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A QUESTION OF B. PLOTKIN ABOUT THE SEMIGROUP OF ENDOMORPHISMS OF A FREE GROUP

EDWARD FORMANEK

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ABSTRACT. Let F be a free group of finite rank $n \geq 2$, let $End(F)$ be the semigroup of endomorphisms of F, and let $Aut(F)$ be the group of automorphisms of F.

Theorem. *If $T : End(F) \to End(F)$ is an automorphism of $End(F)$, then there is an $\alpha \in Aut(F)$ such that $T(\beta) = \alpha \circ \beta \circ \alpha^{-1}$ for all $\beta \in End(F)$.*

For a group G, let $Aut(G)$ denote the group of automorphisms of G, and let $End(G)$ denote the semigroup of endomorphisms of G. Note that $Aut(G)$ is the group of invertible elements of $End(G)$, so any automorphism of $End(G)$ induces an automorphism of $Aut(G)$ by restriction.

In 1975, J. L. Dyer and the author [2] answered a question of G. Baumslag by proving that if F is a free group of finite rank $n \geq 2$, then $Aut(F)$ is a complete group; that is, the center of $Aut(F)$ is trivial and every automorphism of $Aut(F)$ is inner. More recently, new proofs and various generalizations of this theorem have been obtained by M. R. Bridson and K. Vogtmann [1], E. Formanek [3], D. G. Khramtsov [4], and V. Tolstykh [5].

While the author was visiting Israel in May, 2000, B. Plotkin asked: What is the structure of the group of automorphisms of the semigroup $End(F)$? Using the completeness of $Aut(F)$, it is shown below that every automorphism of $End(F)$ is a conjugation by an element of $Aut(F)$.

Notation. Endomorphisms of $F = F\langle x_1, ..., x_n\rangle$ will be regarded as functions acting on the left. Since an endomorphism $\alpha : F \to F$ is completely determined by its values on any free generating set, it can be defined by specifying $\alpha(y_1), ..., \alpha(y_n)$, for some free generating set $\{y_1, ..., y_n\}$ of F. The semigroup operation of $End(F)$ is a composition of functions, denoted "$\circ$". Thus $(\alpha \circ \beta)(x) = \alpha(\beta(x))$, and saying that T is an automorphism of $End(F)$ means that $T : End(F) \to End(F)$ is a bijection satisfying $T(\alpha \circ \beta) = T(\alpha) \circ T(\beta)$, for all $\alpha, \beta \in End(F)$. Multiplication in F will be denoted by juxtaposition, elements of F will be represented by lower case Roman letters, and elements of $End(F)$ will be represented by lower case Greek letters.

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Theorem. *Let $F = F\langle x_1, ..., x_n\rangle$ be a free group of finite rank $n \geq 2$, and suppose that $T : End(F) \to End(F)$ is an automorphism of the semigroup $End(F)$. Then there is an $\alpha \in Aut(F)$ such that $T(\beta) = \alpha \circ \beta \circ \alpha^{-1}$ for all $\beta \in End(F)$.*

Proof. Since T carries $Aut(F)$ to itself, the completeness of $Aut(F)$ [2, Theorem A] implies that there is an $\alpha \in Aut(F)$ such that $T(\beta) = \alpha\circ\beta\circ\alpha^{-1}$ for all $\beta \in Aut(F)$. Replacing T by T', where

$$T'(\beta) = \alpha^{-1} \circ T(\beta) \circ \alpha, \text{ for all } \beta \in End(F),$$

shows that proving the theorem is equivalent to showing

(*) If $T : End(F) \to End(F)$ is an automorphism of $End(F)$ and $T(\beta) = \beta$ for all $\beta \in Aut(F)$, then $T(\beta) = \beta$ for all $\beta \in End(F)$.

Note that if T satisfies the hypotheses of (*), so does T^{-1}, so any property established for T or T^{-1} will also hold for the other.

For $a \in F$, let $\gamma_a \in Aut(F)$ be the inner automorphism of F defined by $\gamma_a(x) = axa^{-1}$, for all $x \in F$. Then for all $\rho \in End(F),\ a, x \in F$,

$$(\rho \circ \gamma_a)(x) = \rho(axa^{-1}) = \rho(a)\rho(x)\rho(a)^{-1} = (\gamma_{\rho(a)} \circ \rho)(x),$$

so $\rho \circ \gamma_a = \gamma_{\rho(a)} \circ \rho$. Now apply T, noting that $T(\gamma_a) = \gamma_a$, by the hypothesis on T in (*). This gives

$$T(\rho) \circ \gamma_a = T(\rho \circ \gamma_a) = T(\gamma_{\rho(a)} \circ \rho) = \gamma_{\rho(a)} \circ T(\rho).$$

Hence for any $x \in F$,

$$\begin{aligned} T(\rho)(a)][T(\rho)(x)][T(\rho)(a)]^{-1} &= T(\rho)(axa^{-1}) = [T(\rho) \circ \gamma_a](x) \\ &= [\gamma_{\rho(a)} \circ T(\rho)](x) = \rho(a)[T(\rho)(x)]\rho(a)^{-1}, \end{aligned}$$

which implies that $\rho(a)^{-1}[T(\rho)(a)]$ centralizes $T(\rho)(F)$, for all $\rho \in End(F),\quad a \in F$. Since any property established for T also holds for T^{-1}, we may replace T by T^{-1}. Then substituting $T(\rho)$ for ρ gives

(1) $[T(\rho)(a)]^{-1}\rho(a)$ centralizes $\rho(F)$, for all $\rho \in End(F),\ a \in F$.

Now suppose that $\rho \in End(F)$ is such that $\rho(F)$ is not abelian. Then the centralizer of $\rho(F)$ in F is trivial, so (1) implies that $[T(\rho)](a) = \rho(a)$ for all $a \in F$; i.e., $T(\rho) = \rho$. Thus we have shown that:

(2) If $\rho(F)$ is not abelian, then $T(\rho) = \rho$.

To establish (*), it remains to show that $T(\rho) = \rho$ for endomorphisms ρ such that $\rho(F)$ is abelian. Abelian subgroups of F are trivial or infinite cyclic. The trivial endomorphism ($\rho(x) = 1$, for all $x \in F$) is characterized by the multiplicative property $\rho \circ \sigma = \rho$ for all $\sigma \in End(F)$, so it is fixed by T. Thus all that remains to be proved is the following:

(3) If $T : End(F) \to End(F)$ satisfies the hypotheses of (*) and $\rho \in End(F)$ is an endomorphism such that $\rho(F)$ is infinite cyclic, then $T(\rho) = \rho$.

To prove (3), consider the endomorphism $\delta : F \to F$ defined by $\delta(x_1) = x_1,\ \delta(x_2) = \delta(x_3) = ... = \delta(x_n) = 1$. The centralizer of $\delta(F) = gp\langle x_1\rangle$ is $gp\langle x_1\rangle$ itself, so (1) implies that $[T(\delta)(a)]^{-1}\delta(a) \in gp\langle x_1\rangle$ for all $a \in F$. Hence there are integers $i_1, ..., i_n$ such that $T(\delta)(x_j) = x_1^{i_j}$, for $j = 1, ..., n$.

For $k = 2, ..., n$, let σ_k be the automorphism of F defined by

$$\sigma_k(x_1) = x_1x_k,\ \sigma_k(x_j) = x_j \ \ (j = 2, ..., n).$$

Then $\delta \circ \sigma_k = \delta$, so

$$T(\delta) \circ \sigma_k = T(\delta) \circ T(\sigma_k) = T(\delta \circ \sigma_k) = T(\delta)$$

and

$$x_1^{(i_1+i_k)} = T(\delta)(x_1 x_k) = [T(\delta) \circ \sigma_k](x_1) = T(\delta)(x_1) = x_1^{i_1},$$

so $i_k = 0$ for $k = 2, ..., n$. Since $\delta \circ \delta = \delta$,

$$x_1^{i_1} = T(\delta)(x_1) = [T(\delta) \circ T(\delta)](x_1) = x_1^{i_1^2}.$$

Thus $i_1^2 = i_1$, so $i_1 = 0$ or $i_1 = 1$. The possibility that $i_1 = 0$ is excluded since $T(\delta)$ would be the trivial endomorphism ($T(\delta)(F) = 1$), which we already know is fixed by T. Thus $i_1 = 1$, so $T(\delta) = \delta$.

Finally, suppose that $\rho \in End(F)$, and that $\rho(F) = gp\langle w \rangle$, an infinite cyclic group. There is a free basis $\{y_1, ..., y_n\}$ for F such that $\rho(y_1) = w$, $\rho(y_2) = \rho(y_3) = ... = \rho(y_n) = 1$. Let σ be the automorphism of F defined by $\sigma(x_i) = y_i$, ($i = 1, ..., n$), and let $\tau \in End(F)$ be defined by $\tau(y_1) = w$, $\tau(y_2) = ... = \tau(y_n) = z$, where z is chosen so that $gp\langle w, z \rangle$ is free of rank two. Computing the images of $y_1, ..., y_n$ shows that $\rho = \tau \circ \sigma \circ \delta \circ \sigma^{-1}$. Since T fixes τ, σ, and δ, it also fixes ρ, which establishes (3) and completes the proof of the theorem.

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Department of Mathematics, The Pennsylvania State University, University Park, Pennsylvania 16802

E-mail address: `formanek@math.psu.edu`