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Bergman Type Spaces and Cesàro Operator

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Abstract In this paper, we give a characterization of Bergman type spaces, and the boundedness of the Cesàro operator on Bergman type spaces.

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1 Introduction

A positive continuous function φ on $[0,1)$ is called normal if there exist two constants β and γ with $0<\gamma<\beta$ such that $\varphi(t)/(1-t^2)^\gamma$ decreases and $\varphi(t)/(1-t^2)^\beta$ increases on $[0,1)$ (*cf.*[1]). The simplest example is $\varphi(t)=(1-t^2)^c$, $c>0$. For any $\alpha>\beta$, let $\psi(t)=(1-t^2)^\alpha/\varphi(t)$. Then $\{\varphi,\psi\}$ is called a normal pair, and ψ is also a normal function with $\alpha-\beta$ and $\alpha-\gamma$. Throughout this paper, we will always assume that φ is a fixed normal function with constants $0<\gamma<\beta$ and $\{\varphi,\psi\}$ is a normal pair with $\varphi(t)\psi(t)=(1-t^2)^\alpha$, $\alpha>\beta$.

Let $D=\{z:|z|<1\}$ denote the open unit disk in the complex plane, and let A denote the normalized area measure on D, and $H(D)$ denote the collection of all analytic functions in D.

Let $0<p<\infty$, and φ be a normal, the space $L^p(\varphi)$ consisit of all Lebesgue measuable functions in D for which

$$\|f\|_{p,\varphi}=\left(\int_D|f(z)|^p\frac{\varphi^p(|z|)}{1-(|z|)}\,dA(z)\right)^{\frac{1}{p}}$$

is finite. Then $L^p(\varphi)$ $(1\le p<\infty)$ is a Banach space with the norm $\|\cdot\|_{p,\varphi}$, and $L^p(\varphi)$ $(0<p<1)$ is a Fréchet space under $\|\cdot\|_{p,\varphi}^p$. Let $A^p(\varphi)=L^p(\varphi)\cap H(D)$. Then the Bergman type space $A^p(\varphi)$ is a closed supspace of $L^p(\varphi)$. Especially, if $\varphi(t)=(1-t^2)^{\frac{1}{p}}$, the $L^p(\varphi)$ is to be the usual Lebesgue space $L^p(D,dA)$, and $A^p(\varphi)$ is to be the usual Bergman space $L^p_a(D,dA)$.

For $f(z)=\sum_{n=0}^\infty a_nz^n\in H(D)$, the Cesàro operator σ is defined by

$$(\sigma f)(z)=\sum_{n=0}^\infty\frac{a_0+a_1+\cdots+a_n}{n+1}z^n.$$

A direct calculation shows that $(\sigma f)(z)=\int_0^1\frac{f(tz)}{1-tz}\,dz$. The characterization of Bergman spaces is described in [2], which we quote as follows.

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Theorem A *Let $n \geq 1$ be an integer, $1 \leq p < \infty$, $f \in H(D)$. Then $f \in L^p_a(D, dA)$ if and only if $(1-|z|^2)^n f^{(n)}(z) \in L^p(D, dA)$.*

In [3], J. Xiao proved

Theorem B *Let $0 < p < \infty$. Then σ is bounded on $L^p_a(D, dA)$.*

In this paper, we will generalize these theorems to Bergman type spaces. We prove the following theorems.

Theorem 1 *Let $n \geq 1$ be an integer, $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\varphi(t) \cdot \psi(t) = (1-t^2)^\alpha$, $\alpha > \beta > \gamma$ and $f \in A^p(\varphi)$. Then we have*

(1) If $1 < p < \infty$, $\alpha > 2q(\beta - \gamma)$, $2\gamma - \beta + n > 0$, then $(1-|z|^2)^n f^{(n)}(z) \in L^p(\varphi)$ and there is a constant $c = c(p, \alpha, n)$ such that

$$(|f(0)| + |f'(0)| + \cdots + |f^{(n-1)}(0)| + \|(1-|z|^2)^n f^{(n)}(z)\|_{p,\varphi}) \leq c\|f\|_{p,\varphi}.$$

(2) If $p = 1$, $\alpha > 2\beta - \gamma$, then $(1-|z|^2)^n f^{(n)}(z) \in L^1(\varphi)$, and there is a constant $c = c(\alpha, n)$ such that

$$(|f(0)| + |f'(0)| + \cdots + |f^{(n-1)}(0)| + \|(1-|z|^2)^n f^{(n)}(z)\|_{1,\varphi}) \leq c\|f\|_{1,\varphi}.$$

Here and elsewhere the constants are denoted by $c, c_1, \ldots$, which may be different at different places.

Theorem 2 *Let $n \geq 1$ be an integer, $1 \leq p < \infty$, $f \in H(D)$, and $(1-|z|^2)^n f^{(n)}(z) \in L^p(\varphi)$. Then $f \in A^p(\varphi)$, and there exists a contant $c = c(p, \alpha, n)$ such that*

$$\|f\|_{p,\varphi} \leq c(|f(0)| + |f'(0)| + \cdots + |f^{(n-1)}(0)| + \|(1-|z|^2)^n f^{(n)}(z)\|_{p,\varphi}).$$

Theorem 3 *Let $1 \leq p < \infty$. Then σ is bounded on $A^p(\varphi)$.*

Theorem 4 *Let $0 < p < 1$. Then σ is bounded on $A^p(\varphi)$.*

2 A Characterization of Bergman Type Spaces

In this section we will establish Theorem 1 and Theorem 2. For this purpose, we need the following lemmas.

Lemma 1[4,1] *Let $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and $\{\varphi, \psi\}$ be a normal pair, $\varphi(t)\psi(t) = (1-t^2)^\alpha$, and let $K_z(\omega) = \dfrac{1}{(1-z\overline{\omega})^{\alpha+1}}$ for $\omega, z \in D$; let the operator P be defined by*

$$(Pf)(z) = \alpha \int_D f(\omega) K_z(\omega)(1-|\omega|^2)^{\alpha-1} dA(\omega), \qquad f \in L^p(\varphi) \cup L^q(\psi).$$

Then P is a bounded projective operator from $L^p(\varphi)$ onto $A^p(\varphi)$, from $L^q(\psi)$ onto $A^q(\psi)$, and we have the reproducing formula

$$(Pf)(z) = f(z), \qquad f \in A^p(\varphi) \cup A^q(\psi),$$

where $L^\infty(\psi)$ is the space of measurable functions f with the norm

$$\|f\|_{\infty,\psi} = \sup_{z \in D}\{\psi(|z|)|f(z)|\}.$$

Lemma 2[5] *Let $z \in D$, c be real, $t > -1$, and $I_{c,t}(z) = \int_D \dfrac{(1-|\omega|^2)^t}{|1-z\overline{\omega}|^{2+t+c}} dA(\omega)$. Then we have*

(1) *If* $c < 0$, *then* $I_{c,t}(z)$ *is bounded in* z;

(2) *If* $c > 0$, *then* $I_{c,t}(z) \sim \dfrac{1}{(1-|z|^2)^c} \qquad (|z| \to 1^-)$;

(3) *If* $c = 0$, *then* $I_{0,t}(z) \sim \log \dfrac{1}{1-|z|^2} \qquad (|z| \to 1^-)$.

Lemma 3 *Let* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\varphi(t)\psi(t) = (1-t^2)^\alpha$, $\alpha > \beta > \gamma$, *and* $\alpha > 2q(\beta - \gamma)$. *Then there is a constant* $c = c(p, \alpha, \beta, \gamma)$ *such that*

$$\int_D h^q(\omega)|K_z(\omega)|\frac{\psi^q(|\omega|)}{1-|\omega|^2}dA(\omega) \le c\frac{(1-|z|^2)^{(\alpha+s)q-\alpha}}{\varphi^q(|z|)},$$

where $h(\omega) = (1-|\omega|^2)^s$, $s \in (\beta - \alpha, 2\gamma - \beta - \frac{\alpha}{p})$.

Proof Fix $z \in D$, and let $\rho_z(\lambda) = \frac{z-\lambda}{1-\bar{z}\lambda}$, $\lambda \in D$. Then there are constants $c_i = c_i(\beta, \gamma)$ $(i = 1, 2)$ such that

$$c_1\frac{\varphi(|z|)(1-|\lambda|^2)^\beta}{|1-\bar{z}\lambda|^{2\gamma}} \le \varphi(|\rho_z(\lambda)|) \le c_2\frac{\varphi(|z|)(1-|\lambda|^2)^\gamma}{|1-\bar{z}\lambda|^{2\beta}}. \qquad (*)$$

In fact, since $\varphi(t)/(1-t^2)^\gamma$ is decreasing and $\varphi(t)/(1-t^2)^\beta$ is increasing, we have

(1) If $0 \le |\rho_z(\lambda)| \le |z|$, then

$$\begin{aligned}\frac{\varphi(|z|)(1-|\lambda|^2)^\beta}{|1-\bar{z}\lambda|^{2\gamma}} &\le \frac{\varphi(|z|)(1-|\lambda|^2)^\gamma}{|1-\bar{z}\lambda|^{2\gamma}} = \frac{\varphi(|z|)(1-|\rho_z(\lambda)|^2)^\gamma}{(1-|z|^2)^\gamma} \le \varphi(|\rho_z(\lambda)|)\\ &\le \frac{\varphi(|z|)(1-|\rho_z(\lambda)|^2)^\beta}{(1-|z|^2)^\beta} = \frac{\varphi(|z|)(1-|\lambda|^2)^\beta}{|1-\bar{z}\lambda|^{2\beta}} \le \frac{\varphi(|z|)(1-|\lambda|^2)^\gamma}{|1-\bar{z}\lambda|^{2\beta}};\end{aligned}$$

(2) If $|z| \le |\rho_z(\lambda)| < 1$ then

$$\begin{aligned}\frac{\varphi(|z|)(1-|\lambda|^2)^\beta}{2^{2(\beta-\gamma)}|1-\bar{z}\lambda|^{2\gamma}} &\le \frac{\varphi(|z|)(1-|\lambda|^2)^\beta}{|1-\bar{z}\lambda|^{2\beta}} = \frac{\varphi(|z|)(1-|\rho_z(\lambda)|^2)^\beta}{(1-|z|^2)^\beta} \le \varphi(|\rho_z(\lambda)|)\\ &\le \frac{\varphi(|z|)(1-|\rho_z(\lambda)|^2)^\gamma}{(1-|z|^2)^\gamma} = \frac{\varphi(|z|)(1-|\lambda|^2)^\gamma}{|1-\bar{z}\lambda|^{2\gamma}} \le 2^{2(\beta-\gamma)}\frac{\varphi(|z|)(1-|\lambda|^2)^\gamma}{|1-\bar{z}\lambda|^{2\beta}};\end{aligned}$$

Therefore $(*)$ holds.

Now letting $\omega = \rho_z(\lambda) = \frac{z-\lambda}{1-\bar{z}\lambda}$ and using $(*)$, we have

$$\begin{aligned}I &= \int_D h^q(\omega)|k_z(\omega)|\frac{\psi^q(|\omega|)}{1-|\omega|^2}dA(\omega)\\ &= \int_D \frac{1}{|1-\bar{z}\omega|^{\alpha+1}}\frac{(1-|\omega|^2)^{(\alpha+s)q-1}}{\varphi^q(|\omega|)}dA(\omega)\\ &= \int_D \frac{|1-\bar{z}\lambda|^{\alpha+1}}{(1-|z|^2)^{\alpha+1}}\frac{(1-|z|^2)^{(\alpha+s)q-1}(1-|\lambda|^2)^{(\alpha+s)q-1}}{|1-\bar{z}\lambda|^{2(\alpha+s)q-2}}\frac{1}{\varphi^q(|\rho_z(\lambda)|)}\frac{(1-|z|^2)^2}{|1-\bar{z}\lambda|^4}dA(\lambda)\\ &= (1-|z|^2)^{(\alpha+s)q-\alpha}\int_D \frac{(1-|\lambda|^2)^{(\alpha+s)q-1}|1-\bar{z}\lambda|^{(\alpha-1)-2(\alpha+s)q}}{\varphi^q(|\rho_z(\lambda)|)}dA(\lambda)\\ &\le c_2\frac{(1-|z|^2)^{(\alpha+s)q-\alpha}}{\varphi^q(|z|)}\int_D (1-|\lambda|^2)^{(\alpha+s-\beta)q-1}(|1-\bar{z}\lambda|)^{(\alpha-1)-2(\alpha+s-\gamma)q}dA(\lambda).\end{aligned}$$

Since $\alpha > 2q(\beta-\gamma)$, and $s \in \left(\beta - \alpha, 2\gamma - \beta - \frac{\alpha}{p}\right)$, Lemma 2 shows that the last integral above is bounded in D, and then Lemma 3 holds.

Lemma 4 *Let $n \geq 1$ be an integer, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\varphi(t)\psi(t) = (1-t^2)^\alpha$, $\alpha > \beta > \gamma > 0$, $\alpha > 2q(\beta - \gamma)$, $2\gamma - \beta + n > 0$, and T be the operator defined by*

$$(Tf)(z) = (1-|z|^2)^n \int_D \frac{\overline{\omega}^n f(\omega)}{(1-z\overline{\omega})^{\alpha+n+1}} (1-|\omega|^2)^{\alpha-1} dA(\omega).$$

Then T is a bounded operator on $L^p(\varphi)$.

Proof Since $\alpha > 2q(\beta-\gamma)$, $2\gamma - \beta + n > 0$, we can choose s such that

$$\max\left(-\frac{\alpha}{p} - n, \beta - \alpha\right) < s < \min\left(0, 2\gamma - \beta - \frac{\alpha}{p}\right).$$

Let $h(\omega) = (1-|\omega|^2)^s$. Using Hölder's inequality and Lemma 3, we have

$$\begin{aligned}
&|(Tf)(z)|^p \\
&\leq \left\{(1-|z|^2)^n \int_D |f(\omega)| h(\omega) h^{-1}(\omega) \frac{\varphi(|\omega|)\psi(|\omega|)}{|1-z\overline{\omega}|^{\alpha+1+n}} \frac{1}{(1-|\omega|^2)^{\frac{1}{p}}} \frac{1}{(1-|\omega|^2)^{\frac{1}{q}}} dA(\omega)\right\}^p \\
&\leq (1-|z|^2)^{np} \left\{\int_D h^q(\omega) \frac{1}{|1-z\overline{\omega}|^{\alpha+1}} \frac{\psi^q(|\omega|)}{1-|\omega|^2} dA(\omega)\right\}^{\frac{p}{q}} \\
&\quad \cdot \left\{\int_D h^{-p}(\omega) |f(\omega)|^p \frac{1}{|1-z\overline{\omega}|^{pn+\alpha+1}} \frac{\varphi^p(|\omega|)}{1-|\omega|^2} dA(\omega)\right\} \\
&\leq c \frac{(1-|z|^2)^{(n+s)p+\alpha}}{\varphi^p(|z|)} \left\{\int_D h^{-p}(\omega)|f(\omega)|^p \frac{1}{|1-z\overline{\omega}|^{pn+\alpha+1}} \frac{\varphi^p(|\omega|)}{1-|\omega|^2} dA(\omega)\right\}.
\end{aligned}$$

Thus, by Fubini's theorem, we have

$$\|Tf\|_{p,\varphi}^p \leq c \cdot \int_D (1-|\omega|^2)^{-sp} |f(\omega)|^p \frac{\varphi^p(|\omega|)}{1-|\omega|^2} \cdot \left\{\int_D \frac{(1-|z|^2)^{(n+s)p+\alpha-1}}{|1-z\overline{\omega}|^{pn+\alpha+1}} dA(z)\right\} dA(\omega).$$

Since $s > -\frac{\alpha}{p} - n$, and $s < 0$, using Lemma 2, we have

$$\|Tf\|_{p,\varphi}^p \leq c \cdot \int_D (1-|\omega|^2)^{-sp} |f(\omega)|^p \frac{\varphi^p(|\omega|)}{1-|\omega|^2} \cdot \frac{1}{(1-|\omega|^2)^{-sp}} dA(\omega) = c(p,\alpha,n,s)\|f\|_{p,\varphi}^p.$$

Thus T is bounded on $L^p(\varphi)$, and the proof is complete.

Lemma 5 *Let $n \geq 1$ be an integer, $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\varphi(t)\psi(t) = (1-t^2)^\alpha$, $\alpha > \beta > \gamma$, and $f \in A^p(\varphi)$. Then we have*

(1) *If $1 < p < \infty$, $\alpha > 2q(\beta - \gamma)$, $2\gamma - \beta + n > 0$, then $(1-|z|^2)^n f^{(n)}(z) \in L^p(\varphi)$, and there is a constant $c = c(p, \alpha, n)$, such that*

$$\|(1-|z|^2)^n f^{(n)}(z)\|_{p,\varphi} \leq c\|f\|_{p,\varphi}.$$

(2) *If $p = 1$, $\alpha > 2\beta - \gamma$, then $(1-|z|^2)^n f^{(n)}(z) \in L^1(\varphi)$, and there is a constant $c = c(p,\alpha,n)$ such that*

$$\|(1-|z|^2)^n f^{(n)}(z)\|_{1,\varphi} \leq c\|f\|_{1,\varphi}.$$

Proof (1) Supposing $f \in A^p(\varphi)$, then by Lemma 1,

$$f(z) = \alpha \int_D \frac{f(\omega)}{(1-z\overline{\omega})^{\alpha+1}} (1-|\omega|^2)^{\alpha-1} dA(\omega), \quad z \in D.$$

Differentiating under the intergral sign, we have

$$(1-|z|^2)^n f^{(n)}(z)=\alpha(\alpha+1)\cdots(\alpha+n)(1-|z|^2)^n\int_D \frac{\overline{\omega}^n f(\omega)}{(1-z\overline{\omega})^{\alpha+1+n}}(1-|\omega|^2)^{\alpha-1}dA(\omega).$$

By Lemma 4, there exists a constant c such that

$$\|(1-|z|^2)^n f^{(n)}(z)\|_{p,\varphi}\le c\|f\|_{p,\varphi}.$$

(2) Supposing $f\in A^1(\varphi)$, then by Lemma 1,

$$f(z)=\alpha\int_D \frac{f(\omega)}{(1-z\overline{\omega})^{\alpha+1}}(1-|\omega|^2)^{\alpha-1}dA(\omega),\quad z\in D.$$

Differentiating under the intergral sign, we have

$$(1-|z|^2)^n f^{(n)}(z)=\alpha(\alpha+1)\cdots(\alpha+n)(1-|z|^2)^n\int_D \frac{\overline{\omega}^n f(\omega)}{(1-z\overline{\omega})^{\alpha+1+n}}(1-|\omega|^2)^{\alpha-1}dA(\omega).$$

Thus, by Fubini's theorem, we have

$$\begin{aligned}\|(1-|z|^2)^n f(n)(z)\|_{1,\varphi}&\le \alpha(\alpha+1)\cdots(\alpha+n)\\ &\cdot\int_D |f(\omega)|(1-|\omega|^2)^{\alpha-1}\left[\int_D \frac{(1-|z|^2)^{n-1}}{|1-z\overline{\omega}|^{\alpha+1+n}}\varphi(|z|)dA(z)\right]dA(\omega).\end{aligned}$$

Let $z=\rho_\omega(\lambda)=\frac{\omega-\lambda}{1-\overline{\omega}\lambda}$. Then

$$\begin{aligned}I_1&=\int_D \frac{(1-|z|^2)^{n-1}}{|1-z\overline{\omega}|^{\alpha+1+n}}\varphi(|z|)dA(z)\\ &=(1-|\omega|^2)^{-\alpha}\int_D \frac{(1-|\lambda|^2)^{n-1}}{|1-\overline{\omega}\lambda|^{n+1-\alpha}}\varphi(|\rho_\omega(\lambda)|)dA(\lambda)\\ &\le c\cdot(1-|\omega|^2)^{-\alpha}\varphi(|\omega|)\int_D \frac{(1-|\lambda|^2)^{n-1+\lambda}}{|1-\overline{\omega}\lambda|^{n+1-\alpha+2\beta}}dA(\lambda)\quad \text{(by (*))}\\ &\le c\cdot(1-|\omega|^2)^{-\alpha}\varphi(|\omega|).\qquad\qquad \text{(by Lemma 2)}\end{aligned}$$

Hence $\|(1-|z|^2)^n f^{(n)}(z)\|_{p,\varphi}\le c\|f\|_{1,\varphi}$.

The lemma is proved.

Remark It is not difficult to see that there is a constant $c=c(p,\alpha,\beta,\gamma)$ such that

$$|f^{(k)}(0)|\le c\|f\|_{p,\varphi}\qquad (k=0,1,2,\ldots,n-1)\ .$$

In fact, since $f^{(k)}(0)=\alpha(\alpha+1)\cdots(\alpha+k)\int_D\overline{\omega}^k f(\omega)(1-|\omega|^2)^{\alpha-1}dA(\omega)$, we have

(1) If $p>1$, by Hölder's inequality, we get $|f^{(k)}(0)|\le c\|f\|_{p,\varphi}$;

(2) If $p=1$, by the fact that $\varphi(t)/(1-t^2)^\beta$ is increasing, then

$$\begin{aligned}|f^{(k)}(0)|&\le \alpha(\alpha+1)\cdots(\alpha+k)\int_D |f(\omega)|\frac{\varphi(|\omega|)}{1-|\omega|^2}\frac{(1-|\omega|^2)^\beta}{\varphi(|\omega|)}(1-|\omega|^2)^{\alpha-\beta}dA(\omega)\\ &\le \frac{\alpha(\alpha+1)\cdots(\alpha+n)}{\varphi(0)}2^{\alpha-\beta}\|f\|_{1,\varphi}\qquad (k=0,1,2,\ldots,n-1)\ .\end{aligned}$$

Lemma 6 *Let $n \geq 1$ be an integer, $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\varphi(t)\psi(t) = (1-t^2)^\alpha$, and $f \in H(D)$, $(1-|z|^2)^n f^{(n)}(z) \in L^p(\varphi)$, $f(0) = f'(0) = \cdots = f^{(2n)}(0) = 0$. Then $f(z) = (Ph)(z)$, where $h(z) = \frac{1}{\alpha(\alpha+1)\cdots(\alpha+n-1)} \frac{(1-|z|^2)^n f^{(n)}(z)}{\overline{z}^n}$, and $h \in L^p(\varphi)$.*

Proof Since $f \in H(D)$, $f(0) = f'(0) = \cdots = f^{(2n)}(0) = 0$, and $(1-|z|^2)^n f^{(n)}(z) \in L^p(\varphi)$, it is easy to see that $h \in L^p(\varphi)$.

Let $g(z) = (Ph)(z)$. Then

$$g(z) = \frac{\alpha}{\alpha(\alpha+1)\cdots(\alpha+n-1)} \int_D \frac{(1-|\omega|^2)^n f^{(n)}(\omega)}{\overline{\omega}^n (1-z\overline{\omega})^{\alpha+1}} (1-|\omega|^2)^{\alpha-1} dA(\omega).$$

Differentiating under the integral sign, we have

$$g^{(n)}(z) = (\alpha+n) \int_D \frac{f^{(n)}(\omega)}{(1-z\overline{\omega})^{\alpha+n+1}} (1-|\omega|^2)^{\alpha+n-1} dA(\omega) .$$

Since $(1-|z|^2)^n f^{(n)}(z) \in L^p(\varphi)$, this shows that $f^{(n)}(z) \in L^p(\varphi_1)$, where $\varphi_1(t) = \varphi(t)(1-t^2)^n$, $\{\varphi_1, \psi\}$ is a normal pair, $\varphi_1(t)\psi(t) = (1-t^2)^{n+\alpha}$.

Thus, by Lemma 1, we have $g^{(n)}(z) = f^{(n)}(z), z \in D$. If $0 \leq k \leq n-1$, the Taylor expansion shows that

$$g^{(k)}(0) = \frac{\alpha(\alpha+1)\cdots(\alpha+k)}{\alpha(\alpha+1)\cdots(\alpha+n-1)} \int_D \frac{f^{(n)}(\omega)}{\overline{\omega}^{n-k}} (1-|\omega|^2)^{\alpha+n-1} dA(\omega) = 0 .$$

Thus we have $g^{(n)}(z) = f^{(n)}(z)$ and $g^{(k)}(0) = f^{(k)}(0)$ for all $0 \leq k \leq n-1$, and the proof is complete.

Corollary 7 *Let $n \geq 1$ be an integer, $1 \leq p < \infty$, and $f \in H(D)$, $(1-|z|^2)^n f^{(n)}(z) \in L^p(\varphi)$. Then $f \in A^p(\varphi)$, and there exists a constant $c = c(p, \varphi, n)$ such that*

$$\|f\|_{p,\varphi} \leq c\big(|f(0)| + |f'(0)| + \cdots + |f^{(n-1)}(0)| + \|(1-|z|^2)^n f^{(n)}(z)\|_{p,\varphi}\big) .$$

Proof Let $f \in H(D)$. Write

$$f(z) = \sum_{k=0}^{2n} \frac{f^{(k)}(0)}{k!} z^k + f_1(z) .$$

Since $(1-|z|^2)^n f^{(n)}(z) \in L^p(\varphi)$, the above equality shows that $(1-|z|^2)^n f_1^{(n)}(z) \in L^p(\varphi)$. Lemma 6 implies that $f_1 = Ph$ for some $h \in L^p(\varphi)$. Since P is bounded on $L^p(\varphi)$ for $1 \leq p < \infty$, we have $f_1 \in A^p(\varphi)$ for $1 \leq p < \infty$; therefore, $f \in A^p(\varphi)$. Now, the closed graph theorem shows that there exists a constant c such that

$$\|f\|_{p,\varphi} \leq c\big(|f(0)| + |f'(0)| + \cdots + |f^{(n-1)}(0)| + \|(1-|z|^2)^n f^{(n)}(z)\|_{p,\varphi}\big) .$$

The proof is complete.

Proof of Theorem 1 This follows from Lemma 5.

Proof of Theorem 2 This follows from Corollary 7.

3 The Cesàro Operator on Bergman Type Spaces

In this section we will prove Theorem 3 and Theorem 4.

Lemma 8 *Let $1 \le p < \infty$, $f \in A^p(\varphi)$, and $(Vf)(z) = \frac{f(z) - f(0)}{z}$. Then there exists a constant c such that $\|Vf\|_{p,\varphi} \le c\|f\|_{p,\varphi}$.*

Proof Since $f \in A^p(\varphi)$, it is not difficult to see that $Vf \in A^p(\varphi)$. Now the closed graph theorem implies that there is a finite constant c such that $\|Vf\|_{p,\varphi} \le c\|f\|_{p,\varphi}$ for all $f \in A^p(\varphi)$.

Lemma 9 *For $s > 0$ and $0 < r < 1$, $\int_0^{2\pi} |1 - re^{it}|^{-1-s} dt = O\big((1-r)^{-s}\big)$.*

For a proof see [6].

Lemma 10 (Hardy-Littlewood) *Let $p \in (0,1)$, $q \in (1,\infty)$ and $f \in H(D)$. Then*

$$\int_{-\pi}^{\pi} \sup_{-0 \le s < 1} |f(sre^{it})^p| dt \le c_1 \int_{-\pi}^{\pi} |f(re^{it})|^p dt,$$

$$\int_0^1 \left\{ \int_{-\pi}^{\pi} |f(sre^{it})^{pq}| dt \right\}^{\frac{1}{q}} (1-s)^{-\frac{1}{q}} ds \le c_2 \int_{-\pi}^{\pi} |f(re^{it})|^p dt,$$

where the constants c_1 and c_2 are independent of $r \in (0,1)$.

For a proof see [6].

Proof of Theorem 3 Suppose $1 \le p < \infty$, $f \in A^p(\varphi)$, and let $g = \sigma f$. Then

$$zg'(z) = \frac{f(z)}{1-z} - \int_0^1 \frac{f(tz)}{1-tz} dt .$$

It is obvious that

$$\|(1-|z|^2)\frac{f(z)}{1-z}\|_{p,\varphi} \le 2\|f\|_{p,\varphi} .$$

Using the Minkowski inequality and the monotonicity of $\int_0^{2\pi} |f(tre^{i\theta})|^p d\theta$ in t, we have

$$\begin{aligned}
\|(1-|z|^2)\int_0^1 \frac{f(tz)}{1-tz} dt\|_{p,\varphi} &\le \int_0^1 \left\{ \int_D \left[(1-|z|^2)|\frac{f(tz)}{1-tz}\right]^p \frac{\varphi^p(|z|)}{1-|z|^2} dA(z) \right\}^{\frac{1}{p}} dt \\
&\le 2\int_0^1 \left[\int_D |f(tz)|^p \frac{\varphi^p(|z|)}{1-|z|^2} dA(z) \right]^{\frac{1}{p}} dt \\
&\le 2\int_0^1 \left[\int_D |f(z)|^p \frac{\varphi^p(|z|)}{1-|z|^2} dA(z) \right]^{\frac{1}{p}} dt \\
&= 2\|f\|_{p,\varphi} .
\end{aligned}$$

Thus $\|(1-|z|^2)zg'(z)\|_{p,\varphi} \le 4\|f\|_{p,\varphi}$.

The above inequality implies that $zg'(z) \in A^p(\varphi_1)$, where $\varphi_1(t) = \varphi(t)(1-t^2)$. By Lemma 8, there exists a constant C_p such that

$$\|(1-|z|^2)g'(z)\|_{p,\varphi} = \|g'(z)\|_{p,\varphi_1} \le C_p\|zg'(z)\|_{p,\varphi_1} = C_p\|(1-|z|^2)zg'(z)\|_{p,\varphi}.$$

Hence, we have $\|(1-|z|^2)g'(z)\|_{p,\varphi} \le 4C_p\|f\|_{p,\varphi}$. Now, by Theorem 1, and noting $g(0) = f(0)$, we have

$$\|g\|_{p,\varphi} \le C_3\big(|f(0)| + \|(1-|z|^2)g'(z)\|_{p,\varphi}\big) \le C_4\|f\|_{p,\varphi},$$

where the constants C_3 and C_4 are independent of f.

This completes the proof.

Proof of Theorem 4 Suppose $p \in (0,1)$, $f \in A^p(\varphi)$, and let $g = \sigma f$, $q \in \left(1, \frac{1}{1-p}\right)$, let $\frac{1}{q'} + \frac{1}{q} = 1$. Then using the Hölder inequality and Lemma 9, we have

$$\begin{aligned}\int_{-\pi}^{\pi} \left|\frac{f(sre^{it})}{1-sre^{it}}\right|^p dt &\le \left[\int_{-\pi}^{\pi} \left|\frac{1}{1-sre^{it}}\right|^{pq'}\right]^{\frac{1}{q'}} \left[\int_{-\pi}^{\pi} |f(sre^{it})|^{pq} dt\right]^{\frac{1}{q}} \\ &\le C_5 (1-s)^{\frac{1-pq'}{q'}} \left[\int_{-\pi}^{\pi} |f(sre^{it})|^{pq} dt\right]^{\frac{1}{q}},\end{aligned}$$

where C_5 is an absolute constant. Now, dividing the interval $[0,1]$ with $s_k = 1 - 2^{-k}$, $k = 0, 1, 2, \ldots$, and using the above inequality and Lemma 10, we have

$$\begin{aligned}\int_{-\pi}^{\pi} |g(re^{it})|^p dt &\le \int_{-\pi}^{\pi} \left[\int_0^1 \left|\frac{f(sre^{it})}{1-sre^{it}}\right| ds\right]^p dt \\ &\le 2^p \sum_{k=1}^{\infty} \int_{-\pi}^{\pi} \left[\int_{s_{k-1}}^{s_k} \left|\frac{f(sre^{it})}{1-sre^{it}}\right| ds\right]^p dt \\ &\le 2^p \sum_{k=1}^{\infty} \frac{1}{2^{kp}} \int_{-\pi}^{\pi} \left[\sup_{0\le s\le s_k} \left|\frac{f(sre^{it})}{1-sre^{it}}\right|^p\right] dt \\ &\le C_6 \sum_{k=1}^{\infty} \left[\int_{s_{k-1}}^{s_k} (1-s)^{p-1} \int_{-\pi}^{\pi} \left|\frac{f(sre^{it})}{1-sre^{it}}\right|^p dt ds\right] \\ &\le C_7 \int_0^1 (1-s)^{-\frac{1}{q}} \left[\int_{-\pi}^{\pi} |f(sre^{it})|^{pq} dt\right]^{\frac{1}{q}} ds \\ &\le C_8 \int_{-\pi}^{\pi} |f(re^{it})|^p dt\,.\end{aligned}$$

Therefore $\|g\|_{A^p(\varphi)} \le C_9 \|f\|_{A^p(\varphi)}$.
This completes the proof.

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