# A Complex Rolle's Theorem

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**1 INTRODUCTION.** It is well known that many results of classical real analysis are consequences of the Rolle and Mean Value Theorems. In the general case of maps from a subset of a Banach space into another (see [**4**], [**5**] for example), the Mean Value Theorem is an inequality which may be adequate in many applications but falls short of establishing a Rolle's Theorem in the form of an equality as this theorem exists in one real variable. Recently, other variations and interesting applications of Rolle's Theorem have also appeared ([**1**], [**2**], [**3**], [**12**], [**14**]).

Concerning the complex case, Jean Dieudonné [**6**] in 1930 published a necessary and sufficient condition for the existence of a zero of $f'(z)$ in the interior of a circle with diameter $ab$ when $f$ is holomorphic and $f(a) = f(b) = 0$. M. Marden ([**10**], [**11**]) furnishes results about the relative locations of the zeros of a complex polynomial and the zeros of its derivative. I. J. Schoenberg [**13**] conjectures an analogue of Rolle's theorem for polynomials with real or complex coefficients.

It is well known that Rolle's Theorem is not valid for holomorphic functions of a complex variable as it is shown by the function $f(z) = e^z - 1$ which takes the value 0 at $z = 2k\pi i$ for every $k \in \mathbb{Z}$, but $f'(z) = e^z$ has no zeros in the complex plane. It is also easy to see that Rolle's Theorem is not valid for real harmonic functions. For example, the zeros of the partial derivatives of $u(x, y) = x^2 - y^2$ do not separate the zeros of $u$. Therefore, there is no hope to establish a Rolle's Theorem about the real part or about the imaginary part of a holomorphic function. To establish our Rolle's Theorem, we will need to use a combination of $\Re(f)$ and $\Im(f)$.

The aim of this paper is to present a generalization of Rolle's Theorem to holomorphic functions of a complex variable and to show how a Mean Value Theorem for holomorphic functions follows from this theorem. To emphasize the main ideas of our results we will give the simplest possible form of the theorems, and will refer to extensive generalizations and applications of these results which will be given elsewhere ([**8**], [**9**]). The basic nature and far reaching consequences of these theorems suggest that they should become standard results for holomorphic functions of a complex variable.

We begin by stating and proving the Complex Rolle's Theorem in Theorem 2.1. In Theorem 2.2 we apply Theorem 2.1 to prove a Complex Mean Value Theorem. In Corollary 2.3 we obtain a standard result in complex analysis as a Corollary of our Complex Mean Value Theorem. We conclude by providing several examples and remarks in 2.4. Throughout this paper, we will use the standard notation $z = x + iy$ for $z \in \mathbb{C}$, where $x = \Re(z)$ and $y = \Im(z)$. If $a$ and $b$ are distinct points in $\mathbb{C}$, we will denote by $]a, b[$ the open line segment joining $a$ and $b$:

$$]a,b[ = \{a + t(b - a) : t \in ]0,1[\}.$$

**2 RESULTS.** The main idea of our complex version of Rolle's Theorem below is to consider the relation between the zeros of a holomorphic function $f$ and the zeros of $\Re(f')$, or between $f$ and $\Im(f')$, knowing that no Rolle's Theorem can be established about $\Re(f)$ only or about $\Im(f)$ only.

**Theorem 2.1.** (Complex Rolle's Theorem). *Let $f$ be a holomorphic function defined on an open convex subset $D_f$ of $\mathbb{C}$. Let $a, b \in D_f$ be such that $f(a) = f(b) = 0$ and $a \neq b$. Then there exists $z_1, z_2 \in ]a, b[$ such that $\Re(f'(z_1)) = 0$ and $\Im(f'(z_2)) = 0$.*

*Proof:* Let $a_1 = \Re(a)$, $a_2 = \Im(a)$, $b_1 = \Re(b)$, $b_2 = \Im(b)$, $u(z) = \Re(f(z))$, $v(z) = \Im(f(z))$ for every $z \in D_f$. Let

$$\phi(t) = (b_1 - a_1)u(a + t(b - a)) + (b_2 - a_2)v(a + t(b - a))$$

for every $t \in [0, 1]$. Then $f(a) = f(b) = 0$ implies that $u(a) = u(b) = v(a) = v(b) = 0$. Consequently, $\phi(0) = 0$ and $\phi(1) = 0$. Therefore, by Rolle's Theorem, there exists $t_1 \in ]0, 1[$ such that $\phi'(t_1) = 0$. Let $z_1 = a + t_1(b - a)$. Then

$$0 = \phi'(t_1) = (b_1 - a_1)\left[\frac{\partial u}{\partial x}(z_1)(b_1 - a_1) + \frac{\partial u}{\partial y}(z_1)(b_2 - a_2)\right]$$

$$+(b_2 - a_2)\left[\frac{\partial v}{\partial x}(z_1)(b_1 - a_1) + \frac{\partial v}{\partial y}(z_1)(b_2 - a_2)\right].$$

By the Cauchy-Riemann equations it follows that

$$0 = \frac{\partial u}{\partial x}(z_1)\left[(b_1 - a_1)^2 + (b_2 - a_2)^2\right].$$

Therefore,

$$\Re(f'(z_1)) = \frac{\partial u}{\partial x}(z_1) = 0.$$

By applying this first part of the theorem to the function $g = -if$ we obtain that there exists a $z_2 \in ]a, b[$ such that

$$0 = \Re(g'(z_2)) = \frac{\partial v}{\partial x}(z_2) = -\frac{\partial u}{\partial y}(z_2) = \Im(f'(z_2)). \qquad \blacksquare$$

An important application of Theorem 2.1 is the generalization of the real Mean Value Theorem.

**Theorem 2.2.** (Complex Mean Value Theorem). *Let $f$ be a holomorphic function defined on an open convex subset $D_f$ of $\mathbb{C}$. Let $a$ and $b$ be two distinct points in $D_f$. Then there exist $z_1, z_2 \in ]a, b[$ such that*

$$\Re(f'(z_1)) = \Re\left(\frac{f(b) - f(a)}{b - a}\right) \quad \textit{and} \quad \Im(f'(z_2)) = \Im\left(\frac{f(b) - f(a)}{b - a}\right).$$

*Proof:* Let

$$g(z) = f(z) - f(a) - \frac{f(b) - f(a)}{b - a}(z - a) \tag{1}$$

for every $z \in D_f$. Clearly, $g(a) = g(b) = 0$. Therefore, by Theorem 2.1, there exist $z_1, z_2 \in ]a, b[$ such that $\Re(g'(z_1)) = 0$ and $\Im(g'(z_2)) = 0$. But by (1)

$$g'(z) = f'(z) - \frac{f(b) - f(a)}{b - a}$$

for every $z \in D_f$. Therefore,

$$0 = \Re(g'(z_1)) = \Re(f'(z_1)) - \Re\left(\frac{f(b) - f(a)}{b - a}\right),$$

and

$$0 = \Im(g'(z_2)) = \Im(f'(z_2)) - \Im\left(\frac{f(b) - f(a)}{b - a}\right). \qquad \blacksquare$$

Let us show that our Complex Mean Value Theorem (2.2) is strong enough to imply the following basic result in complex analysis.

**Corollary 2.3.** *Let $f$ be a holomorphic function defined on an open connected subset $D_f$ of $\mathbb{C}$ such that $f'(z) = 0$ for every $z \in D_f$. Then $f$ is constant.*

*Proof:* By Lemma 2.1 in [7], or by the Analytic Continuation Theorem of complex analysis, it is sufficient to show $f$ is locally constant. Let $z_0$ be arbitrary in $D_f$ and let $U_{z_0}$ be a convex neighborhood of $z_0$ contained in $D_f$. Let $z$ be a point of $U_{z_0}$, $z \neq z_0$. By Theorem 2.2, there exist $z_1, z_2 \in ]z_0, z[$ such that

$$\Re\left(\frac{f(z) - f(z_0)}{z - z_0}\right) = \Re(f'(z_1)) = 0,$$

and

$$\Im\left(\frac{f(z) - f(z_0)}{z - z_0}\right) = \Im(f'(z_2)) = 0.$$

Therefore, $f(z) = f(z_0)$. Thus $f$ is constant in $U_{z_0}$. $\blacksquare$

We conclude this note by providing several examples. These examples shed light on the Complex Rolle's Theorem and illustrate the assertion that the zeros of the real and imaginary parts of the derivative of a holomorphic function separate the zeros of that holomorphic function.

**Examples and Remarks 2.4.** (i) Let $f(z) = e^z - 1$ and note that $f(z) = 0$ for $z = 2k\pi i$ for every integer $k$. Since $f'(z) = e^z = e^x \cos y + ie^x \sin y$, $\Re(f'(z)) = 0$ if $y = (2k + 1)\pi/2$, and $\Im(f'(z)) = 0$ if $y = k\pi$. Therefore the zeros of the real and imaginary parts of $f'$ are straight lines both separating the zeros of $f$.
(ii) If $f(z) = (z - a)(z - b), a \neq b$, then $f(z) = 0$ when $z = a$ or $z = b$. Since $f'(z) = 2z - a - b$, $\Re(f'(z)) = 0$ if $x = \Re(a + b)/2$, $\Im(f'(z)) = 0$ if $y = \Im(a + b)/2$. So again the zeros of the real and imaginary parts of $f'$ are lines both separating the zeros of $f$.

(iii) Note that in general the zero set of $\Re(f'(z))$ and $\Im(f'(z))$ need not be straight lines as it may be seen by considering $f(z) = z^3 + z^2 + z + 1$; the zero set of $\Re(f')$ is a hyperbola in this case.

We provide many extensions and applications of these theorems in a separate paper [**9**].

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Logic is the hygiene the mathematician practices to keep his ideas healthy and strong.

—*H. Weyl*