it):

$$\delta\sigma_H=-4\sigma_D\left(\frac{T}{E_F}\right)[-\nu\mathcal{D}(2k_F)] \tag{3.45}$$

[for weak coupling $\mathcal{D}(2k_F)\equiv-V(2k_F)$]. The factor of 4 in Eq. (3.45) can be interpreted as a result of a summation over four spin configurations. Although correct for weak coupling, this factor should be modified when stronger interaction is considered, see discussion above.

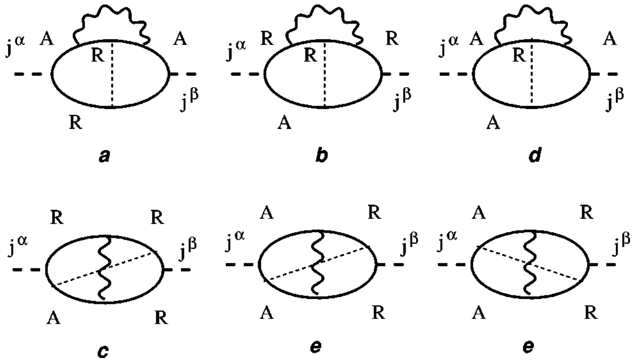

FIG. 20. Single impurity diagrams for Fock channel with the impurity line connecting a retarded and an advanced Green's functions across the diagram.

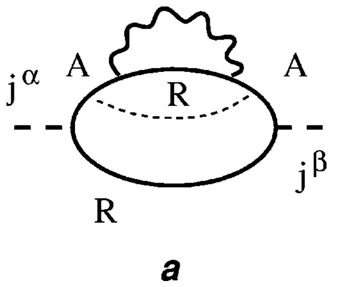

FIG. 21. The single impurity diagram for Fock channel with the impurity line connecting two advanced Green's functions.

2. Fock contribution

In the similar manner one can discuss the single impurity contribution to the Fock term. Again, for weak interaction we could simply expand the result of disorder averaging for the Fock term Eq. (3.11) to the leading order in $1/T\tau$. For Coulomb interaction we would expand Eq. (3.26), since in Eq. (3.11) the special form of the delta-function potential was utilized to eliminate the diagram "b3" on Fig. 9. Diagrammatically, such expansions equivalent to direct evaluation of diagrams without impurity lines (but with averaged electron Green's functions) shown in Fig. 3 and diagrams with only one impurity line shown on Figs. 19–21.

The evaluation of the single impurity diagrams for the Fock term is straightforward and is completely analogous to the Hartree term discussed in the previous subsection. The result can be written as

$$\sigma_F=\frac{e^2\tau T}{\pi}. \tag{3.46}$$

This result contradicts (even in sign) that of Ref. 27. Here we briefly discuss the reason for this contradiction. We notice that one has to be careful to keep track of gauge invariance while evaluating diagrams for the Fock term. Gauge invariance manifests itself in the fact that any interaction at zero momentum gives no contribution to physical quantities, which are expressed diagrammatically as closed loops, see Eq. (3.27). This is indeed the case for Eq. (3.26), where we summed up all the diagrams. On the other hand, any individual diagram is not gauge invariant. In particular, each subset of diagrams in Figs. 18–21 is not gauge invariant. Therefore to obtain the result Eq. (3.46) from these diagrams one has to disregard terms which contain higher than second powers of the scattering rate $1/\tau$. As we already mentioned, the contribution from the plasmon pole is small due to the condition (3.27). However, in Ref. 27 it was claimed otherwise. Namely, diagrams in Fig. 18 were claimed to be important for the plasmon correction and to give a large result, while diagrams in Figs. 19–21 were alleged to be not important for the plasmon correction. This claim explicitly violates gauge invariance and leads to incorrect conclusions. In particular, the plasmon contribution to the conductivity was overestimated by a factor of order of $(v_F/d_sT)\simeq(E_F/T)$.

IV. KINETIC EQUATION APPROACH

Our purpose in this section is to put the treatment of the interaction effects in disordered systems into the framework of the kinetic equation. Even though at this point this will not produce any further physical results, this proves to be more convenient for practical calculations of more sophisticated quantities, such as the Hall coefficient, the thermal conductivity, energy relaxation, etc. The kinetic equation approach is also applicable for the description of nonlinear effects. The main technical advantage of the kinetic equation is that it operates with gauge invariant quantities from the very beginning, unlike the perturbation theory, where each diagram taken separately is not gauge invariant (and may produce nonphysical divergences).

We will present the final form of the kinetic equation in Sec. IV A, and show how to operate with this equation for the conductivity calculation in Sec. IV B. The derivation of this equation based on the Keldysh technique for nonequilibrium systems[28] is presented in Sec. IV C.

A. Final form of the kinetic equation

As usual in the kinetic equation approach, averages of observable quantities are expressed as certain integrals of the distribution function $f(t;\epsilon,\vec{r},\vec{n})$. For instance, the averaged density is

$$\rho(t,\vec{r})=\nu\int_{-\infty}^{\infty}d\epsilon\langle f(t;\epsilon,\vec{r},\vec{n})\rangle_n \tag{4.1a}$$

and the average current is

$$\vec{J}(t,\vec{r})=e\nu v_F\int_{-\infty}^{\infty}d\epsilon\langle\vec{n}f(t;\epsilon,\vec{r},\vec{n})\rangle_n \tag{4.1b}$$

and so on. Here ν is the density of states (entering into linear specific heat of the clean system) at the Fermi surface and v_F is the Fermi velocity, $\vec{n}=(\cos\theta,\sin\theta)$ is the unit vector in the direction of the electron momentum and angular averaging is introduced as

$$\langle\cdots\rangle_n=\int\frac{d\theta}{2\pi}\cdots.$$

The Boltzmann-like equation for the distribution function has the form

$$\left[\partial_t+v_F\vec{n}\vec{\nabla}+ev_F(\vec{n}\vec{E})\frac{\partial}{\partial\epsilon}+\vec{\omega}_c\left(\vec{n}\times\frac{\partial}{\partial\vec{n}}\right)\right]f=\mathrm{St}\{f\}, \tag{4.2}$$

where $\vec{E}$ denotes the external electric field and $\vec{\omega}_c$ is a vector with the magnitude equal to the cyclotron frequency corresponding to an external magnetic field perpendicular to the plane and the direction along the field.

Equations (4.1) and (4.2) neglect energy dependence of the velocity of electrons, which makes it inapplicable for quantities associated with electron-hole asymmetry, such as the thermopower. On the other hand, any component of the thermal and electrical conductivities is still within our description.

All of the interaction effects are taken into account in the collision integral

$$\mathrm{St}\{f\}=\mathrm{St}_{\mathrm{el}}\{f\}+\mathrm{St}_{\mathrm{in}}\{f\}. \tag{4.3}$$

The elastic part of the collision integral describes scattering of electrons by static impurities (we assume pointlike scattering; generalization to the finite range is straightforward) as well as by the self-consistent field generated by all the other electrons:

$$\mathrm{St}_{\mathrm{el}}\{f\}=-\frac{f(t;\epsilon,\vec{r},\vec{n})-\langle f(t;\epsilon,\vec{r},\vec{n})\rangle_n}{\tau}+I_0(t;\epsilon,\vec{n},\vec{r})\times\langle f(t;\epsilon,\vec{r},\vec{n})\rangle_n+n_\alpha I_1^{\alpha\beta}(t;\epsilon,\vec{r})\langle n_\beta f(t;\epsilon,\vec{r},\vec{n})\rangle_n. \tag{4.4}$$

The effect of the self-consistent field is described by the last two terms, where we introduce notations

$$I_0(t;\epsilon,\vec{n},\vec{r})=-\frac{8}{\tau}\int\frac{d\omega}{2\pi}\left\{n_\alpha K_0^{\alpha\beta}(\omega)\langle n_\beta f(t;\epsilon-\omega,\vec{r},\vec{n})\rangle_n+\frac{n_\alpha L_0^{\alpha\beta}(\omega)}{2}\left[\nabla_\beta+eE_\beta\frac{\partial}{\partial\epsilon}\right]\langle f(t;\epsilon-\omega,\vec{r},\vec{n})\rangle_n\right\} \tag{4.5a}$$

$$I_1^{\alpha\beta}(t;\epsilon,\vec{r})=-\frac{8}{\tau}\int\frac{d\omega}{2\pi}K_1^{\alpha\beta}(\omega)\langle f(t;\epsilon-\omega,\vec{r},\vec{n})\rangle_n. \tag{4.5b}$$

The collision integral Eq. (4.4) preserves the number of particles on a given energy shell: integrating $\mathrm{St}_{\mathrm{el}}\{f(t;\epsilon,\vec{r},\vec{n})\}$ over directions of $\vec{n}$ gives zero for any value of ϵ [see also Eq. (4.58)].

The term I_1 expresses enhanced momentum relaxation due to static disorder. The physics of this term was discussed in detail in Sec. II. The term I_0 describes electron scattering by nonequilibrium nonlocal Focklike potential created by all other electrons. This process is responsible for generation of the finite drift velocity of electrons. One can easily see that I_0 vanishes in the equilibrium situation $f(\epsilon,\vec{n},\vec{r})=f[\epsilon+e\varphi(\vec{r})]$, $\nabla_\alpha\varphi=-E_\alpha$.

The kernels K_0, K_1, and L_0 entering into Eqs. (4.5) can be expressed in terms of interaction propagators and the propagators describing semiclassical dynamics of noninteracting electrons. Explicitly,

$$K_1^{\alpha\beta}(\omega)=\mathrm{Im}\int\frac{d^2q}{(2\pi)^2}\mathcal{D}^R(\omega,\vec{q})\times\left\{\langle n_\alpha D\rangle\langle Dn_\beta\rangle-\frac{\delta_{\alpha\beta}}{2}\left(\langle D\rangle\langle D\rangle+i\frac{\partial}{\partial\omega}\langle D\rangle\right)\right\}, \tag{4.6a}$$

$$K_0^{\alpha\beta}(\omega)=\mathrm{Im}\int\frac{d^2q}{(2\pi)^2}\mathcal{D}^R(\omega,\vec{q})\times\left\{\langle n_\alpha D n_\beta\rangle\langle D\rangle-\frac{i}{v_F}\frac{\partial}{\partial q_\alpha}\langle D n_\beta\rangle-\langle D n_\alpha\rangle\langle D n_\beta\rangle\right\},\tag{4.6b}$$

$$L_0^{\alpha\beta}(\omega)=-\mathrm{Re}\int\frac{d^2q}{(2\pi)^2}\mathcal{D}^R(\omega,\vec{q})\left\{\langle D\rangle\frac{\partial}{\partial q_\beta}\langle n_\alpha D\rangle-\langle D n_\alpha\rangle\frac{\partial}{\partial q_\beta}\langle D\rangle-\left\langle D n_\alpha\frac{\partial}{\partial q_\beta}D\right\rangle\right\}.\tag{4.6c}$$

Here, $\mathcal{D}^R(\omega)=\mathcal{D}^A(-\omega)$ denotes the retarded interaction propagator [see, i.e., Eq. (3.30)] and we introduce the short hand notation for the angular averaging

$$\langle aDb\rangle\equiv\int\frac{d\theta d\theta'}{(2\pi)^2}a(\vec{n})D(\vec{n},\vec{n}';\omega,q)b(\vec{n}'),$$

$$\langle aDbDc\rangle\equiv\int\frac{d\theta d\theta' d\theta''}{(2\pi)^3}\times a(\vec{n})D(\vec{n},\vec{n}')b(\vec{n}')D(\vec{n}',\vec{n}'')c(\vec{n}'')$$

for arbitrary functions a,b. The function $D(\vec{n},\vec{n}';\omega,\vec{q})$ describes the classical motion of a particle on the energy shell ϵ_F in a magnetic field

$$\left[-i\omega+iv_F\vec{n}\vec{q}+\vec{\omega}_c\left(\vec{n}\times\frac{\partial}{\partial\vec{n}}\right)\right]D(\vec{n},\vec{n}';\omega,\vec{q})+\frac{1}{\tau}[D(\vec{n},\vec{n}';\omega,\vec{q})-\langle D(\vec{n},\vec{n}';\omega,\vec{q})\rangle_n]=2\pi\delta(\widehat{\vec{n}\vec{n}'}).\tag{4.7}$$

As we have already mentioned, the elastic part of the collision integral is nulled by a distribution function of the form $f[\epsilon+e\varphi(\vec{r})]$ for an arbitrary *f*. It is the inelastic term that is responsible for establishing the local thermal equilibrium and it has the standard form

$$\mathrm{St}_{\mathrm{in}}\{f\}=\int d\omega\int d\epsilon_1 A(\omega)f(\epsilon_1)[1-f(\epsilon_1-\omega)]\times\{-f(\epsilon)[1-f(\epsilon+\omega)]+[1-f(\epsilon)]f(\epsilon-\omega)\}\tag{4.8a}$$

$$f(\epsilon)=\langle f(t;\epsilon,\vec{r},\vec{n})\rangle_n.$$

The kernel $A(\omega)$ describes matrix elements for inelastic processes in both ballistic and diffusive limits. The explicit expression for this kernel is

$$A(\omega)=\frac{2\nu}{\pi}\int\frac{d^2q}{(2\pi)^2}[\mathrm{Re}\langle D\rangle]^2|\mathcal{D}^R(\omega,\vec{q})|^2,\tag{4.8b}$$

where $\langle D\rangle$ is given by the solution to Eq. (4.7) averaged over angles.

The above equations are written for the interaction in the singlet channel only. In a situation where both triplet and singlet channels are present, but the distribution function does not have a spin structure (no Zeeman splitting or nonequilibrium spin occupation present), one has to replace

$$\mathcal{D}^R\to\mathcal{D}^R+\mathrm{Tr}\hat{\mathcal{D}}_T^R\tag{4.9}$$

in Eqs. (4.6) and

$$|\mathcal{D}^R|^2\to|\mathcal{D}^R|^2+\mathrm{Tr}\{\hat{\mathcal{D}}_T^R[\hat{\mathcal{D}}_T^R]^\dagger\}\tag{4.10}$$

in Eq. (4.8b).

Equations (4.2)–(4.8) constitute the complete system of transport equations with the leading interaction corrections taken into account. They may be used to study both linear and nonlinear response. We reiterate that they do not include effects of electron-hole asymmetry and in this form cannot produce finite thermopower. The Hall effect, the thermal conductivity, and energy relaxation, however, are included and will be studied in a subsequent publication.[14] In the following subsection we apply the kinetic equation approach to study the interaction correction to the conductivity at intermediate and low temperatures and reproduce the results obtained in Sec. III by means of diagrammatic technique. The reason for doing so is to show how the kinetic equation works and to demonstrate explicitly that both approaches are equivalent.

Closing our description of the structure of the kinetic equation, we discuss the range of its applicability. Any kinetic equation implies that the distribution function changes slowly on the spatial scale of the Fermi wavelength λ_F and on the time scale $1/\epsilon_F$. In our case, the conditions are more restrictive. First, in the interaction correction to the elastic collision integral we take into account only the effect of the interaction on the zeroth and first angular harmonics of the distribution function. This implies that the equation gives the correct description for the interaction effects on the conductivity and diffusion, whereas it is not correct for description of the quantities involving higher angular harmonics. Second, we made a gradient expansion in Eq. (4.5a) and only took into account terms linear in the electric field. This implies that the distribution function changes slowly on the spatial scale $L_T=\min[\hbar v_F/T, v_F(\hbar\tau/T)^{1/2}]$, and on the time scale of the order of $\hbar/T$. The electric field expansion is justified by the condition $eEL_T\ll T$. One can check that both these conditions are satisfied, if the energy relaxation time is much longer than the time for the elastic collisions. We also did not include quantum effects of the magnetic field. This is justified at $\omega_c\ll\max(T/\hbar,\tau^{-1})$.

Finally, the interaction part of Eq. (4.4) is calculated in the first loop approximation. It means, that it has to be considered as the first order correction to $1/\tau$. If one is interested in the next order interaction correction to the elastic part, one should take into account the second loop correction, which is not considered in the present paper. On the contrary, the inelastic part (4.8) can be considered in all orders to find the zero angular momentum part of the distribution function; the only assumption here is the validity of the Fermi-liquid description at energies smaller than ϵ_F.

B. Conductivity calculation

In order to calculate the conductivity at zero magnetic field $\omega_c=0$ we look for the solution of Eqs. (4.2)–(4.4) in the form

$$f(\vec{n},\epsilon)=f_F(\epsilon)+\vec{n}\vec{\Gamma}(\epsilon), \tag{4.11}$$

where $f_F(\epsilon)=1/(e^{\epsilon/T}+1)$ is the Fermi distribution function (all the energies are counted from the Fermi level), and Γ is the quantity to be found and it is proportional to the electric field.

We substitute Eq. (4.11) into Eqs. (4.2), (4.4), (4.5), and (4.8). The inelastic part of the collision integral [see Eq. (4.8)] obviously vanishes, as effects of the heating are proportional to at least the second power of the electric field. As a result, we obtain an equation for Γ:

$$\begin{aligned} ev_F E_\alpha \frac{\partial f_F(\epsilon)}{\partial\epsilon} = &-\frac{\Gamma_\alpha(\epsilon)}{\tau} - \frac{4}{\tau}\int\frac{d\omega}{2\pi}[K_1^{\alpha\beta}(\omega)f_F(\epsilon-\omega) \\ &\times\Gamma_\beta(\epsilon)+K_0^{\alpha\beta}(\omega)f_F(\epsilon)\Gamma_\beta(\epsilon-\omega)] \\ &-\frac{4f_F(\epsilon)}{\tau}\int\frac{d\omega}{2\pi}L_0^{\alpha\beta}(\omega)eE_\beta\frac{\partial}{\partial\epsilon}f_F(\epsilon-\omega). \end{aligned} \tag{4.12}$$

We solve Eq. (4.12) by iterations. As usual for kinetic equations, the solution is expressed in terms of the unperturbed distribution function $f_F(\epsilon)$ and the kernels, which in this case are given by Eq. (4.6):

$$\begin{aligned} \Gamma_\alpha(\epsilon) = &-ev_F\tau E_\alpha\frac{\partial f_F(\epsilon)}{\partial\epsilon}+4ev_F\tau\int\frac{d\omega}{2\pi}\bigg[K_1^{\alpha\beta}(\omega)f_F(\epsilon-\omega) \\ &\times\frac{\partial f_F(\epsilon)}{\partial\epsilon}+K_0^{\alpha\beta}(\omega)f_F(\epsilon)\frac{\partial f_F(\epsilon-\omega)}{\partial\epsilon}\bigg]E_\beta \\ &-4f_F(\epsilon)\int\frac{d\omega}{2\pi}L_0^{\alpha\beta}(\omega)eE_\beta\frac{\partial}{\partial\epsilon}f_F(\epsilon-\omega). \end{aligned} \tag{4.13}$$

Substituting Eqs. (4.13) into Eq. (4.11) and the result into Eq. (4.1b), we integrate over ϵ and find the conductivity

$$\sigma=\sigma_D+\delta\sigma, \tag{4.14a}$$

$$\frac{\delta\sigma}{\sigma_D}=\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\frac{\partial}{\partial\omega}\left(\omega\coth\frac{\omega}{2T}\right)\left[K_0(\omega)-K_1(\omega)-\frac{L_0(\omega)}{v_F\tau}\right], \tag{4.14b}$$

where the Drude conductivity is $\sigma_D=e^2\nu v_F^2\tau/2$. Here we used the fact that in the absence of the magnetic field all the kernels are diagonal, $K_i^{\alpha\beta}=\delta_{\alpha\beta}K_i$, $L_0^{\alpha\beta}=\delta_{\alpha\beta}L_0$. We also used the identities

$$2\int_{-\infty}^{\infty}d\epsilon f_F(\epsilon)\frac{\partial f_F(\epsilon-\omega)}{\partial\epsilon}=-1+\frac{\partial}{\partial\omega}\left(\omega\coth\frac{\omega}{2T}\right),$$

$$\int_{-\infty}^{\infty}d\omega K_i(\omega)=\int_{-\infty}^{\infty}d\omega L_0(\omega)=0.$$

In order to derive explicit expressions for the kernels K_i and L_0 we have to solve Eq. (4.7) for the function D in the absence of the magnetic field. The result can be written as

$$\begin{aligned} D(\vec{n},\vec{n}';\omega,\vec{q}) = &2\pi\delta(\widehat{\vec{n}\vec{n}'})D_0(\vec{n},\omega,\vec{q}) \\ &+D_0(\vec{n},\omega,\vec{q})D_0(\vec{n}',\omega,\vec{q})\frac{C}{C\tau-1}, \end{aligned} \tag{4.15}$$

where D_0 denotes the solution of Eq. (4.7) without the angular averaged term (and in the absence of the magnetic field)

$$D_0(\vec{n},\omega,\vec{q})=\frac{1}{-i\omega+iv_F\vec{n}\vec{q}+1/\tau}.$$

Here we used the short-hand notation

$$C=\sqrt{(-i\omega+1/\tau)^2+v_F^2q^2},$$

which is similar to the notation S used in Sec. III [in fact, $C=S^*$, see Eq. (3.9b)]. Substituting Eq. (4.15) into Eqs. (4.6a)–(4.6c) and performing the angular integration we arrive to

$$\begin{aligned} K_1(\omega) = &-\mathrm{Im}\int\frac{qdq}{4\pi}\mathcal{D}^R(\omega,q) \\ &\times\left\{\frac{1}{v_F^2q^2}\left(\frac{C-(-i\omega+1/\tau)}{C-1/\tau}\right)^2+\frac{C-(-i\omega+1/\tau)}{C(C-1/\tau)^2}\right\}, \end{aligned} \tag{4.16a}$$

$$\begin{aligned} K_0(\omega) = &\,\mathrm{Im}\int\frac{qdq}{4\pi}\mathcal{D}^R(\omega,q) \\ &\times\left\{\frac{C-(-i\omega+1/\tau)}{C(C-1/\tau)^2}+\frac{[C-(-i\omega+1/\tau)]^2}{C(C-1/\tau)}\frac{1}{v_F^2q^2}\right\}, \end{aligned} \tag{4.16b}$$

$$\begin{aligned} \frac{L_0(\omega)}{v_F\tau} = &-\mathrm{Im}\int\frac{qdq}{4\pi}\mathcal{D}^R(\omega,q) \\ &\times\left\{\frac{3}{2\tau}\frac{v_F^2q^2}{C^3(C-1/\tau)^2}+\frac{v_F^2q^2}{C^3}\frac{1/\tau^2}{(C-1/\tau)^3}\right\}. \end{aligned} \tag{4.16c}$$

Together with the conductivity correction Eq. (4.14b) the above expressions Eq. (4.16) are identical to Eq. (3.26) obtained in Sec. III by means of the standard perturbation

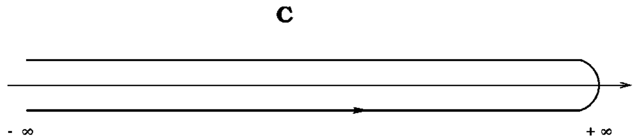

FIG. 22. The Keldysh contour.

theory. Thus the kinetic equation approach is completely equivalent to such diagrammatic calculation.

Integration over the wave vector q requires the knowledge of the interaction propagator. Substituting Eq. (3.30) for the singlet channel and Eq. (3.40) for the triplet channel and performing the straightforward integration we arrive to the results in Sec. II B.

C. Derivation of the kinetic equation

In this section we derive the kinetic equation discussed in Sec. IV A. For simplicity we show the derivation for the case of the singlet channel interaction Eq. (3.17). The case of the triplet channel can be treated in the same manner with minor differences (introduction of extra spin indices) described in the end of this section. To keep the discussion at the same level as in Sec. III, we treat the Fermi-liquid parameter F^{ρ} in Eq. (3.17) as a constant, similar to our treatment of the triplet channel in Sec. III E.

1. Keldysh formalism

Here we summarize the results originally obtained by Keldysh[28] that enable us to calculate correlation functions for any nonequilibrium distribution. Let us first consider a Green's function of electrons before disorder averaging. The electron-electron interaction is described by the Hamiltonian Eq. (3.17). In the path-integral formulation it can be decoupled from fermion operators using an auxiliary bosonic field ϕ. Then the Green's function can be written as

$$\hat{G}(x_1,x_2)=\int[\mathcal{D}\phi]\hat{G}(x_1,x_2|\phi)e^{-iS_B[\phi]}, \tag{4.17}$$

with the action defined as

$$S_B[\phi]=\int_{-\infty}^{\infty}dtd^2r\left\{\frac{1}{2}\phi^T V_0^{-1}\sigma_3\phi\right\}+i\ln Z[\phi], \tag{4.18}$$

where $-V_0$ is the (bare; following Eq. (3.17) $V_0=V(q)+F_0^{\rho}/\nu$) electron-electron interaction propagator and Z is the partition function

$$Z[\phi]=\langle\mathbf{T}_C e^{-iS_F[\phi,\psi]}\rangle \tag{4.19}$$

$$S_F[\phi,\psi]=\int_{-\infty}^{\infty}dtd^2r\{\psi^{\dagger}\phi_{\alpha}\hat{\gamma}^{\alpha}\psi\}, \tag{4.20}$$

where $\hat{\sigma}_z=\mathrm{diag}(-1,1)$ is the Pauli matrix in the Keldysh space.

In the above expressions all the fields are defined on the Keldysh time contour shown in Fig. 22. In particular, the fermionic fields $\psi^{\dagger}$ and ψ (as well as the bosonic field ϕ) can be treated as doublets

$$\psi=\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}, \tag{4.21}$$

where we adopt the notation that fields with a $-$ $(+)$ subscript (also referred to by Greek letters in this section) reside on the lower (upper) part of the contour on Fig. (22). The time dependent fermionic operators ψ are taken in the interaction represntation

$$-i\partial_t\psi(t)=[\hat{H}_1(t);\psi(t)],$$

where $\hat{H}_1$ is the one-electron Hamiltonian which includes the static disorder potential as well as external fields.

Consequently, the Green's function in Eq. (4.17) is a 2×2 matrix. Time ordering along the contour is denoted in Eq. (4.19) by $\mathbf{T}_C$. Matrices $\hat{\gamma}^{\alpha}$ in Eq. (4.20) are defined as

$$\hat{\gamma}^{+}=\begin{pmatrix}-1&0\\0&0\end{pmatrix},\qquad \hat{\gamma}^{-}=\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

The Green's function $\hat{G}(x_1,x_2|\phi)$ in Eq. (4.17) is given by

$$\hat{G}(x_1,x_2|\phi)=\frac{1}{Z[\phi]}\langle T_C\psi_{\alpha}^{\dagger}(x_1)\psi_{\beta}(x_2)e^{-iS_F[\phi,\psi]}\rangle. \tag{4.22}$$

Here, as well as in Eq. (4.19) the angular brackets $\langle\cdots\rangle$ denote quantum-mechanical averaging. In this section we will use the short hand notation

$$x_i\equiv(t_i,\vec{r}_i).$$

The bosonic action Eq. (4.18) can be treated in the saddle point approximation

$$\langle e^{-iS_B[\phi]}\rangle=e^{-iF[\phi]}, \tag{4.23a}$$

$$F[\phi]=F[\phi=0]+\frac{1}{2}\phi^T\hat{\Pi}\phi+\mathcal{O}(\phi^3), \tag{4.23b}$$

where Π is the electronic polarization operator, defined as

$$\Pi_{\alpha\beta}(x_1,x_2)=\left.\frac{\delta^2F}{\delta\phi_{\alpha}(x_1)\delta\phi_{\beta}(x_2)}\right|_{\phi=0}. \tag{4.24}$$

The quadratic expansion in Eq. (4.23b) is justified, provided that the fields ϕ are slowly changing on the scale much larger than λ_F.

Let us now average the Green's function Eq. (4.17) over disorder:

$$\langle\hat{G}(x_1,x_2)\rangle_{\mathrm{dis}}=\int[\mathcal{D}\phi]\langle\hat{G}(x_1,x_2|\phi)\rangle_{\mathrm{dis}}e^{-i\langle S_B[\phi]\rangle_{\mathrm{dis}}}, \tag{4.25}$$

where $\langle\cdots\rangle_{\mathrm{dis}}$ hereafter denotes averaging over disorder. Here we average the electronic Green's function Eq. (4.22) separately from the bosonic action Eq. (4.23). This approximation means that we neglect correlations between mesoscopic fluctuations of the polarizability in Eq. (4.23b) and the fermionic operators in Eq. (4.22) (which describe the motion of conduction electrons). This is the same approximation we used in Sec. III. It is justified by the well known fact that mesoscopic fluctuations are smaller than average quantities by a factor of the order $1/(E_F\tau)^2$.

It is convenient[31] to rotate the Keldysh basis as follows:

$$\hat{G}\rightarrow\frac{1}{2}\hat{\sigma}_x\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\hat{G}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}. \tag{4.26}$$

In the new basis the Green's function Eq. (4.22) has the form

$$\hat{G}(x_1,x_2|\phi)=\begin{pmatrix}G^R(x_1,x_2|\phi) & G^K(x_1,x_2|\phi)\\ G^Z(x_1,x_2|\phi) & G^A(x_1,x_2|\phi)\end{pmatrix}. \tag{4.27}$$

After the averaging over the bosonic field and over the disorder according to Eq. (4.25) the entries in Eq. (4.27) acquires the following meaning/where after integrating over the bosonic field ϕ the diagonal elements $G^{R(A)}$ become the retarded (advanced) Green's functions of the electron system

$$\langle G^R(t_1,t_2)\rangle=-i\,\eta(t_1-t_2)\langle\psi(t_1)\psi^\dagger(t_2)+\psi^\dagger(t_2)\psi(t_1)\rangle,$$

$$\langle G^A(t_1,t_2)\rangle=i\,\eta(t_2-t_1)\langle\psi(t_1)\psi^\dagger(t_2)+\psi^\dagger(t_2)\psi(t_1)\rangle,$$

where $\eta(t)$ is the Heaviside step function. The lower diagonal element vanishes due to the causality,

$$\langle G^Z(t_1,t_2)\rangle=0,$$

even before the disorder averaging. Finally, the upper off-diagonal element (the so-called Keldysh Green's function) is related to the one particle density matrix

$$\langle G^K(t_1,t_2)\rangle=-i\langle\psi(t_1)\psi^\dagger(t_2)-\psi^\dagger(t_2)\psi(t_1)\rangle, \tag{4.28}$$

the quantum mechanical averaging is performed with an arbitrary distribution function to be found from the solution of the kinetic equation.

The bosonic field in the rotated basis has the two components

$$\phi_{1(2)}=\frac{1}{2}(\phi_+\pm\phi_-) \tag{4.29}$$

which are described by the propagators

$$\langle\phi_1(t_1,\vec{r}_1)\phi_1(t_2,\vec{r}_2)\rangle=\frac{i}{2}\mathcal{D}^K(t_1,t_2;\vec{r}_1,\vec{r}_2), \tag{4.30a}$$

$$\langle\phi_1(t_1,\vec{r}_1)\phi_2(t_2,\vec{r}_2)\rangle=\frac{i}{2}\mathcal{D}^R(t_1,t_2;\vec{r}_1,\vec{r}_2), \tag{4.30b}$$

$$\langle\phi_2(t_1,\vec{r}_1)\phi_1(t_2,\vec{r}_2)\rangle=\frac{i}{2}\mathcal{D}^A(t_1,t_2;\vec{r}_1,\vec{r}_2), \tag{4.30c}$$

$$\langle\phi_2(t_1,\vec{r}_1)\phi_2(t_2,\vec{r}_2)\rangle=0. \tag{4.30d}$$

The coupling Eq. (4.20) between the fermionic and bosonic fields in the rotated basis has the form

$$\psi^\dagger\phi_\alpha\hat{\gamma}^\alpha\psi\rightarrow\psi^\dagger\begin{pmatrix}\phi_1 & \phi_2\\ \phi_2 & \phi_1\end{pmatrix}\psi. \tag{4.31}$$

The propagators Eq. (4.30) are solutions of the Dyson equations

$$\hat{\mathcal{D}}(1,2)=\hat{\mathcal{D}}_0(1,2)+\int d3d4\hat{\mathcal{D}}_0(1,3)\hat{\Pi}(3,4)\hat{\mathcal{D}}(4,2) \tag{4.32}$$

$$\hat{\mathcal{D}}=\begin{pmatrix}\mathcal{D}^R & \mathcal{D}^K\\ 0 & \mathcal{D}^A\end{pmatrix},\qquad \hat{\Pi}=\begin{pmatrix}\Pi^R & \Pi^K\\ 0 & \Pi^A\end{pmatrix}$$

and we introduced the short hand notation $(i)\equiv(t_i,\vec{r}_i)$. The bare interaction propagators are

$$\mathcal{D}_0^R=\mathcal{D}_0^A=-[V(\mathbf{r}_1-\mathbf{r}_2)+F_0^\rho\delta(\mathbf{r}_1-\mathbf{r}_2)]\delta(t_1-t_2),$$

$$\mathcal{D}_0^K=0. \tag{4.33}$$

Any classical external field takes identical values on the two branches of the contour and, hence, in the rotated basis has only a diagonal component.

The matrix Green's function (4.27) satisfies the equation

$$\left\{i\partial_{t_1}+E_F-\frac{[-i\vec{\nabla}_{r_1}+\vec{A}_{\text{ext}}(x_1)]^2}{2m}-\hat{\phi}(x_1)-U(\vec{r}_1)-\varphi_{\text{ext}}(x_1)\right\}\hat{G}(x_1,x_2|\phi)=\hat{I}\delta(x_1-x_2), \tag{4.34}$$

where $U(\vec{r})$ is the potential due to the static disorder, $\vec{A}_{\text{ext}}(x_1)$ and $\varphi_{\text{ext}}(x_1)$ are the vector and scalar potential due to the external electric and magnetic fields.

$$e\vec{E}=\partial_t\vec{A}_{\text{ext}}-\vec{\nabla}\varphi_{\text{ext}},\qquad e\vec{B}=-\frac{1}{c}\vec{\nabla}\times\vec{A}_{\text{ext}}. \tag{4.35}$$

Equation (4.34) is the basis for the further consideration. One can perform the disorder average in Eq. (4.34) in the leading in $1/(E_F\tau)$ approximation, which amounts to summation over all the nonintersecting impurity lines one obtains

$$\left\{i\partial_{t_1}+E_F-\frac{[-i\vec{\nabla}_{r_1}+\vec{A}_{\text{ext}}(x_1)]^2}{2m}-\hat{\phi}(x_1)-\varphi_{\text{ext}}(x_1)\right\}\times\hat{G}(x_1,x_2|\phi)=\hat{I}\delta(x_1-x_2)+\int dx_3\hat{\Sigma}(x_1,x_3|\phi)\hat{G}(x_3,x_2|\phi); \tag{4.36}$$

$$\hat{\Sigma}(x_1,x_2|\phi)=\frac{\delta(r_1-r_2)}{2\pi\nu\tau}\hat{G}(x_1,x_2|\phi).$$

Equation (4.36) allows for semiclassical treatment introduced in Refs. 29,30, and described in great details in Ref. 31. Since we have already averaged the equation of motion over disorder, the semiclassical approximation now amounts to averaging the Green's function $\hat{G}(x_1,x_2|\phi)$ over the distance from the Fermi surface. This is done in two steps:

$$\hat{G}(t_1,t_2;\vec{p};\vec{R})=\int d^2 r e^{i\vec{P}\cdot\vec{r}}\hat{G}(x_1,x_2|\phi), \qquad (4.37)$$

$$\vec{r}=\vec{r}_1-\vec{r}_2; \qquad \vec{R}=\frac{1}{2}(\vec{r}_1+\vec{r}_2),$$

$$\vec{P}=\vec{p}-\frac{1}{2}[\vec{A}_{\mathrm{ext}}(t_1,\vec{R})+\vec{A}_{\mathrm{ext}}(t_2,\vec{R})];$$

$$\hat{g}(t_1,t_2;\vec{n},\vec{r})=\frac{i}{\pi}\int_{-\infty}^{\infty}d\xi\hat{G}\left(t_1,t_2;\vec{n}\left[p_F+\frac{\xi}{v_F}\right];\vec{r}\right). \qquad (4.38)$$

Since we follow the avenue of Ref. 31, we will skip further intermediate steps, and use the semiclassical equation written in the next subsection.

2. Eilenberger equation

The dynamics of the electron matrix Green's function is then described by the Eilenberger equation[29]

$$\left[\tilde{\partial}_t+v_F\vec{n}\tilde{\vec{\nabla}}+\vec{\omega}_c\left(\vec{n}\times\frac{\partial}{\partial\vec{n}}\right)\right]\hat{g}=\frac{\hat{g}\langle\hat{g}\rangle_n-\langle\hat{g}\rangle_n\hat{g}}{2\tau}, \qquad (4.39)$$

where angular averaging is defined as before

$$\langle\cdots\rangle_n=\int\frac{d\theta}{2\pi}\cdots, \qquad \vec{n}=(\cos\theta,\sin\theta),$$

and the covariant derivatives in Eq. (4.39) are defined as

$$\tilde{\partial}_t\hat{g}=\partial_{t_1}\hat{g}+\partial_{t_2}\hat{g}+i\hat{\varphi}(t_1,r)\hat{g}-i\hat{g}\hat{\varphi}(t_2,r), \qquad (4.40a)$$

$$\tilde{\vec{\nabla}}\hat{g}=\vec{\nabla}\hat{g}+i\hat{\vec{A}}(t_1,\vec{r})\hat{g}-i\hat{g}\hat{\vec{A}}(t_2,\vec{r}). \qquad (4.40b)$$

Here $\hat{g}$ is a matrix in Keldysh space,

$$\hat{g}(t_1,t_2;\vec{n},\vec{r})=\begin{pmatrix} g^R & g^K \\ g^Z & g^A \end{pmatrix}, \qquad (4.41)$$

and we will suppress the coordinate and the time arguments unless otherwise is stated. A product of such matrices should be understood as a matrix product in Keldysh space and a convolution in time:

$$[\hat{g}(\vec{n},\vec{r})\hat{g}(\vec{n}_1,\vec{r})]_{ij}$$
$$\equiv\int_{-\infty}^{\infty}dt_3\sum_k[\hat{g}(t_1,t_3;\vec{n},\vec{r})]_{ik}[\hat{g}(t_3,t_2;\vec{n}_1,\vec{r})]_{kj}, \qquad (4.42)$$

and solutions of the homogeneous equation (4.39) are subject to the constraints

$$\hat{g}(\vec{n},\vec{r})\hat{g}(\vec{n},\vec{r})=\hat{I}^K, \qquad \int_{-\infty}^{\infty}dt \ \mathbf{Tr} \ \hat{g}(t,t;\vec{n},\vec{r})=0, \qquad (4.43)$$

where

$$[I^K]_{ij}=\delta_{ij}\delta(t_1-t_2).$$

The scalar and vector potentials in Eq. (4.40) have the following structure in the Keldysh space

$$\hat{\vec{A}}(t,\vec{r})=\begin{pmatrix}\vec{A}_{\mathrm{ext}}(t,\vec{r}) & 0\\ 0 & \vec{A}_{\mathrm{ext}}(t,\vec{r})\end{pmatrix}, \qquad (4.44)$$

$$\hat{\varphi}(t,\vec{r})=\begin{pmatrix}\varphi_{\mathrm{ext}}(t,\vec{r})+\phi_1(t,\vec{r}) & \phi_2(t,\vec{r})\\ \phi_2(t,\vec{r}) & \varphi_{\mathrm{ext}}(t,\vec{r})+\phi_1(t,\vec{r})\end{pmatrix},$$

where φ_{ext} and $\vec{A}_{\mathrm{ext}}$ are the external (classical) potentials due to the electric field $\vec{E}$,

$$e\vec{E}=\partial_t\vec{A}_{\mathrm{ext}}-\vec{\nabla}\varphi_{\mathrm{ext}} \qquad (4.45)$$

acting on the electron system, and $\phi_{1,2}(t,\vec{r})$ are the auxiliary fluctuating fields decoupling the interaction in the singlet channel. Because the singlet channel describes processes with small momentum transfers (smaller than q^*, see Sec. III B), the fields $\phi_{1,2}(t,\vec{r})$ vary slowly on the scale of the $1/q^*$.

The condition (4.30d) enforces causality of the physical response functions. It is worth noticing that the decoupling of interaction can be performed also using a fluctuating vector potential; our choice is strictly a matter of taste.

In this formalism any observable quantity described by one electron operator $\mathcal{O}(\hat{p},\hat{r})$ is given by [see Eqs. (4.28) and (4.38)]

$$\mathcal{O}(t,\vec{r})=-\nu\int\frac{d\theta}{2\pi}\mathcal{O}(p_F\vec{n},\vec{r})$$
$$\times\lim_{t_1\to t}\left[\frac{\pi}{2}\langle g^K(t_1,t;\vec{n},\vec{r})\rangle_\phi+\varphi_{ext}(t,\vec{r})\right], \qquad (4.46)$$

where $\langle\cdots\rangle_\phi$ stands for averaging over both auxiliary fields $\phi_{1,2}$ fluctuating according to Eqs. (4.30). The last term in brackets is a consequence of the ultraviolet anomaly, and its form is enforced by the requirement of the gauge invariance.

Finally, the electronic polarization operators are determined [see Eqs. (4.24) and (4.38)] as variational derivatives of the solutions to the Eilenberger equation (4.39):

$$\Pi^R(1,2)=\Pi^A(2,1)=\nu\int\frac{d\theta}{2\pi}\left(\delta_{12}+\frac{\pi\langle\delta g^K(t_1,t_1;\vec{n},\vec{r}_1)\rangle_\phi}{2\delta\phi_1(t_2,\mathbf{r}_2)}\right), \qquad (4.47)$$

$$\Pi^K(1,2)=\nu\pi\int\frac{d\theta}{2\pi}\times\frac{\langle\delta g^K(t_1,t_1;\vec{n},\vec{r}_1)+\delta g^Z(t_1,t_1;\vec{n},\vec{r}_1)\rangle_\phi}{2\,\delta\phi_2(t_2,\vec{r}_2)}.$$

3. Derivation of the kinetic equation

Our goal now is to obtain an equation for the Keldysh function averaged over the fluctuating fields, $\langle g^K(t_1,t_1;\vec{n},\vec{r}_1)\rangle_\phi$. It is this quantity that determines physical observables, see Eq. (4.46). We will do this using the noncrossing approximation for bosonic propagators (i.e., the first loop approximation for the collision integral), see Fig. 16. This approximation is justified provided that the resulting dynamics for the electrons (characterized by time τ_ϵ) is slow in comparison with motion of relevant bosonic mode $T\tau_\epsilon \gg 1$.

To do so, we notice that only two components of the matrix $\hat{g}$ are independent, and the other two are fixed by the constraint (4.43). For our purposes, we choose to fix the diagonal components

$$g^R=\sqrt{1-g^Kg^Z};\qquad g^A=-\sqrt{1-g^Zg^K},\tag{4.48}$$

where the square root should be understood in operator sense: as a sum of its Taylor series, with all arising products hereafter being time convolutions, similar to Eq. (4.42). The two remaining independent components of the Eilenberger equation have the explicit form

$$\begin{aligned}\left[\tilde{\partial}_t+v_F\vec{n}\tilde{\vec{\nabla}}+\vec{\omega}_c\left(\vec{n}\times\frac{\partial}{\partial\vec{n}}\right)\right]g^Z&=-i[\phi_1(t_1,\vec{r})-\phi_1(t_2,\vec{r})]g^Z-i\phi_2(t_1,\vec{r})g^R\\&\quad+ig^A\phi_2(t_2,\vec{r})+\frac{1}{2\tau}[g^Z\langle g^R\rangle_n\\&\quad-\langle g^Z\rangle_ng^R+g^A\langle g^Z\rangle_n-\langle g^A\rangle_ng^Z],\end{aligned}\tag{4.49a}$$

$$\begin{aligned}\left[\tilde{\partial}_t+v_F\vec{n}\tilde{\vec{\nabla}}+\vec{\omega}_c\left(\vec{n}\times\frac{\partial}{\partial\vec{n}}\right)\right]g^K&=-i[\phi_1(t_1,\vec{r})-\phi_1(t_2,\vec{r})]g^K-i\phi_2(t_1,\vec{r})g^A\\&\quad+ig^R\phi_2(t_2,\vec{r})+\frac{1}{2\tau}[g^K\langle g^A\rangle_n-\langle g^K\rangle_ng^A+g^R\langle g^K\rangle_n\\&\quad-\langle g^R\rangle_ng^K],\end{aligned}\tag{4.49b}$$

and we redefine the covariant derivatives Eq. (4.40) to include only the external scalar and vector potentials

$$\tilde{\partial}_tg=\partial_{t_1}g+\partial_{t_2}g+i[\varphi_{\rm ext}(t_1,r)-\varphi_{\rm ext}(t_2,r)]g,\tag{4.50}$$

$$\tilde{\vec{\nabla}}g=\vec{\nabla}g+i[\vec{A}_{\rm ext}(t_1,r)-\vec{A}_{\rm ext}(t_2,r)]g.$$

Now we are prepared to derive the collision integral. We notice that due to causality $\langle g_Z\rangle_\phi=0$ in all orders of the perturbation theory. We separate slow and fast degrees of freedom as follows:

$$g_K=\langle g_K\rangle_\phi+\delta g_K;\quad g_Z=\delta g_Z,\tag{4.51}$$

where δg is the contribution fluctuating with the auxiliary fields and we calculate it to first order in ϕ. In the same approximation Eq. (4.48) becomes

$$g^R=\delta(t_1-t_2)-\frac{1}{2}g^K\delta g^Z,$$

$$g^A=-\delta(t_1-t_2)+\frac{1}{2}\delta g^Zg^K\tag{4.52}$$

[expansion up to the second order in δg^Z is unnecessary because terms of such kind vanish due to Eq. (4.30d)].

We now substitute Eqs. (4.51) and (4.52) into Eqs. (4.49) and obtain equations governing the behavior of the fluctuating parts

$$\left[\tilde{\partial}_t+v_F\vec{n}\tilde{\vec{\nabla}}+\vec{\omega}_c\left(\vec{n}\times\frac{\partial}{\partial\vec{n}}\right)\right]\delta g^Z-\frac{1}{\tau}[\delta g^Z-\langle\delta g^Z\rangle_n]=-2i\phi_2(t_1,\mathbf{r})\delta(t_1-t_2),\tag{4.53a}$$

$$\begin{aligned}\left[\tilde{\partial}_t+v_F\vec{n}\tilde{\vec{\nabla}}+\vec{\omega}_c\left(\vec{n}\times\frac{\partial}{\partial\vec{n}}\right)\right]\delta g^K+\frac{1}{\tau}[\delta g^K-\langle\delta g^K\rangle_n]&=2i\phi_2(t_1,\mathbf{r})\delta(t_1-t_2)-i[\phi_1(t_1,\mathbf{r})-\phi_1(t_2,\mathbf{r})]\langle g^K\rangle_\phi\\&\quad+\frac{1}{4\tau}[\langle g^K\rangle_\phi\langle\delta g^Z\langle g^K\rangle_\phi\rangle_n-\langle\langle g^K\rangle_n\rangle_\phi\delta g^Z\langle g^K\rangle_\phi]\\&\quad-\frac{1}{4\tau}[\langle g^K\rangle_\phi\delta g^Z\langle\langle g^K\rangle_n\rangle_\phi-\langle\langle g^K\rangle_\phi\delta g^Z\rangle_n\langle g^K\rangle_\phi].\end{aligned}\tag{4.53b}$$

Solutions to Eqs. (4.53) should be substituted into Eq. (4.49b) for the smooth part. Than the equation for the smooth part should be averaged over the fluctuating fields $\phi_{1,2}$ with the help of Eq. (4.30). As a result

$$\left[\tilde{\partial}_t+v_F\vec{n}\tilde{\vec{\nabla}}+\vec{\omega}_c\left(\vec{n}\times\frac{\partial}{\partial\vec{n}}\right)\right]\langle g^K\rangle_\phi={\rm St}_{in}\{\langle g^K\rangle_\phi\}+{\rm St}_{el}\{\langle g^K\rangle_\phi\}.\tag{4.54}$$

Here we separate the collision integrals into two contributions. The physical meaning of such separation will be discussed shortly. The first, inelastic part has the structure

$$\begin{aligned}{\rm St}_{\rm in}\{\langle g^K\rangle_\phi\}(t_1,t_2;\vec{n},\vec{r})\\=-i\langle[\phi_1(t_1,\vec{r})-\phi_1(t_2,\vec{r})]\delta g^K(t_1,t_2;\vec{n},\vec{r})\rangle_\phi.\end{aligned}\tag{4.55}$$

The second, elastic contribution has the form

$$
\begin{aligned}
&\mathrm{St}_{el}\{\langle g^K\rangle_\phi\}(t_1,t_2;\vec{n};\vec{r})\\
&\quad=\frac{1}{\tau}[\langle\langle g^K(t_1,t_2;\vec{n};\vec{r})\rangle_\phi\rangle_n-\langle g^K(t_1,t_2;\vec{n};\vec{r})\rangle_\phi]\\
&\qquad+\int dt_3\int\frac{d\theta_1}{2\pi}\\
&\qquad\times\Big[\langle\langle g^K(t_1,t_3;\vec{n}_1,\vec{r})\rangle_\phi F^A(t_3,t_2;\vec{n}_1,\vec{n};\vec{r})\\
&\qquad-\langle\langle g^K(t_1,t_3;\vec{n},\vec{r})\rangle_\phi F^A(t_3,t_2;\vec{n},\vec{n}_1;\vec{r})]\\
&\qquad+\int dt_3\int\frac{d\theta_1}{2\pi}[F^R(t_1,t_3;\vec{n},\vec{n}_1;\vec{r})\langle\langle g^K(t_3,t_2;\vec{n}_1,\vec{r})\rangle_\phi\\
&\qquad-F^R(t_1,t_3;\vec{n}_1,\vec{n};\vec{r})\langle\langle g^K(t_3,t_2;\vec{n},\vec{r})\rangle_\phi],
\end{aligned}
\tag{4.56}
$$

where the first term is just the ordinary impurity scattering and the remaining terms characterize interaction effects. The kernels in Eq. (4.56) are defined as

$$
\begin{aligned}
&F^R(t_1,t_2;\vec{n},\vec{n}_1;\vec{r})\\
&\quad=\frac{1}{4\tau}\int dt_3\langle\,\delta g^K(t_1,t_3;\vec{n},\vec{r})[\,\delta g^Z(t_3,t_2;\vec{n}_1,\vec{r})\\
&\qquad-\delta g^Z(t_3,t_2;\vec{n},\vec{r})]\rangle_\phi,\\
&F^A(t_1,t_2;\vec{n},\vec{n}_1;\vec{r})=\frac{1}{4\tau}\int dt_3\langle[\,\delta g^Z(t_1,t_3;\vec{n}_1,\vec{r})\\
&\qquad-\delta g^Z(t_1,t_3;\vec{n},\vec{r})]\,\delta g^K(t_3,t_2;\vec{n},\vec{r})\rangle_\phi.
\end{aligned}
\tag{4.57}
$$

Equations (4.54), (4.55), (4.56), (4.53), (4.30), and (4.32) constitute a closed system of kinetic equations. Although sufficient for description of interaction effects in disorder systems, these equations are inconvenient for analytical calculations because the expressions for the collision integral are nonlocal in space and time. To simplify further calculations we will use the assumption that $\langle g^K(t_1,t_2;\mathbf{n},\mathbf{r})\rangle_\phi$ is a smooth function so that a gradient expansion will be possible.

Before embarking on such calculation we pause to discuss the physical distinction between the elastic (4.56) and inelastic (4.55) collision terms. One immediately notices from Eq. (4.56) that

$$\int d\theta\ \ \mathrm{St}_{el}(t_1,t_2;\vec{n};\vec{r})=0,\tag{4.58}$$

for any t_1 and t_2. This indicates that this part of the collision integral preserves the number of particles on a given energy shell [see below for explicit connections between time representation and energy representation Eq. (4.65)].

The inelastic term (4.55) does not vanish after angular averaging. Therefore this part does promote electrons between energy shells. However, we notice that

$$\mathrm{St}_{\mathrm{in}}\{\langle g^K\rangle_\phi\}(t_1,t_1;\vec{n},\mathbf{r})=0,\tag{4.59}$$

for any direction $\vec{n}$. Taking coinciding time arguments is equivalent to integrating over the whole energy spectrum [see Eq. (4.65)], so that not only the total number of particles is conserved, but the total number of particles moving along a given direction $\vec{n}$ is conserved (i.e., inelastic forward scattering).

Let us now perform the actual calculation of the collision integrals. We solve Eq. (4.53a) and obtain

$$
\begin{aligned}
\delta g^Z(t_1,t_2;\vec{n},\vec{r})=2i\,\delta(t_1-t_2)\int d\mathbf{r}_1dt_3\phi_2(\mathbf{r}_1,t_3)\\
\times\int\frac{d\theta'}{2\pi}D(t_3-t_1,\vec{n}',\vec{n};\vec{r}_1,\vec{r})
\end{aligned}
\tag{4.60}
$$

$$D(t;\vec{n},\vec{n}';\vec{r}_1,\vec{r}_2)=\int\frac{d\omega d^2\mathbf{q}}{(2\pi)^3}e^{i\vec{q}(\vec{r}_1-\vec{r}_2)-i\omega t}D(\vec{n},\vec{n}';\omega,\vec{q}),$$

where the diffuson propagator D is defined in Eq. (4.7).

To simplify the analytic solution of Eq. (4.53b), we assume without loss of generality that $\langle g^K\rangle_\phi$ varies slowly on the spatial scale $L_T=v_F\min(1/T,\sqrt{\tau/T})$, and also a slow function of t_1+t_2 on the time scale $\simeq 1/T$. These assumptions are consistent with the first loop approximation we already invoked.

In what follows we will keep only the zeroth and first angular harmonics (which is consistent with assumption about the spatial smoothness) in the direction dependence of the Keldysh function

$$
\begin{aligned}
\langle g(t_1,t_2;\vec{n},\vec{r})\rangle_\phi\approx\langle g(t_1,t_2;\vec{n},\vec{r})\rangle_n\\
+2\vec{n}\langle\vec{n}'g(t_1,t_2;\vec{n}',\vec{r})\rangle_{n'}.
\end{aligned}
\tag{4.61}
$$

This approximation does not affect results for any relevant quantities. From now on we will suppress the explicit sign of averaging over the fluctuating fields because we will not be dealing with nonaveraged quantities anymore.

We now substitute Eq. (4.61) into the right-hand side of Eq. (4.53b) and obtain

$$\delta g^K(t_1,t_2;\vec{n},\vec{r})=\delta g_1^K(t_1,t_2;\vec{n},\vec{r})+\delta g_2^K(t_1,t_2;\vec{n},\vec{r}).\tag{4.62}$$

The first term in Eq. (4.62) is proportional to the field ϕ_1 and gives contributions to both the elastic and the inelastic parts of the collision integral. To obtain nonvanishing contribution to the latter we have to do each one of the following: (i) take into account the first angular harmonic, (ii) perform the first order gradient expansion, (iii) expand up to the first order in external fields $\vec{A}_{\mathrm{ext}}$. The result is

$$\delta g_1^K(t_1,t_2;\vec{n},\vec{r})$$

$$=-i\int dt[\phi_1(\vec{r}_1,t_1-t)-\phi_1(\vec{r}_1,t_2-t)]\int\frac{d\vec{n}'}{2\pi}$$

$$\times D(t,\vec{n},\vec{n}';\vec{r},\vec{r}_1)\{\langle g^K(t_1-t,t_2-t;\vec{n}_1,\vec{r})\rangle_{\vec{n}_1} \tag{4.63a}$$

$$+2\vec{n}'\langle\vec{n}_1 g^K(t_1-t,t_2-t;\vec{n}_1,\vec{r})\rangle_{\vec{n}_1} \tag{4.63b}$$

$$+(\vec{r}_1-\vec{r})\tilde{\vec{\nabla}}\langle g^K(t_1-t,t_2-t;\vec{n}_1,\vec{r})\rangle_{\vec{n}_1}\}, \tag{4.63c}$$

where the covariant derivative is defined in Eq. (4.50) and we neglected higher order derivatives of the external fields. Expansion in the time coordinate t_1+t_2 (using the covariant derivative $\tilde{\partial}_t$) is not necessary because it produces a negligible correction to the inelastic collision integral and does not affect the elastic one.

The second term in the right-hand side of Eq. (4.62) is proportional to the field ϕ_2, and according to Eqs. (4.57) and (4.30d) it does not contribute to the elastic collision integral. Therefore, it is sufficient to keep only the zeroth angular component and neglect gradient terms at all. This yields

$$\begin{aligned}\delta g_2^K(t_1,t_2;\vec{n},\vec{r})\\ =\int\frac{d\theta'}{2\pi}\frac{d\theta''}{2\pi}\int d\vec{r}_1 dt D(t,\vec{n},\vec{n}';\vec{r},\vec{r}_1)\\ \times\Bigg\{2i\phi_2(\vec{r}_1,t_1-t)\,\delta(t_1-t_2)\\ +\frac{i}{\tau}\langle g(t_1-t,t_3;\vec{n}_1,\vec{r})\rangle_{\vec{n}_1}\langle g(t_3,t_2-t;\vec{n}_1,\vec{r})\rangle_{\vec{n}_1}\\ \times[\langle D(t_4-t_3,\vec{n}'',\vec{n}_1;\vec{r}_2,\vec{r}_1)\rangle_{\vec{n}_1}\\ -D(t_4-t_3,\vec{n}'',\vec{n}';\vec{r}_2,\vec{r}_1)]\phi_2(\vec{r}_2,t_4)\Bigg\}.\end{aligned} \tag{4.64}$$

As we already mentioned, $g(t_1,t_2)$ has a much faster dependence on the difference t_1-t_2 then on the sum t_1+t_2. Therefore it is more convenient to use a temporal transformation of the Green's function

$$g^K(t_1,t_2;\vec{n},\vec{r})=\int\frac{d\epsilon}{2\pi}g\left(\frac{t_1+t_2}{2},\epsilon;\vec{n},\vec{r}\right)e^{i\epsilon(t_2-t_1)}, \tag{4.65}$$

which defines the precise notion of energy ϵ in this context. We introduce the same transformation for the propagators of auxiliary fields (4.30)

$$\mathcal{D}(t_1,t_2)=\int\frac{d\omega}{2\pi}\mathcal{D}\left(\frac{t_1+t_2}{2},\omega\right)e^{i\omega(t_2-t_1)} \tag{4.66}$$

The transformed functions have the symmetry property (hereafter we omit the K superscript for brevity since we are only dealing with the Keldysh function)

$$g(t,\epsilon)=-g(t,-\epsilon), \tag{4.67}$$

$$\mathcal{D}^K(t,\omega;\vec{r}_1,\vec{r}_2)=\mathcal{D}^K(t,-\omega;\vec{r}_2,\vec{r}_1),$$

$$\mathcal{D}^R(t,\omega;\vec{r}_1,\vec{r}_2)=\mathcal{D}^A(t,-\omega;\vec{r}_2,\vec{r}_1).$$

Now, we are ready to obtain the explicit form of the collision integral. We start with the inelastic contribution and perform the following three steps: (1) substitute Eq. (4.63a) and (4.64) into Eq. (4.8a); (2) average over the fields $\phi_{1,2}$ with the help of Eq. (4.30); (3) perform the temporal transformation (4.65) of the result. As a result we obtain with the help of Eqs. (4.66) and (4.67) the following form of the collision integral:

$$\begin{aligned}\mathrm{St}_{\mathrm{in}}\{g^K\}(t,\epsilon;\vec{r})=-\frac{i}{2}\int d^2r_1\int\frac{d\omega}{2\pi}\mathcal{D}^K(t,\omega;\vec{r},\vec{r}_1)\\ \times[\langle D(\omega;\vec{r},\vec{r}_1)\rangle+\langle D(-\omega;\vec{r}_1,\vec{r})\rangle]\\ \times[\langle g(t,\epsilon;\vec{n},\vec{r})\rangle_{\vec{n}}-\langle g(t,\epsilon-\omega;\vec{n},\vec{r})\rangle_{\vec{n}}]\\ +\frac{i}{2\tau}\int d\vec{r}_1 d\vec{r}_2\int\frac{d\omega}{2\pi}[\mathcal{D}^R(t,\omega;\vec{r}_1,\vec{r}_2)\\ -\mathcal{D}^A(t,\omega;\vec{r}_2,\vec{r}_1)][\langle D(\omega;\vec{r},\vec{r}_1)\rangle\langle D\\ (-\omega;\vec{r},\vec{r}_2)\rangle-\langle D(\omega;\vec{r},\vec{r}_1)D(-\omega;\vec{r},\vec{r}_2)\rangle]\\ \times\langle g(t,\epsilon+\omega;\vec{n},\vec{r})\rangle_{\vec{n}}\langle g(t,\epsilon;\vec{n},\vec{r})\rangle_{\vec{n}},\end{aligned} \tag{4.68}$$

where the angular averaging of the diffusons is defined after Eqs. (4.6).

Now, we have to express the bosonic propagator in terms of the fermionic polarization operators. The polarization operators are given by Eqs. (4.47), where we now substitute Eqs. (4.60), (4.63a), and (4.64). After the temporal transformation (4.65) we find

$$\begin{aligned}\Pi^R(\omega;t,\vec{r}_1,\vec{r}_2)=\Pi^A(-\omega;t,\vec{r}_2,\vec{r}_1)\\ =\nu\Bigg[\delta(\vec{r}_1-\vec{r}_2)+\frac{i}{4}\langle D(\omega;\vec{r}_1,\vec{r}_2)\rangle\\ \times\int d\epsilon[\langle g(t,\epsilon;\vec{n},\vec{r})\rangle_{\vec{n}}\\ -\langle g(t,\epsilon-\omega;\vec{n},\vec{r})\rangle_{\vec{n}}]\Bigg]\end{aligned} \tag{4.69a}$$

$$=\nu[\delta(\vec{r}_1-\vec{r}_2)+i\omega\langle D(\omega;\vec{r}_1,\vec{r}_2)\rangle], \tag{4.69b}$$

$$\begin{aligned}\Pi^K(t,\omega;\vec r_1,\vec r_2)=\frac{i\nu}{4\tau}\int & d\vec r[\langle D(\omega;\vec r,\vec r_1)\rangle\langle D(-\omega;\vec r,\vec r_2)\rangle\\ &-\langle D(\omega;\vec r,\vec r_1)D(-\omega;\vec r,\vec r_2)\rangle]\\ &\times\int d\epsilon[\langle g(t,\epsilon+\omega;\vec n,\vec r)\rangle_{\vec n}\\ &\times\langle g(t,\epsilon;\vec n,\vec r)\rangle_{\vec n}-4].\end{aligned}\tag{4.70}$$

The last step in the calculation of the interaction propagators is to solve Eq. (4.32) with the polarization operators Eq. (4.69a). This gives $\mathcal{D}^{R,A}$ in the form given by Eq. (3.29a) and for the Keldysh component we obtain

$$\mathcal{D}^K=\mathcal{D}^R\Pi^K\mathcal{D}^A.\tag{4.71a}$$

Also we can relate the difference of the retarded and advanced propagators which enters the collision integral Eq. (4.68) to the polarization operators

$$\mathcal{D}^R-\mathcal{D}^A=\mathcal{D}^R[\Pi^R-\Pi^A]\mathcal{D}^A.\tag{4.71b}$$

To obtain the final form of the inelastic part of the collision integral Eqs. (4.8) we need to substitute Eq. (4.71) into Eq. (4.68), while using Eq. (4.69a) for $\Pi^R-\Pi^A$. In addition, we note that

$$\begin{aligned}&\frac{2}{\tau}[\langle D(\omega;q)D(-\omega;-q)\rangle-\langle D(\omega;q)\rangle\langle D(-\omega;-q)\rangle]\\ &\quad=\langle D(\omega;q)\rangle+\langle D(-\omega;-q)\rangle.\end{aligned}$$

Finally, we introduce the gauge invariant distribution function f as

$$f(\epsilon,t;\vec n,\vec r)=\frac{1}{2}-\frac{1}{4}g(\epsilon+\varphi_{ext}(\vec r),t;\vec n,\vec r)\tag{4.72}$$

and obtain Eqs. (4.8).

The calculation of the elastic part of the collision integral is completely analogous. We substitute Eqs. (4.63a)–(4.63c) and Eq. (4.60) into Eqs. (4.57) and average over fluctuating fields with the help of Eq. (4.30). After the temporal transformation (4.65) we find

$$\begin{aligned}&F^R(\epsilon,t;\vec n_1,\vec n_2,\vec r)\\ &\quad=F^A(\epsilon,t;\vec n_1,\vec n_2,\vec r)^*\\ &\quad=\frac{i}{4\tau}\int\frac{d\omega}{2\pi}\int d\vec r_1d\vec r_2\mathcal{D}^R(\omega,\vec r_1,\vec r_2)\int\frac{d\vec n_3}{2\pi}\frac{d\vec n_4}{2\pi}\\ &\qquad\times[D(\omega;\vec n_3,\vec n_2,\vec r_2,\vec r)-D(\omega;\vec n_3,\vec n_1,\vec r_2,\vec r)]\\ &\qquad\times D(\omega;\vec n_1,\vec n_4,\vec r,\vec r_1)\{\langle g(t,\epsilon-\omega;\vec n',\vec r)\rangle_{\vec n'}\qquad(4.73\text{a})\\ &\qquad+2\vec n_4\langle\vec n' g(t,\epsilon-\omega;\vec n_1,\vec r)\rangle_{\vec n'}\qquad(4.73\text{b})\\ &\qquad+(\vec r_1-\vec r)\left(\vec\nabla+\frac{\partial\vec A_{ext}}{\partial t}\frac{\partial}{\partial\epsilon}\right)\langle g(t,\epsilon-\omega;\vec n',\vec r)\rangle_{\vec n'}\}.\qquad(4.73\text{c})\end{aligned}$$

Deriving Eq. (4.73) we use the fact that $\int d\omega\mathcal{D}^R(\omega)D(\omega)=0$.

We substitute Eqs. (4.73) and (4.61) into Eq. (4.58). We expand the result into angular harmonics. The zeroth angular harmonic vanishes because of the conservation law (4.58), and the first harmonic gives (we write only interaction correction to the collision integral)

$$\begin{aligned}&\mathrm{St}_{\rm el}(t;\epsilon;\vec r)\\ &\quad=\frac{2}{\tau}\int\frac{d\omega}{2\pi}[n_\alpha K_1^{\alpha\beta}(\omega)\langle n_\beta g(t;\epsilon,\vec r,\vec n)\rangle_n\\ &\qquad\times\langle g(t;\epsilon-\omega,\vec r,\vec n)\rangle_n\qquad(4.74\text{a})\\ &\qquad+n_\alpha K_0^{\alpha\beta}(\omega)\langle n_\beta g(t;\epsilon-\omega,\vec r,\vec n)\rangle_n\langle g(t;\epsilon,\vec r,\vec n)\rangle_n\qquad(4.74\text{b})\\ &\qquad+\frac{n_\alpha L_0^{\alpha\beta}(\omega)}{2}\langle g(t;\epsilon,\vec r,\vec n)\rangle_n\\ &\qquad\times\left(\nabla_\beta+\frac{\partial[A_{\rm ext}]_\beta}{\partial t}\frac{\partial}{\partial\epsilon}\right)\langle g(t;\epsilon-\omega,\vec r,\vec n)\rangle_n].\qquad(4.74\text{c})\end{aligned}$$

Here the kernels K and L are given by Eq. (4.6). Each labeled separately term in Eqs. (4.74) corresponds to ones in Eqs. (4.73) and in Eqs. (4.63) so the origin of terms can be easily traced.

Finally, we use the gauge invariant distribution function (4.72) instead of g and we arrive to Eqs. (4.5). Closing this section, we remark that the above treatment can be easily generalized to include other channels as well as the higher angular harmonics of the Fermi-liquid constant. For the latter task one has to introduce angle dependent auxiliary fields $\phi_{1,2}(\vec r,\vec n,t)$ and use $F_0^\rho\to F^\rho(\widehat{\vec n_1\vec n_2})$.

The triplet channel requires introduction of the coupling of the form $\vec h_{1,2}(\vec r,t)\hat{\vec\sigma}$, where $\hat\sigma^j$, $j=x,y,z$ are the Pauli matrices in the spin space, and $\vec h_{1,2}(\vec r,t)$ are the auxiliary fields. Accordingly, each bosonic propagator from Eq. (4.30) becomes a 3×3 matrix. Equation (4.32) retains the same form with the matrix multiplication in Keldysh and spin spaces implied. Equation (4.33) becomes

$$[\mathcal{D}_0^R]_{ij}=[\mathcal{D}_0^A]_{ij}=-F_0^\sigma\delta_{ij}\delta(\vec r_1-\vec r_2)\delta(t_1-t_2),\tag{4.75}$$

and Eq. (4.47) is modified to

$$\Pi^R_{ij}(1,2)=\Pi^A_{ji}(2,1)=\nu\int\frac{d\theta}{2\pi}\left(\delta_{12}\delta_{ji}+\frac{\pi\langle \mathrm{Tr}\sigma_i\delta g^K(t_1,t_1;\vec{n},\vec{r}_1)\rangle_\phi}{4\delta h_1^j(t_2,\vec{r}_2)}\right), \tag{4.76}$$

$$\Pi^K(1,2)=\pi\nu=\int\frac{d\theta}{2\pi}\frac{\langle \mathrm{Tr}\sigma_i\delta g^K(t_1,t_1;\vec{n},\vec{r}_1)+\mathrm{Tr}\sigma_i\delta g^Z(t_1,t_1;\vec{n},\vec{r}_1)\rangle_\phi}{4\delta h_2^j(t_2,\vec{r}_2)},$$

where trace is performed in spin space.

Further derivation consists of a repetition of the steps described in this section, and in the absence of the spin structure of the distribution function, $f_{ij}=\delta_{ij}f$, results in Eqs. (4.9) and (4.10). The spin-orbit interaction or Zeeman splitting by external magnetic field slightly changes the results, but we will postpone the corresponding analysis until a future publication.[14]

Finally, the Cooper channel interaction (3.21) can be treated in the same manner by introducing auxiliary fields in the Gorkov-Nambu space. We will not discuss this question further in the present paper.

V. CONCLUSIONS

This paper is an attempt to consistently describe the effect of electron-electron interaction on longitudinal conductivity of disordered 2D electron gas at $T\ll E_F$. Our results are valid for an arbitrary relation between T and $\hbar/\tau$ and are summarized in Sec. II B. At low temperatures $T\tau\ll\hbar$ we reproduce the logarithmically divergent Altshuler-Aronov correction. At higher temperatures $T\tau>\hbar$, i.e., in the ballistic region, we found the linear temperature dependence in accord with Refs. 6,27. However, even the sign of the slope of this dependence depends on the strength of electron-electron interaction in contradiction to the results of Refs. 6,27 (see Secs. II and III F for discussions of this discrepancy).

We deliberately did not compare the theory with experimental data, postponing this comparison until the publication of theoretical results for Hall conductivity and magnetoresistance in the parallel field. For comparison with data obtained for Si-MOSFET samples the valley degeneracy should be taken into account [the degeneracy may increase the numerical factor in Eq. (2.16c) by as much as a factor of 5 in the case of low intervalley scattering]. We also relegate the corresponding discussion to a separate publication.

Finally, we derived a kinetic equation to describe the effect of electron-electron interaction at arbitrary $T\tau$. The advantage of this approach is that it turns out to be more convenient for practical calculations of transport properties in magnetic field as well as thermal transport properties.

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APPENDIX

In this appendix we show in some detail the procedure of analytical continuation that leads to the expression for the interaction correction Eq. (3.3) to the conductivity in terms of exact Green's function of noninteracting disordered system. The structure of the current correlator is

$$\int_0^{1/T}d\tau\langle \mathbf{T}_\tau\hat{j}_\alpha(\tau)\hat{j}_\beta(0)\rangle e^{i\Omega_n\tau}$$

$$-T\sum_{\epsilon_n}J_\alpha G(i\epsilon_n+i\Omega_n)J_\beta G(i\epsilon_n) \tag{A1a}$$

$$-T\sum_{\epsilon_n}J_\alpha G(i\epsilon_n+i\Omega_n)\Gamma_\beta(i\epsilon_n+i\Omega_n,i\epsilon_n)G(i\epsilon_n), \tag{A1b}$$

where $\epsilon_n=\pi T(2n+1)$ is the fermionic Matsubara frequency, $G(i\epsilon_n)$ is the exact Green's function of the interacting system (diagrams 1 and 2 on Fig. 9 are the first order correction to the Green's function), $\Gamma_\beta(i\epsilon_n+i\Omega_n,i\epsilon_n)$ is the vertex function (not to be confused with disorder averaged interaction vertex Γ from Sec. III). Diagrams 3 on Fig. 9 are the first order correction to the vertex function. The current operator is defined in Eq. (3.6). Note, that we omit the spatial coordinates and the integration whenever it should cause no confusion.

We perform the analytic continuation in each term (A1a) and (A1b) separately. In Eq. (A1a) we use the standard procedure

$$T\sum_{\epsilon_n}(\cdots)=\frac{1}{4\pi i}\int_{\mathcal{C}}d\epsilon\tanh\frac{\epsilon}{2T}(\cdots), \tag{A2}$$

where integration contour is shown on Fig. 23. We deform this contour to form C_1, use the facts that $\tanh(\epsilon+i\Omega_n)/2T=\tanh\epsilon/2T$, and $G(\epsilon\pm i0)=G^{R(A)}(\epsilon)$ and we obtain

$$M_1(i\Omega_n)=T\sum_{\epsilon_n}J_\alpha G(i\epsilon_n+i\Omega_n)J_\beta G(i\epsilon_n)$$

$$=\int\frac{d\epsilon}{4\pi i}\tanh\frac{\epsilon}{2T}\{J_\alpha G(\epsilon+i\Omega_n)J_\beta[G^R(\epsilon)-G^A(\epsilon)]+J_\alpha[G^R(\epsilon)-G^A(\epsilon)]J_\beta G(\epsilon-i\Omega_n)\}. \tag{A3}$$

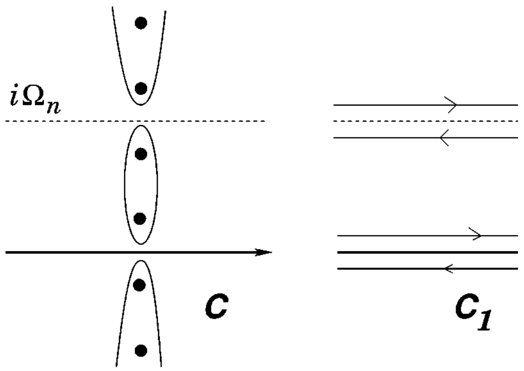

FIG. 23. Integration contours for analytic continuation of Eqs. (A1).

In the form (A3) frequency Ω is present only in functions which may have singularities only on the real axis, so that the required analytic continuation can be easily performed:

$$\begin{aligned}M_1(\omega)&=M_1(i\Omega_n\to\omega+i0)\\&=\int\frac{d\epsilon}{4\pi i}\tanh\frac{\epsilon}{2T}\{J_\alpha G^R(\epsilon+\omega)J_\beta[G^R(\epsilon)-G^A(\epsilon)]\\&\quad+J_\alpha[G^R(\epsilon)-G^A(\epsilon)]J_\beta G^A(\epsilon-\omega)\}.\end{aligned}\tag{A4}$$

Thus one obtains for the quantity entering into conductivity (3.1)

$$\begin{aligned}N_1&=-\lim_{\omega\to0}\mathrm{Im}\left(\frac{M_1(\omega)}{\omega}\right)\\&=\mathrm{Re}\int\frac{d\epsilon}{2\pi}\tanh\frac{\epsilon}{2T}J_\alpha G^A(\epsilon)J_\beta\partial_\epsilon G^A(\epsilon)\\&\quad+\int\frac{d\epsilon}{4\pi}\left(\frac{d}{d\epsilon}\tanh\frac{\epsilon}{2T}\right)J_\alpha G^R(\epsilon)J_\beta G^A(\epsilon).\end{aligned}\tag{A5}$$

Equation (A5) can be further simplified for the calculation of the symmetric part of the conductivity

$$\begin{aligned}N_1^{\mathrm{sym}}=\mathrm{Re}\int\frac{d\epsilon}{8\pi}\left(\frac{d}{d\epsilon}\tanh\frac{\epsilon}{2T}\right)&[-J_\alpha G^A(\epsilon)J_\beta G^A(\epsilon)\\&+J_\alpha G^R(\epsilon)J_\beta G^A(\epsilon)+(\alpha\leftrightarrow\beta)].\end{aligned}\tag{A6}$$

Term (A1b) is considered analogously. We find similarly to Eq. (A3)

$$\begin{aligned}M_2(i\Omega_n)&=T\sum_{\epsilon_n}J_\alpha G(i\epsilon_n+i\Omega_n)\Gamma_\beta(i\epsilon_n+i\Omega_n,i\epsilon_n)G(i\epsilon_n)\\&=\int\frac{d\epsilon}{4\pi i}\tanh\frac{\epsilon}{2T}\{J_\alpha G(\epsilon+i\Omega_n)[\Gamma_\beta(\epsilon+i\Omega_n,\epsilon+i0)G^R(\epsilon)-\Gamma_\beta(\epsilon+i\Omega_n,\epsilon-i0)G^A(\epsilon)]\\&\quad+J_\alpha[G^R(\epsilon)\Gamma_\beta(\epsilon+i0,\epsilon-i\Omega_n)-G^A(\epsilon)\Gamma_\beta(\epsilon-i0,\epsilon-i\Omega_n)]G(\epsilon-i\Omega_n)\}.\end{aligned}\tag{A7}$$

Using analytic properties of the Green's function and the vertex function, we perform the analytic continuation and obtain

$$\begin{aligned}M_2(\omega)&=M_2(i\Omega_n\to\omega+i0)\\&=\lim_{\delta_1\to0^+}\lim_{\delta_2\to0^+}\int\frac{d\epsilon}{4\pi i}\tanh\frac{\epsilon}{2T}\{J_\alpha G^R(\epsilon+\omega)[\Gamma_\beta(\epsilon+\omega+i\delta_1,\epsilon+i\delta_2)G^R(\epsilon)-\Gamma_\beta(\epsilon+\omega+i0,\epsilon-i0)G^A(\epsilon)]\\&\quad+J_\alpha[G^R(\epsilon)\Gamma_\beta(\epsilon+i0,\epsilon-\omega-i0)-G^A(\epsilon)\Gamma_\beta(\epsilon-i\delta_2,\epsilon-\omega-i\delta_1)]G^A(\epsilon-\omega)\}.\end{aligned}\tag{A8}$$

In the appropriate frequency limit, we find

$$\begin{aligned}N_2=-\lim_{\omega\to0}\mathrm{Im}\left(\frac{M_2(\omega)}{\omega}\right)&=\mathrm{Re}\int\frac{d\epsilon}{2\pi}\tanh\frac{\epsilon}{2T}J_\alpha G^A(\epsilon)\left.\frac{\partial}{\partial\epsilon_1}\right|_{\epsilon_1=\epsilon}[\lim_{\delta_1\to0^+}\lim_{\delta_2\to0^+}\Gamma_\beta(\epsilon-i\delta_2,\epsilon_1-i\delta_1)G^A(\epsilon_1)]\\&\quad+\int\frac{d\epsilon}{4\pi}\left(\frac{d}{d\epsilon}\tanh\frac{\epsilon}{2T}\right)J_\alpha G^R(\epsilon)\Gamma_\beta(\epsilon+i0,\epsilon-i0)G^A(\epsilon).\end{aligned}\tag{A9}$$

Further calculation requires specification of the form of the self-energy and the vertex function. We have to find both in the first order in interaction propagator, and expand Eq. (A6) up to the first order:

$$\delta N_1^{\rm sym}={\rm Re}\int\frac{d\epsilon}{4\pi}\left(\frac{d}{d\epsilon}\tanh\frac{\epsilon}{2T}\right)[-J_\alpha G^A(\epsilon)\Sigma^A(\epsilon)G^A(\epsilon)J_\beta G^A(\epsilon)+J_\alpha G^R(\epsilon)J_\beta G^A(\epsilon)\Sigma^A(\epsilon)G^A(\epsilon)+(\alpha\leftrightarrow\beta)]. \tag{A10}$$

For brevity we consider only the "Fock" contribution of Fig. 9(b):

$$\Sigma(i\epsilon_n)_{12}=T\sum_{\Omega_m}\mathcal{D}_{12}(i\Omega_m)G_{12}(i\epsilon_n-i\Omega_m), \tag{A11}$$

where $\mathcal{D}$ is the bosonic propagator, and we restored the notation for spatial coordinates. In the same order

$$[\Gamma(i\epsilon_n,i\epsilon_m)_\beta]_{12}=T\sum_{\Omega_m}\mathcal{D}_{12}(\Omega_m)[G(i\epsilon_n-i\Omega_m)J_\beta G(i\epsilon_m-i\Omega_m)]_{12}. \tag{A12}$$

After analytic continuation similar to that in the derivation of Eq. (A4) we find

$$\Sigma_{12}^A(\epsilon)=-\int\frac{d\Omega}{2\pi}\coth\frac{\Omega}{2T}[{\rm Im}\mathcal{D}_{12}^A(\Omega)]G_{12}^A(\epsilon-\Omega)+i\int\frac{d\Omega}{4\pi}\tanh\frac{\epsilon-\Omega}{2T}\mathcal{D}_{12}^A(\Omega)[G_{12}^A(\epsilon-\Omega)-G_{12}^R(\epsilon-\Omega)] \tag{A13}$$

and for the vertex function we have two cases

$$\begin{aligned}\lim_{\delta_1\to0^+}\lim_{\delta_2\to0^+}\Gamma_\beta(\epsilon-i\delta_2,\epsilon_1-i\delta_1)=&-\int\frac{d\Omega}{2\pi}\coth\frac{\Omega}{2T}[{\rm Im}\mathcal{D}_{12}^A(\Omega)][G^A(\epsilon-\Omega)J_\beta G^A(\epsilon_1-\Omega)]_{12}+i\int\frac{d\Omega}{4\pi}\tanh\frac{\epsilon-\Omega}{2T}\mathcal{D}_{12}^A(\Omega)\\&\times[(G^A(\epsilon-\Omega)-G^R(\epsilon-\Omega))J_\beta G^A(\epsilon_1-\Omega)]_{12}+i\int\frac{d\Omega}{4\pi}\tanh\frac{\epsilon_1-\Omega}{2T}\mathcal{D}_{12}^A(\Omega)\\&\times[G^R(\epsilon-\Omega)J_\beta(G^A(\epsilon_1-\Omega)-G^R(\epsilon_1-\Omega))]_{12},\end{aligned} \tag{A14a}$$

$$\begin{aligned}\Gamma_\beta(\epsilon+i0,\epsilon-i0)&\\=&-\int\frac{d\Omega}{2\pi}\coth\frac{\Omega}{2T}[{\rm Im}\mathcal{D}_{12}^A(\Omega)]\\&\times[G^R(\epsilon-\Omega)J_\beta G^A(\epsilon-\Omega)]_{12}\\&+i\int\frac{d\Omega}{4\pi}\tanh\frac{\epsilon-\Omega}{2T}\mathcal{D}_{12}^R(\Omega)\\&\times[(G^A(\epsilon-\Omega)-G^R(\epsilon-\Omega))J_\beta G^A(\epsilon-\Omega)]_{12}\\&+i\int\frac{d\Omega}{4\pi}\tanh\frac{\epsilon-\Omega}{2T}\mathcal{D}_{12}^A(\Omega)\\&\times[G^R(\epsilon-\Omega)J_\beta(G^A(\epsilon-\Omega)-G^R(\epsilon-\Omega))]_{12}.\end{aligned} \tag{A14b}$$

We now substitute Eq. (A13) into Eq. (A10). We use the fact that the combination containing only retarded or only advanced Green's functions vanish upon the disorder averaging. Moreover, the average of the combinations like $G(\epsilon)G(\epsilon-\Omega_1)\cdots G(\epsilon-\Omega_N)$ does not depend on the energy ϵ, which enable us to perform the integration over ϵ using

$$\int d\epsilon\tanh\frac{\epsilon-\Omega}{2T}\frac{d}{d\epsilon}\tanh\frac{\epsilon}{2T}=-2\frac{d}{d\Omega}\left(\Omega\coth\frac{\Omega}{2T}\right).$$

We find using $D^A(\Omega)=[D^A(-\Omega)]^*$

$$\begin{aligned}\delta N_1^{\rm sym}={\rm Im}\int&\frac{d\Omega}{8\pi^2}\left[\frac{d}{d\Omega}\left(\Omega\coth\frac{\Omega}{2T}\right)\right]\mathcal{D}_{12}^A(\Omega)\\&\times[J_\alpha G^A(\epsilon)G_{12}^R(\epsilon-\Omega)G^A(\epsilon)J_\beta G^A(\epsilon)\\&-J_\alpha G^A(\epsilon)G_{12}^R(\epsilon-\Omega)G^A(\epsilon)J_\beta G^R(\epsilon)\\&-J_\alpha G^R(\epsilon)G_{12}^R(\epsilon-\Omega)G^R(\epsilon)J_\beta G^A(\epsilon)]\\&-\int\frac{d\Omega}{4\pi^2}\left[\frac{\Omega}{2T\sinh^2\dfrac{\Omega}{2T}}{\rm Im}\mathcal{D}_{12}^A(\Omega)\right]\\&\times{\rm Re}[J_\alpha G^A(\epsilon)G_{12}^A(\epsilon-\Omega)G^A(\epsilon)J_\beta G^R(\epsilon)]\\&+(\alpha\leftrightarrow\beta).\end{aligned} \tag{A15a}$$

The same manipulations are performed with substituition of Eqs. (A14) into Eq. (A9). One finds for the symmetrized part

$$\begin{aligned}\delta N_2^{\rm sym}={\rm Im}\int&\frac{d\Omega}{8\pi^2}\left[\frac{d}{d\Omega}\left(\Omega\coth\frac{\Omega}{2T}\right)\right]\mathcal{D}_{12}^A(\Omega)\\&\times\{[G^A(\epsilon)J_\alpha G^A(\epsilon)]_{12}[G^R(\epsilon-\Omega)J_\beta G^R(\epsilon-\Omega)]_{21}\\&-2[G^R(\epsilon)J_\alpha G^A(\epsilon)]_{12}[G^R(\epsilon-\Omega)J_\beta G^R(\epsilon-\Omega)]_{21}\}\\&-\int\frac{d\Omega}{8\pi^2}\left[\frac{\Omega}{2T\sinh^2\dfrac{\Omega}{2T}}{\rm Im}\mathcal{D}_{12}^A(\Omega)\right]\\&\times\{[G^R(\epsilon)J_\alpha G^A(\epsilon)]_{12}[G^A(\epsilon-\Omega)J_\beta G^R(\epsilon-\Omega)]_{21}\}\\&+(\alpha\leftrightarrow\beta).\end{aligned} \tag{A16a}$$

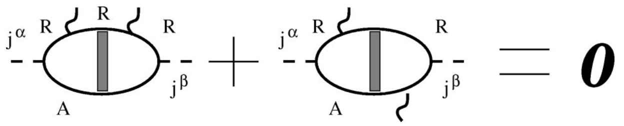

FIG. 24. Cancellation of inelastic term (A17) in the leading ladder approximation.

Total correction to the conductivity is just N_1+N_2. In the Hartree-Fock approximation $D^A=-V(q)$ and we obtain Eq. (3.3). In the case for the stronger interaction terms (A15a) and (A16a) are added to produce Eq. (3.25) and terms (A15a), (A16a) give rise to the inelastic or so-called dephasing term

$$\begin{aligned}\delta\sigma^{\mathrm{deph}}_{\alpha\beta}=&-\int\frac{d\Omega}{8\pi^2}\left[\frac{\Omega}{2T\sinh^2\dfrac{\Omega}{2T}}\mathrm{Im}\mathcal{D}^A_{12}(\Omega)\right]\\&\times\mathrm{Re}\{[2J_\alpha G^A(\epsilon)G^A_{12}(\epsilon-\Omega)G^A(\epsilon)J_\beta G^R(\epsilon)]\\&+[G^R(\epsilon)J_\alpha G^A(\epsilon)]_{12}[G^A(\epsilon-\Omega)J_\beta G^R(\epsilon-\Omega)]_{21}\}\\&+(\alpha\leftrightarrow\beta).\end{aligned}\tag{A17}$$

In our leading approximation in $1/E_F\tau$ this term vanishes, see Fig. 24. The role of this term in the temperature dependence of weak localization correction is discussed in detail in Ref. 16.

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[1] B. L. Altshuler and A. G. Aronov, in *Electron-Electron Interactions in Disordered Systems*, edited by A.L. Efros and M. Pollak (North-Holland, Amsterdam, 1985).

[2] A.M. Finkelstein, Zh. Éksp. Teor. Fiz. **84**, 168 (1983) [Sov. Phys. JETP **57**, 97 (1983)]; Z. Phys. B: Condens. Matter **56**, 189 (1984).

[3] C. Castellani, C. Di Castro, P.A. Lee, and M. Ma, Phys. Rev. B **30**, 527 (1984); **30**, 1596 (1984); C. Castellani, C. Di Castro, and P.A. Lee, *ibid.* **57**, R9381 (1998).

[4] F. Stern, Phys. Rev. Lett. **44**, 1469 (1980).

[5] F. Stern and S. Das Sarma, Solid-State Electron. **28**, 158 (1985).

[6] A. Gold and V.T. Dolgopolov, Phys. Rev. B **33**, 1076 (1986).

[7] S. Das Sarma and E.H. Hwang, Phys. Rev. Lett. **83**, 164 (1999).

[8] B.L. Altshuler, A.G. Aronov, M.E. Gershenson, and Yu.V. Sharvin, Sov. Sci. Rev., Sect. A **9**, 223 (1987).

[9] G. Bergman, Phys. Rep. **107**, 1 (1984).

[10] K.M. Cham and R.G. Wheeler, Phys. Rev. Lett. **44**, 1472 (1980).

[11] For a review see B.L. Altshuler, D.L. Maslov, and V.M. Pudalov, Physica E **9**, 209 (2001).

[12] For a recent review and references see E. Abrahams, S.V. Kravchenko, and M.P. Sarachik, Rev. Mod. Phys. **73**, 251 (2001).

[13] Logarithmic renormalization of F_0^σ, see Refs. 2,3, will not be relevant for the further discussion.

[14] G. Zala, B.N. Narozhny, and I.L. Aleiner (unpublished).

[15] A.M. Rudin, I.L. Aleiner, and L.I. Glazman, Phys. Rev. B **55**, 9322 (1997).

[16] I.L. Aleiner, B.L. Altshuler, and M.E. Gershenson, Waves Random Media **9**, 201 (1999).

[17] C. Kittel, *Quantum Theory of Solids* (Wiley, New York, 1963).

[18] L. D. Landau and I.M. Lifshits, *Quantum Mechanics* (Pergamon, New York, 1977).

[19] I. S. Gradstein and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, San Diego, 1994).

[20] K.A. Matveev, D. Yue, and L.I. Glazman, Phys. Rev. Lett. **71**, 3351 (1993); D. Yue, L.I. Glazman, and K.A. Matveev, Phys. Rev. B **49**, 1966 (1994).

[21] L.D. Landau, Zh. Eksp. Teor. Fiz. **30**, 1058 (1956); **32**, 59 (1957).

[22] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Prentice-Hall, Englewood Cliffs, NJ, 1963).

[23] B. Tanatar and D.M. Ceperley, Phys. Rev. B **39**, 5005 (1989).

[24] This statement contradicts conclusions of Q. Si and C.M. Varma, Phys. Rev. Lett. **81**, 4951 (1998). We believe that this paper mishandles the Fermi liquid renormalizations, which amounts to omission of F_0^ρ in the numerator of Eq. (3.30), see also comment by P. Schwab and C. Castellani, *ibid.* **84**, 4779 (2000).

[25] D. Belitz, T.R. Kirkpatrick, and T. Vojta, Phys. Rev. B **55**, 9452 (1997); D. Belitz, T.R. Kirkpatrick, A.J. Millis, and T. Vojta, *ibid.* **58**, 14 155 (1998).

[26] We are grateful to A.I. Larkin for calling our attention to the momentum dependence of F_0^σ.

[27] M.Yu. Reizer, Phys. Rev. B **57**, 12 338 (1998).

[28] L.V. Keldysh, Zh. Éksp. Teor. Fiz. **47**, 1945 (1964) [Sov. Phys. JETP **20**, 1018 (1964)].

[29] G. Eilenberger, Z. Phys. B **214**, 195 (1968).

[30] A.I. Larkin and Y.N. Ovchinnikov, Zh. Éksp. Teor. Fiz. **55**, 2262 (1968) [Sov. Phys. JETP **28**, 1200 (1969)].

[31] A.I. Larkin and Y.N. Ovchinnikov, Zh. Éksp. Teor. Fiz. **73**, 299 (1977) [Sov. Phys. JETP **46**, 155 (1977)].