MATHEMATICAL METHODS IN THE APPLIED SCIENCES
Math. Meth. Appl. Sci. 2006; **29**:1435–1450
Published online 21 April 2006 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/mma.735
MOS subject classification: 46 F 10; 30 G 35

On the interplay between the Hilbert transform and conjugate harmonic functions

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Communicated by W. Sprößig

SUMMARY

As is well-known, there is a close and well-defined connection between the notions of Hilbert transform and of conjugate harmonic functions in the context of the complex plane. This holds e.g. in the case of the Hilbert transform on the real line, which is linked to conjugate harmonicity in the upper (or lower) half plane. It also can be rephrased when dealing with the Hilbert transform on the boundary of a simply connected domain related to conjugate harmonics in its interior (or exterior). In this paper, we extend these principles to higher dimensional space, more specifically, in a Clifford analysis setting. We will show that the intimate relation between both concepts remains, however giving rise to a range of possibilities for the definition of either new Hilbert-like transforms, or specific notions of conjugate harmonicity. Copyright © 2006 John Wiley & Sons, Ltd.

KEY WORDS: Hilbert transform; conjugate harmonic functions; Clifford analysis

1. INTRODUCTION

The motivation for writing this paper comes from the observation that there is an intimate relationship between the Hilbert transform on the one hand and conjugate harmonic functions on the other. Let us explain this relationship in the classical setting of the complex plane.

In the first place, it is well-known that given a real-valued function $f \in L_2(\mathbb{R})$, where the real axis is to be understood as the boundary of the upper half plane $\mathbb{C}^+ = \{z = x + iy \in \mathbb{C} : y > 0\}$, there exists a unique harmonic function $F(x, y) = \mathscr{P}[f](x, y)$ in $\mathbb{C}^+$, called the Poisson

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Contract/grant sponsor: FWO-Vlaanderen

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transform of f, such that, in L_2-sense, $\lim_{y\to 0+} F(x,y)=f(x)$. Moreover, this function $F(x,y)$ admits a unique harmonic function $G(x,y)$ in $\mathbb{C}^+$ vanishing at infinity and such that $F+\mathrm{i}G$ is holomorphic in the upper half plane. This function $G(x,y)$ is known as the conjugate harmonic function to $F(x,y)$ in $\mathbb{C}^+$. It is usually called the conjugate Poisson transform of f and denoted as $\mathscr{Q}[f](x,y)$. The Hilbert transform $\mathscr{H}[f](x)$ may then be defined as the boundary value $\lim_{y\to 0+} G(x,y)$, taken in L_2-sense. In other words: $\mathscr{Q}[f](x,y)=\mathscr{P}[\mathscr{H}[f]](x,y)$.

So, more explicitly, the Hilbert transform on the real line $\mathscr{H}: L_2(\mathbb{R})\to L_2(\mathbb{R})$, given by the convolution integral

$$\mathscr{H}[f](x)=\frac{1}{\pi}\,\mathrm{Pv}\frac{1}{\cdot}*f(\cdot)(x)=\frac{1}{\pi}\,\mathrm{Pv}\int_{-\infty}^{+\infty}\frac{f(t)}{x-t}\,\mathrm{d}t$$

and the Poisson transform $\mathscr{P}: L_2(\mathbb{R})\to \mathrm{Harm}(\mathbb{C}^+)$, given by

$$\mathscr{P}[f](x)=P(\cdot,y)*f(\cdot)(x)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{y}{(x-t)^2+y^2}\,f(t)\,\mathrm{d}t$$

are related by the property that for a real-valued function f, $\mathscr{P}[f]$ and $\mathscr{P}[\mathscr{H}[f]]$ are conjugate harmonic functions in the upper half plane $\mathbb{C}^+$. Moreover they constitute the real and imaginary parts of the holomorphic Cauchy integral of f, given in $\mathbb{C}^+$ by

$$\mathscr{C}[f](z)=-\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{f(t)}{(x-t)+\mathrm{i}y}\,\mathrm{d}t=\frac{1}{2}\mathscr{P}[f]+\frac{\mathrm{i}}{2}\mathscr{P}[\mathscr{H}[f]]$$

Sometimes, instead of $\mathscr{H}$, the Hilbert operator $H=\mathrm{i}\mathscr{H}$ is considered; this bounded linear operator on $L_2(\mathbb{R})$ squares to unity (i.e. $H^2=\mathbf{1}$), it is self-adjoint and unitary.

The above construction may be summarized in the following scheme:

$$\begin{array}{ccc} f(x)\in L_2(\mathbb{R}) & \xrightarrow{\mathscr{P}} & \mathscr{P}[f]\in \mathrm{Harm}(\mathbb{C}^+) \\ \downarrow & & \downarrow \\ \mathscr{H}[f](x)\in L_2(\mathbb{R}) & \xleftarrow{\mathrm{BV}} & \mathscr{Q}[f]\in \mathrm{Harm}(\mathbb{C}^+) \end{array}$$

Next, if Ω is a bounded, simply connected domain in the complex plane, with C^∞ smooth boundary, then, according to Reference [1], the analogue of this scheme is precisely used to define the Hilbert transform on $\partial\Omega$. Indeed, take $u\in C^\infty(\partial\Omega)$ real-valued, then there exists a real-valued harmonic function $U\in C^\infty(\overline{\Omega})$ for which the restriction to the boundary $\partial\Omega$ is the given function u. Let $V\in C^\infty(\overline{\Omega})$ be the conjugate harmonic to U for which $V(a)=0$, $a\in\Omega$ and let v be the restriction to the boundary $\partial\Omega$ of V, then v is called the Hilbert transform of u. This Hilbert transform maps $C^\infty(\partial\Omega)$ into itself and extends uniquely to a bounded linear operator on $L_2(\partial\Omega)$.

This paper treats a similar relationship between the notions of Hilbert transform and of conjugate harmonic functions in higher dimensional Euclidean space. Section 3 deals with the case of Euclidean space $\mathbb{R}^m$, embedded in $\mathbb{R}^{m+1}$ as the boundary of both upper and lower half space; the definition of the Hilbert transform is recalled, as it was used in Reference [2] in a Clifford setting, and its relation to conjugate harmonic functions, now adding up to a monogenic one, is recalled. In Section 4, we pass to the case of a bounded and simply

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connected domain Ω in $\mathbb{R}^{m+1}$, with a C^∞ smooth boundary. We show how every well-chosen 'Hilbert-like' operator on $\partial\Omega$ may give rise to a specific notion of conjugate harmonicity in Ω and we give some explicit examples. Finally, in Section 5, the rôles are inverted, starting from a well-defined notion of conjugate harmonic functions in the interior (or even the exterior) of the domain Ω, to arrive at new Hilbert transforms on the boundary by means of a construction as indicated in the scheme above. As an example, we revert to the explicit construction of a Hilbert-like integral transform on the unit sphere S^m, see Reference [3], based upon a particular notion of conjugate harmonicity in the unit ball and in its exterior, as introduced in Reference [4]. In order to make the paper self-contained, in Section 2 a quick introduction is given to those notions of Clifford analysis which are necessary for our purpose.

For an overview of the historical background of the higher dimensional Hilbert transform and some of its properties, we refer to Reference [5].

2. THE BASICS OF CLIFFORD ANALYSIS

We investigate how the above relationship between the Hilbert transform and the notion of conjugate harmonic functions behaves in higher dimension within the framework of Clifford analysis.

Clifford analysis is a function theory which constitutes a generalization to higher dimension of the theory of holomorphic functions in the complex plane. It is centred around the notion of a monogenic function, i.e. a null solution to the Dirac operator $\underline{\partial}$. For a thorough introduction to Clifford algebra and Clifford analysis we refer the reader to References [6–8]. Here, we only recall those notions which are explicitly used in the paper.

Let $\mathbb{R}^{0,m}$ be the real vector space $\mathbb{R}^m$, endowed with a non-degenerate quadratic form of signature $(0,m)$, let $(e_1,\ldots,e_m)$ be an orthonormal basis for $\mathbb{R}^{0,m}$, and let $\mathbb{R}_{0,m}$ be the universal Clifford algebra constructed over $\mathbb{R}^{0,m}$. The non-commutative multiplication in $\mathbb{R}_{0,m}$ is governed by the rules

$$e_ie_j + e_je_i = -2\delta_{i,j} \quad \forall i,j \in \{1,\ldots,m\}$$

For a set $A=\{i_1,\ldots,i_h\}\subset\{1,\ldots,m\}$ with $1\leqslant i_1<i_2<\cdots<i_h\leqslant m$, let $e_A=e_{i_1}e_{i_2}\cdots e_{i_h}$. Moreover, we put $e_\phi=1$, the latter being the identity element. Then $(e_A : A\subset\{1,\ldots,m\})$ is a basis for the Clifford algebra $\mathbb{R}_{0,m}$. Any $a\in\mathbb{R}_{0,m}$ may thus be written as $a=\sum_A a_Ae_A$ with $a_A\in\mathbb{R}$ or still as $a=\sum_{k=0}^m [a]_k$ where $[a]_k=\sum_{|A|=k} a_Ae_A$ is a so-called k-vector ($k=0,1,\ldots,m$). If we denote the space of k-vectors by $\mathbb{R}^k_{0,m}$, then $\mathbb{R}_{0,m}=\bigoplus_{k=0}^m \mathbb{R}^k_{0,m}$.

We will also identify an element $\underline{x}=(x_1,\ldots,x_m)\in\mathbb{R}^m$ with the one-vector (or vector for short) $\underline{x}=\sum_{j=1}^m x_je_j$. The product of any two Clifford vectors $\underline{x}$ and $\underline{y}$ is given by

$$\underline{x}\,\underline{y} = -\langle\underline{x},\underline{y}\rangle + \underline{x}\wedge\underline{y}$$

where

$$\langle\underline{x},\underline{y}\rangle = \sum_{j=1}^m x_jy_j = -\frac{1}{2}\left(\underline{x}\,\underline{y}+\underline{y}\,\underline{x}\right)$$

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is a scalar and

$$\underline{x} \wedge \underline{y} = \sum_{i<j} e_{ij}(x_i y_j - x_j y_i) = \frac{1}{2}(\underline{x}\,\underline{y} - \underline{y}\,\underline{x})$$

is a 2-vector (also called bivector). In particular $\underline{x}^2 = -\langle \underline{x}, \underline{x} \rangle = -|\underline{x}|^2 = -\sum_{j=1}^m x_j^2$.

Conjugation in $\mathbb{R}_{0,m}$ is defined as the anti-involution for which $\overline{e}_j = -e_j$, $j = 1, \ldots, m$. In particular for a vector $\underline{x}$ we have $\overline{\underline{x}} = -\underline{x}$.

A central notion in Clifford analysis is that of monogenicity, introduced by means of the Dirac operator, i.e. the first-order Clifford-vector valued differential operator in $\mathbb{R}^m$ given by

$$\underline{\partial} = \sum_{j=1}^m e_j \partial_{x_j}$$

with fundamental solution

$$E(\underline{x}) = \frac{1}{a_m} \frac{\overline{\underline{x}}}{|\underline{x}|^m}$$

where $a_m = 2\pi^{m/2}/\Gamma(m/2)$ is the area of the unit sphere S^{m-1} in $\mathbb{R}^m$.

Now let f be a function defined on $\mathbb{R}^m$ and taking values in $\mathbb{R}_{0,m}$, then we say that f is left monogenic in the open region Ω of $\mathbb{R}^m$ if and only if f is continuously differentiable in Ω and satisfies in Ω the equation $\underline{\partial} f = 0$. Similarly, a continuously differentiable function f satisfying in Ω the equation $f\underline{\partial} = 0$ is called right monogenic in Ω. As $\overline{\underline{\partial} f} = \overline{f}\,\overline{\underline{\partial}} = -\overline{f}\,\underline{\partial}$, the left monogenicity of a function f is equivalent to the right monogenicity of $\overline{f}$. As moreover the Dirac operator factorizes the Laplace operator Δ, $-\underline{\partial}^2 = \underline{\partial}\,\overline{\underline{\partial}} = \overline{\underline{\partial}}\,\underline{\partial} = \Delta$, a monogenic function in Ω is harmonic and hence C_∞ in Ω, and so are its components.

Introducing spherical co-ordinates $(r, \theta_1, \ldots, \theta_{m-1}) \in \mathbb{R}^m$, an arbitrary Clifford vector may be written as $\underline{x} = r\underline{\omega}$, with $r = |\underline{x}|$ and $\underline{\omega} \in S^{m-1}$. Then the Dirac operator $\underline{\partial}$ takes the form

$$\underline{\partial} = \underline{\omega}\partial_r + \frac{1}{r}\partial_{\underline{\omega}} = \underline{\omega}\left(\partial_r - \frac{1}{r}\underline{\omega}\partial_{\underline{\omega}}\right) \tag{1}$$

while we may write the Laplace operator as

$$\Delta = \partial_r^2 + \frac{m-1}{r}\partial_r + \frac{1}{r^2}\Delta^*$$

Δ^* being the Laplace–Beltrami operator acting on S^{m-1}, and explicitly given by $\Delta^* = \underline{\omega}\partial_{\underline{\omega}} - \partial^2_{\underline{\omega}\underline{\omega}}$. Form (1) of the Dirac operator can easily be rewritten as

$$\underline{\partial} = \frac{\underline{\omega}}{r}(r\partial_r - \underline{\omega}\partial_{\underline{\omega}}) = \frac{\underline{\omega}}{r}(E + \Gamma)$$

where

$$E = r\partial_r = \sum_{j=1}^m x_j \partial_{x_j}$$

$$\Gamma = -\underline{\omega}\partial_{\underline{\omega}} = -\underline{\omega} \wedge \partial_{\underline{\omega}} = -\sum_{i<j} e_i e_j (x_i \partial_{x_j} - x_j \partial_{x_i})$$

are the (scalar) Euler operator and the (bivector-valued) so-called spherical Dirac operator, respectively.

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In this paper, we will also use the so-called inner and outer spherical monogenics. Starting from a homogeneous polynomial $P_k(\underline{x})$ of degree $k \in \mathbb{N}$ which we take to be left monogenic, it is clear through so-called spherical inversion that the functions $Q_k(\underline{x}) = (\overline{\underline{x}}/|\underline{x}|^{m+2k})P_k(\underline{x})$ are left monogenic functions in the complement of the origin, which are moreover homogeneous of degree $-(m+k-1)$. By taking restrictions to the unit sphere S^{m-1} of both $P_k(\underline{x})$ and $Q_k(\underline{x})$ we obtain the inner spherical monogenics $P_k(\underline{\omega})$ and the outer spherical monogenics $Q_k(\underline{\omega}) = \underline{\omega} P_k(\underline{\omega})$, respectively. Together, both notions constitute a refinement of the concept of a spherical harmonic, i.e. the restriction to the unit sphere of a homogeneous harmonic polynomial of degree k. Indeed, taking an arbitrary spherical harmonic $S_k(\underline{\omega})$, one may consider its unique orthogonal decomposition into an inner and an outer spherical monogenic, viz

$$S_k(\underline{\omega}) = P_k(\underline{\omega}) + Q_{k-1}(\underline{\omega}) \tag{2}$$

where obviously $Q_{-1}(\underline{\omega}) \equiv 0$.

3. A CARTESIAN APPROACH FOR THE HALF SPACE IN $m+1$ DIMENSIONS

From $\mathbb{R}^m$ we pass to $\mathbb{R}^{m+1}$ by adding one more basis vector e_0, which satisfies similar multiplication rules as the others, i.e.

$$e_0^2 = -1, \quad e_0 e_j + e_j e_0 = 0, \quad j = 1, \ldots, m$$

In other words, denoting $x \in \mathbb{R}^{m+1}$ as

$$x = x_0 e_0 + \underline{x}$$

with $\underline{x} = \sum_{j=1}^{m} e_j x_j \in \mathbb{R}^m$, $\mathbb{R}^m$ is embedded in $\mathbb{R}^{m+1}$ as the hyperplane $x_0 = 0$. Clearly, this approach also leads to a cartesian splitting of the Clifford algebra $\mathbb{R}_{0,m+1}$, i.e.

$$\mathbb{R}_{0,m+1} = \mathbb{R}_{0,m} \oplus \overline{e_0}\mathbb{R}_{0,m}$$

In this setting, the observations of Section 1 may be rephrased quite literally.

To this end, take a function $f \in L_2(\mathbb{R}^m)$, which is $\mathbb{R}_{0,m}$-valued and consider its Hilbert transform, defined by the convolution integral

$$\mathscr{H}[f](\underline{x}) = \frac{2}{a_{m+1}} \operatorname{Pv} \frac{\overline{\underline{\cdot}}}{|\underline{\cdot}|^{m+1}} * f(\cdot)(\underline{x}) = \frac{2}{a_{m+1}} \operatorname{Pv} \int_{\mathbb{R}^m} \frac{\overline{\underline{x}} - \overline{\underline{y}}}{|\underline{x} - \underline{y}|^{m+1}} f(\underline{y}) \, \mathrm{d}V(\underline{y})$$

Similarly as in the complex plane, instead of $\mathscr{H}$, one may consider the Hilbert operator $H = \overline{e_0}\mathscr{H}$, which then is a bounded linear operator on $L_2(\mathbb{R}^m)$, which squares to unity, is self-adjoint, and hence also unitary.

The Poisson transform to upper half space ($x_0 > 0$) of the same function f is the unique harmonic extension of f to $\mathbb{R}_+^{m+1}$ given by

$$\mathscr{P}[f](x) = P(x_0, \cdot) * f(\cdot)(\underline{x}) = \frac{1}{a_{m+1}} \int_{\mathbb{R}^m} \frac{2x_0}{|x_0 e_0 + \underline{x} - \underline{y}|^{m+1}} f(\underline{y}) \, \mathrm{d}V(\underline{y})$$

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where the limit

$$\lim_{\substack{x_0 \to 0 \\ >}} \mathscr{P}[f](x) = f(\underline{x})$$

has to be interpreted in the sense of L_2 non-tangential boundary values.

Explicitly, one has that

$$\mathscr{Q}[f](x) = \mathscr{P}[\mathscr{H}[f]] = Q(x_0, \cdot) * f(\cdot)(\underline{x}) = -\frac{1}{a_{m+1}} \int_{\mathbb{R}^m} \frac{2(\underline{x} - \underline{y})}{|x_0 e_0 + \underline{x} - \underline{y}|^{m+1}} f(\underline{y}) \, \mathrm{d}V(\underline{y})$$

and

$$\mathscr{P}[f] + \overline{e_0} \mathscr{P}[\mathscr{H}[f]]$$

is a monogenic function in $\mathbb{R}^{m+1}_+$, with respect to the Cauchy–Riemann operator $\partial_{x_0} + \underline{\partial}$, consequent to which one says that $U = \mathscr{P}[f]$ and $V = \mathscr{Q}[f] = \mathscr{P}[\mathscr{H}[f]]$ are conjugate harmonic functions in $\mathbb{R}^{m+1}_+$ in the sense of [2], i.e. they satisfy the system

$$\begin{cases} \partial_{x_0} V + \underline{\partial} U = 0 \\ \underline{\partial} V + \partial_{x_0} U = 0 \end{cases}$$

Moreover, up to a factor $\frac{1}{2}$, the monogenic function to which they add up is exactly the Cauchy integral $\mathscr{C}[f]$, which is given by

$$\mathscr{C}[f](x) = \frac{1}{a_{m+1}} \int_{\mathbb{R}^m} \frac{x_0 - \overline{e_0}(\underline{x} - \underline{y})}{|x - \underline{y}|^{m+1}} f(\underline{y}) \, \mathrm{d}V(\underline{y})$$

and indeed decomposes as

$$\mathscr{C}[f] = \tfrac{1}{2} \mathscr{P}[f] + \tfrac{1}{2} \overline{e_0} \mathscr{P}[\mathscr{H}[f]]$$

Here, note that the additional basis vector e_0 plays a similar, although not identical, rôle as the imaginary unit i in the complex plane. Indeed, both may be interpreted as unit normal vectors with respect to the boundary (either $\mathbb{R}$ or $\mathbb{R}^m$) of the domain in which the conjugate harmonicity is defined (either $\mathbb{C}^+$ or $\mathbb{R}^{m+1}_+$).

Hence we may conclude that in the context of $\mathbb{R}^m$ embedded in $\mathbb{R}^{m+1}$, a scheme completely similar to the one in the complex plane is valid: the Hilbert transform of a function $f \in L_2(\mathbb{R}^m)$ is the L_2 non-tangential boundary value of the function in $\mathbb{R}^{m+1}_+$ which is conjugate harmonic to the Poisson transform of the original function f.

4. THE CASE OF A DOMAIN Ω IN $\mathbb{R}^{m+1}$: FROM HILBERT-LIKE OPERATORS TO CONJUGATE HARMONICITY

Now, let $\Omega \subset \mathbb{R}^{m+1}$ be a bounded and simply connected domain, with a C^∞ smooth boundary, denoted by $\partial\Omega$. In what follows, a crucial rôle will be played by the so-called Hardy space on $\partial\Omega$. This is the function space defined as the closure in $L_2(\partial\Omega)$ of the L_2 non-tangential boundary values (NTBVs) on $\partial\Omega$ of all monogenic functions in Ω, having such a NTBV in $L_2(\partial\Omega)$. This space is denoted as $H^2(\partial\Omega; \mathbb{R}_{0,m+1})$, or shortly $H^2(\partial\Omega)$.

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A first characterization of the space is given in Proposition 4.1. To this end, we define the Cauchy integral of $f \in L_2(\partial\Omega)$ by

$$\mathscr{C}[f](x) = \frac{1}{a_{m+1}} \int_{\partial\Omega} \frac{x-\zeta}{|x-\zeta|^{m+1}} N(\zeta) f(\zeta)\, \mathrm{d}S(\zeta), \quad x \notin \partial\Omega \tag{3}$$

where $N(\zeta)$ denotes the unit Clifford vector, outward normal to $\partial\Omega$ at the point $\zeta \in \partial\Omega$. As above, again note the appearance of this normal vector to the boundary, however now no longer being constant and thus playing its rôle in the integral. One has

Proposition 4.1
A function $h \in L_2(\partial\Omega)$ belongs to the Hardy space $H^2(\partial\Omega)$ if and only if

$$\mathscr{C}^{\mathrm{int}}[h](\xi) \equiv \lim_{\substack{x\to\xi \\ x\in\Omega,\, \xi\in\partial\Omega}} \mathscr{C}[h](x) = h(\xi)$$

where the limit has to be understood as an L_2 NTBV.

Introducing also the so-called Poisson transform $\mathscr{P}[f]$ of $f \in L_2(\partial\Omega)$ as the unique harmonic function in Ω for which the NTBV is precisely f, i.e.

$$\Delta\mathscr{P}[f](x) = 0 \text{ in } \Omega \quad \text{and} \quad \lim_{\substack{x\to\xi \\ x\in\Omega,\, \xi\in\partial\Omega}} \mathscr{P}[f](x) = f(\xi)$$

we are immediately led to a second characterization of this Hardy space.

Proposition 4.2
A function $h \in L_2(\partial\Omega)$ belongs to the Hardy space $H^2(\partial\Omega)$ if and only if its Cauchy integral and its Poisson transform coincide, i.e.

$$\mathscr{C}[h](x) = \mathscr{P}[h](x) \quad \text{in } \Omega$$

Proof
On account of Proposition 4.1 we have that $\mathscr{C}[h]$ is a monogenic, hence harmonic, function with the same boundary value as $\mathscr{P}[h]$, if and only if $h \in H^2(\partial\Omega)$. $\square$

We will now introduce a so-called 'Hilbert-like' operator on $\partial\Omega$, denoted as $\mathscr{B} : L_2(\partial\Omega) \to L_2(\partial\Omega)$, and satisfying the following properties:

$$\text{(i)} \quad \mathscr{B} \text{ is a bounded linear operator} \tag{4}$$

$$\text{(ii)} \quad \frac{1+\mathscr{B}}{2} \text{ is a (skew) projection operator on } H^2(\partial\Omega) \tag{5}$$

One may then easily prove the following result.

Proposition 4.3
Let the operator $\mathscr{B} : L_2(\partial\Omega) \to L_2(\partial\Omega)$ satisfy conditions (4)–(5). Then

- $\mathscr{B}^2 = 1$,
- $h \in H^2(\partial\Omega) \iff \mathscr{B}[h] = h$.

Observe that the second statement in the above proposition may be seen as a third characterization of $H^2(\partial\Omega)$.

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With each operator $\mathscr{B}$ satisfying conditions (4)–(5), one may now easily associate a corresponding notion of conjugate harmonicity. Indeed, for any $f \in L_2(\partial\Omega)$, consider the function

$$h = \tfrac{1}{2} f + \tfrac{1}{2}\mathscr{B}[f]$$

On account of Proposition 4.3, it directly follows that $\mathscr{B}[h] = h$, or equivalently, h belongs to $H^2(\partial\Omega)$. Hence, on account of Proposition 4.2 one has that $\mathscr{C}[h] = \mathscr{P}[h]$, or, using the linearity of the Poisson transform

$$\mathscr{C}[h] \equiv \mathscr{C}[\tfrac{1}{2} f + \tfrac{1}{2}\mathscr{B}[f]] = \tfrac{1}{2}\mathscr{P}[f] + \tfrac{1}{2}\mathscr{P}[\mathscr{B}[f]]$$

This implies that the harmonic functions $\mathscr{P}[f]$ and $\mathscr{P}[\mathscr{B}[f]]$ add up to the monogenic Cauchy integral of $\frac{1}{2} f + \frac{1}{2}\mathscr{B}[f]$. These observations lead to the following definition.

Definition 4.4
Let $\mathscr{B}$ be an operator satisfying properties (4)–(5) and let $f \in L_2(\partial\Omega)$, then $\mathscr{P}[f]$ and $\mathscr{P}[\mathscr{B}[f]]$ are called $\mathscr{B}$-conjugate harmonic functions in Ω.

The above ideas are now illustrated in three concrete examples.

Example 4.5
For the case of a bounded and simply connected domain Ω in $\mathbb{R}^{m+1}$ with a C^∞ smooth simply connected boundary $\partial\Omega$, the Hilbert (or short: H-) transform has been defined in Reference [8]. We recall this definition: take $f \in L_2(\partial\Omega)$ and put

$$H[f](\xi) = \frac{2}{a_{m+1}} \operatorname{Pv} \int_{\partial\Omega} \frac{\overline{\eta} - \overline{\xi}}{|\eta - \xi|^{m+1}} N(\eta) f(\eta) \, \mathrm{d}S(\eta), \quad \xi \in \partial\Omega$$

where again, $N(\eta)$ is the outward unit normal Clifford-vector to $\partial\Omega$ at the point $\eta \in \partial\Omega$.

This H-transform arises as part of the NTBV of $\mathscr{C}[f]$, viz

$$\mathscr{C}^{\text{int}}[f](\xi) \equiv \lim_{\substack{x \to \xi \\ x \in \Omega,\, \xi \in \partial\Omega}} \mathscr{C}[f](x) = \tfrac{1}{2} f(\xi) + \tfrac{1}{2} H[f](\xi)$$

As, clearly, H is bounded and $(1 + H)/2$ is a (skew) projection on $H^2(\partial\Omega)$, we immediately have that $\mathscr{P}[f]$ and $\mathscr{P}[H[f]]$ are H-conjugate harmonics in Ω in the sense of Definition 4.4. Moreover

$$\mathscr{C}[\tfrac{1}{2} f + \tfrac{1}{2} H[f]] = \tfrac{1}{2}\mathscr{P}[f] + \tfrac{1}{2}\mathscr{P}[H[f]]$$

a relation which for this particular operator H further reduces to

$$\mathscr{C}[f] = \tfrac{1}{2}\mathscr{P}[f] + \tfrac{1}{2}\mathscr{P}[H[f]]$$

since in this case $\mathscr{C}[\frac{1}{2} f + \frac{1}{2} H[f]]$ and $\mathscr{C}[f]$ are two monogenic, hence harmonic functions with the same L_2 NTBV, which consequently have to coincide. Note however that, as opposed to the Cauchy integral $\mathscr{C}[f]$, the kernel of the Poisson integral $\mathscr{P}[f]$ is not explicitly known for a general domain Ω.

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Example 4.6
Consider the orthogonal decomposition of $L_2(\partial\Omega)$ w.r.t. the Hardy space $H^2(\partial\Omega)$, i.e. let

$$f = \mathbb{P}[f] + \mathbb{P}^{\perp}[f]$$

where $\mathbb{P}[f] \in H^2(\partial\Omega)$ and $\mathbb{P}^{\perp}[f] \in H^{2^{\perp}}(\partial\Omega)$ are the so-called Szegö projections of $f \in L_2(\partial\Omega)$. One has that

$$\mathbb{P}^{\perp}[f](\xi) = -N(\xi)\mathbb{P}[N(\cdot)f(\cdot)](\xi)$$

The above result allows to decompose any $f \in L_2(\partial\Omega)$ as

$$f(\xi) = \mathbb{P}[f](\xi) - N(\xi)\mathbb{P}[Nf](\xi), \quad \xi \in \partial\Omega \tag{6}$$

Consequent to this decomposition, we now define the operator $K: L_2(\partial\Omega) \to L_2(\partial\Omega)$ by

$$K[f](\xi) = \mathbb{P}[f](\xi) + N(\xi)\mathbb{P}[Nf](\xi), \quad \xi \in \partial\Omega \tag{7}$$

Roughly speaking, the K-transform $K[f]$ is obtained by orthogonal reflection of the $H^{2^{\perp}}$-component of f. It turns out that K is a bounded linear operator on $L_2(\partial\Omega)$, enjoying the following properties:

- $K^2 = 1$,
- $K[1] = 1$,
- $K^* = K$,
- $h \in H^2(\partial\Omega) \Longleftrightarrow K[h] = h$,
- $g \in H^{2^{\perp}}(\partial\Omega) \Longleftrightarrow K[g] = -g$.

Furthermore, combination of (6) and (7) yields

$$\mathbb{P}[f] = \tfrac{1}{2}f + \tfrac{1}{2}K[f]$$

$$\mathbb{P}^{\perp}[f] = \tfrac{1}{2}f - \tfrac{1}{2}K[f]$$

Hence, in particular, K is a 'Hilbert-like' operator in the sense of Definition 4.4, inducing a corresponding concept of conjugate harmonicity: the functions $\mathscr{P}[f]$ and $\mathscr{P}[K[f]]$ are qualified as being K-conjugate harmonic in Ω.

Note that if f is real-valued then $\mathscr{P}[f]$ is real-valued, while $\mathscr{P}[K[f]]$ is a para-bivector, i.e. a scalar plus a bivector. Furthermore, we have

$$\mathscr{C}[\tfrac{1}{2}f + \tfrac{1}{2}K[f]] = \tfrac{1}{2}\mathscr{P}[f] + \tfrac{1}{2}\mathscr{P}[K[f]]$$

Comparing both examples, we will in general have that $K \neq H$, since $K^* = K$, while one may check that $H^* = NHN$, an equality to be interpreted in the sense that

$$H^*[f](\xi) = N(\xi)H[N(\cdot)f(\cdot)](\xi)$$

Furthermore, $(1 + H)/2 = \mathscr{C}^{\text{int}}$ is a skew projection on $H^2(\partial\Omega)$, while $(1 + K)/2 = \mathbb{P}$ is an orthogonal projection.

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However, in the case of the unit sphere (i.e. $\Omega = B_{m+1}(O,1)$ and $\partial\Omega = S^m$) the K- and the H-transform do coincide, since here it has been shown that $\mathscr{C}^{\text{int}}$ constitutes an orthogonal projection, forcing it to coincide with $\mathbb{P}$. In addition, a straightforward calculation reveals that here $H^* = H$ as well. as it should, since we already know that $K^* = K$.

The Poisson kernel for the unit ball being explicitly known, we treat this particular case in the following example.

Example 4.7
Let $\Omega = B_{m+1}(O,1)$ and $\partial\Omega = S^m$, and take $f \in L_2(S^m)$. Then we have that

$$\mathscr{P}[f](x) = \int_{S^m} P(x,\xi) f(\xi)\,\mathrm{d}S(\xi) = \frac{1}{a_{m+1}} \int_{S^m} \frac{1-|x|^2}{|x-\xi|^{m+1}} f(\xi)\,\mathrm{d}S(\xi)$$

$$\mathscr{Q}[f](x) = \int_{S^m} Q(x,\xi) f(\xi)\,\mathrm{d}S(\xi) = \frac{1}{a_{m+1}} \int_{S^m} \frac{1+|x|^2+2x\xi}{|x-\xi|^{m+1}} f(\xi)\,\mathrm{d}S(\xi)$$

where $Q(x,\xi)$ is conjugate harmonic to the Poisson kernel $P(x,\xi)$ and $\mathscr{Q}[f](x)$ denotes $\mathscr{P}[H[f]]$ $(=\mathscr{P}[K[f]])$. Furthermore

$$\begin{aligned}\mathscr{C}[f](x) &= \frac{1}{2}\mathscr{P}[f](x) + \frac{1}{2}\mathscr{Q}[f](x)\\ &= \frac{1}{a_{m+1}} \int_{S^m} \frac{1+x\xi}{|x-\xi|^{m+1}} f(\xi)\,\mathrm{d}S(\xi) = \frac{1}{a_{m+1}} \int_{S^m} \frac{x-\xi}{|x-\xi|^{m+1}} \xi f(\xi)\,\mathrm{d}S(\xi)\end{aligned}$$

the last expression being in correspondence with (3).

5. THE CASE OF A DOMAIN Ω IN $\mathbb{R}^{m+1}$: CONJUGATE HARMONICS INDUCING NEW HILBERT TRANSFORMS

We now turn our attention to the inverse case, starting from a well-defined notion of conjugate harmonic functions in the interior (or even the exterior) of a domain Ω, to arrive at new Hilbert transforms on the boundary by means of a construction as indicated in the scheme of Section 1.

To this end, we revert to the notion of angular conjugate harmonic function, as introduced in Reference [4] for a given real-valued harmonic function $u(x)$. We recall the basic definitions and main results established there. Note in particular that the explicit construction of an angular conjugate harmonic holds in a broad class of the so-called radially normal domains, the definition of which is given below.

Definition 5.1
Let $\Omega \subset \mathbb{R}^{m+1}$ be open and let $\widetilde{\Omega}$ be its projection on S^m along the radial direction. Then Ω is called a radially normal domain if there exists a constant $c \in \mathbb{R}_+$ such that for all $\omega \in \widetilde{\Omega}$ the set $\Omega \cap \{t\omega : t \geqslant 0\}$ is non-empty, connected and moreover contains the point $c\underline{\omega}$.

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Clearly, the most simple, yet important examples of radially normal domains are the unit ball and its complement.

We first reformulate the main result of Reference [4] concerning the construction of a tangential conjugate to a given real-valued harmonic function in such a radially normal domain.

Proposition 5.2
Let $\Omega \subset \mathbb{R}^{m+1}$ be an open and radially normal domain, and let $u(x) = u(r\omega)$ be a real-valued harmonic function in Ω. Furthermore, let $C(\underline{\omega})$ be a real-valued C_∞-function satisfying

$$\Delta^* C(\omega) = -c^m (\partial_r u)_{r=c}, \quad \omega \in \widetilde{\Omega}$$

with c as in Definition 5.1 and Δ^* the Laplace–Beltrami operator on S^m. Put

$$h(r\omega) = \frac{1}{r^{m-1}} \left(\int_c^r \rho^{m-2} u(\rho\omega) \, \mathrm{d}\rho + C(\omega) \right)$$

Then

1. $h(r\omega)$ is a real-valued harmonic function in Ω.
2. $w(r\omega) = -\Gamma[h(r\omega)]$ is a $\mathscr{W}$-valued harmonic function in Ω, with

$$\mathscr{W} = \operatorname{span}\{\omega e_{\theta_1}, \ldots, \omega e_{\theta_{m-1}}\} \subset \mathbb{R}^{(2)}_{0,m+1}$$

   the subspace of bivectors spanned by products of $\omega \in S^m$ with the tangent unit vectors $e_{\theta_1}, \ldots, e_{\theta_{m-1}}$ to the sphere S^m at ω.
3. $f(r\omega) = u(r\omega) + w(r\omega) = (E - \Gamma + m - 1)[h]$ is left monogenic in Ω.

In Reference [4], the function $v(r\omega) = -e_r w(r\omega)$ is called the tangential conjugate to u in Ω; note that this function is not harmonic, since the decomposition of the monogenic function f in the harmonic terms $u + w = u + e_r v$ involves the non-constant basis vector e_r (or ω). Hence, for the present purpose, we will use the actual harmonic function w and call it 'the harmonic complement' of u.

It is clear that, in principle, the following scheme allows for the introduction of a new Hilbert operator $\mathscr{B}$ on the boundary of any radially normal domain Ω,

$$\begin{array}{ccll} f \in L_2(\partial\Omega, \mathbb{R}) & \longrightarrow & \mathscr{P}[f] & \text{harmonic in } \Omega \\ \big\downarrow \mathscr{B} & & \big\downarrow & \\ \mathscr{B}[f] \in L_2(\partial\Omega; W) & \overset{\text{NTBV}}{\longleftarrow} & \mathscr{Q}[f] & \text{harmonic complement in } \Omega \end{array}$$

However, up to now, only on S^m the explicit construction of suitable Hilbert-like integral transforms, based on the above concept of conjugate harmonicity and its constructive determination, has been established, see Reference [3]. This case is treated in the examples below. Here, we first sketch the main lines in the method.

In a first step, we start from a function $f \in L_2(S^m)$ of which we determine the Poisson transform $\mathscr{P}[f]$ in $\mathbb{R}^{m+1} \backslash S^m$, given by the following integral (see also Section 4), here

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expressed in spherical co-ordinates:

$$\mathscr{P}[f](r\omega) = \int_{S^m} P(r\omega,\eta) f(\eta)\, \mathrm{d}S(\eta) = \frac{1}{a_{m+1}} \int_{S^m} \frac{1-r^2}{|r\omega-\eta|^{m+1}}\, f(\eta)\, \mathrm{d}S(\eta)$$

The obtained function $\mathscr{P}[f]$ is harmonic in $\mathbb{R}^{m+1}\backslash S^m$ and vanishes at infinity. Restriction of $\mathscr{P}[f]$ to the unit ball $B_{m+1}(O,1)$ yields the unique harmonic function $\mathscr{P}^{\mathrm{int}}[f](x)$, of which the $L_2(S^m)$ non-tangential boundary value for $r\to 1-$ is precisely $f(\omega)$. In order to obtain the analogue in $\mathbb{R}^{m+1}\backslash \overline{B}_{m+1}(O,1)$, we need the Kelvin inversion, given by

$$\mathscr{K}(u(x)) = \mathscr{K}^{-1}(u(x)) = \frac{1}{|x|^{m-1}}\, u\left(\frac{x}{|x|^2}\right)$$

and leading to the function $\mathscr{P}^{\mathrm{ext}}[f]$, viz

$$\mathscr{P}^{\mathrm{ext}}[f](x) = \mathscr{K}(\mathscr{P}^{\mathrm{int}}[f])(x)$$

which is the unique harmonic function in $\mathbb{R}^{m+1}\backslash \overline{B}_{m+1}(O,1)$, vanishing at infinity, of which the $L_2(S^m)$ non-tangential boundary value for $r\to 1+$ is $f(\omega)$.

Next, we will determine the harmonic complements of the functions $\mathscr{P}^{\mathrm{int}}[f](x)$ in $B_{m+1}(O,1)$ and $\mathscr{P}^{\mathrm{ext}}[f](x)$ in $\mathbb{R}^{m+1}\backslash \overline{B}_{m+1}(O,1)$, respectively, denoted by $\mathscr{Q}_\theta^{\mathrm{int}}[f]$ and $\mathscr{Q}_\theta^{\mathrm{ext}}[f]$.

The constructive procedure is concluded by taking the non-tangential boundary values in $L_2(S^m)$-sense of the harmonic functions $\mathscr{Q}_\theta^{\mathrm{int}}[f]$ and $\mathscr{Q}_\theta^{\mathrm{ext}}[f]$, hence obtaining the Hilbert-like transforms $\mathscr{D}^{\mathrm{int}}[f](\underline{\omega})$ and $\mathscr{D}^{\mathrm{ext}}[f](\underline{\omega})$ according to the scheme

$$\begin{array}{ccc} f(\underline{\omega}) \in L_2(S^m) & \xrightarrow{\mathscr{P}^{\mathrm{int}}} & \mathscr{P}^{\mathrm{int}}[f] \in \mathrm{Harm}(B_{m+1}(O,1)) \\ \downarrow & & \downarrow \\ \mathscr{D}^{\mathrm{int}}[f](\underline{\omega}) \in L_2(S^m) & \xleftarrow{\mathrm{BV}} & \mathscr{Q}_\theta^{\mathrm{int}}[f] \in \mathrm{Harm}(B_{m+1}(O,1)) \end{array}$$

and similarly for $\mathscr{P}^{\mathrm{ext}}[f]$.

Example 5.3 (*the interior $\mathscr{D}$-transform*)
We first consider the above method for $\mathscr{P}^{\mathrm{int}}$, i.e. in the interior of the unit ball, where we will obtain an explicit form for the corresponding $\mathscr{D}^{\mathrm{int}}$-transform of a function $f\in L_2(S^m)$.

To this end, a first strategy is to construct the harmonic complement of the Poisson kernel $P(r\omega,\eta)$. This construction was established in Reference [4], leading to the conjugate Poisson kernel $Q(r\omega,\eta)$, given by

$$\widetilde{Q}(r\omega,\eta) = \frac{1}{a_{m+1}}\left(\frac{2}{|r\omega-\eta|^{m+1}} - \frac{m-1}{r^m} F(r,\langle\omega,\eta\rangle)\right) r\omega\wedge\eta$$

where $F(0,\langle\omega,\eta\rangle)=0$ and $\partial_r F = r^{(m+1)/2}/(1-2r\langle\omega,\eta\rangle+r^2)^{(m+1)/2}$.

There is however an alternative way to proceed, where we start from the decomposition of the given function f into spherical harmonics, viz

$$f(\omega) = \sum_{k=0}^{\infty} S_k[f](\omega)$$

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where S_k denotes the projection of f onto the vector space $\mathscr{H}(k)$ of spherical harmonics of degree k in $\mathbb{R}^{m+1}$, given by

$$S_k[f](\omega) = \frac{\mathcal{N}(m+1,k)}{a_{m+1}} \int_{S^m} P_{k,m+1}(\langle \omega, \eta \rangle) f(\eta)\, \mathrm{d}S(\eta)$$

in which $\mathcal{N}(m+1,k)$ is the dimension of $\mathscr{H}(k)$ and $P_{k,m+1}(t)$ are the Legendre polynomials of degree k in $m+1$ dimensions.

We then may write

$$\mathscr{P}^{\mathrm{int}}[f](x) = \sum_{k=0}^{\infty} S_k[f](x)$$

since clearly the uniquely determined harmonic function in $\mathbb{R}^{m+1}$ with non-tangential boundary value $S_k(\omega)$ on the unit sphere, is $S_k(x) = r^k S_k(\omega)$. Applying Proposition 5.2, we arrive at the harmonic potential function h, given by

$$h(x) = \frac{1}{r^{m-1}} \sum_{k=0}^{\infty} \int_0^r \rho^{m+k-2} S_k[f](\omega)\, \mathrm{d}\rho = \sum_{k=0}^{\infty} \frac{S_k[f](x)}{m+k-1}$$

The harmonic complement to $\mathscr{P}^{\mathrm{int}}[f](x)$ then is

$$\mathscr{Q}_\theta^{\mathrm{int}}[f](x) = -\sum_{k=1}^{\infty} \frac{\Gamma(S_k[f])(x)}{m+k-1}$$

Strictly speaking, this construction is only valid for *real-valued* functions f, see Reference [4]. However, decomposing a Clifford algebra-valued function f into its real components $f(\omega) = \sum_A f_A(\omega) e_A$ and applying the theorem to each of them, the result is easily extendable to arbitrary Clifford algebra-valued functions.

The $L_2(S^m)$ non-tangential boundary value of $\mathscr{Q}^{\mathrm{int}}(x)$ is given by

$$\mathscr{Q}^{\mathrm{int}}(\omega) = -\sum_{k=1}^{\infty} \frac{\Gamma(S_k[f])(\omega)}{m+k-1}$$

an expression which may be refined by means of the orthogonal decomposition of a spherical harmonic into an inner and an outer spherical monogenic

$$S_k[f] = P_k[f] + Q_{k-1}[f] = \frac{k+m-1-\Gamma}{2k+m-1} S_k[f] + \frac{k+\Gamma}{2k+m-1} S_k[f]$$

As moreover

$$\Gamma P_k[f] = -k P_k[f] \quad \text{and} \quad \Gamma Q_{k-1}[f] = (m+k-1) Q_{k-1}[f]$$

we obtain

$$\mathscr{Q}_\theta^{\mathrm{int}}[f](\omega) = \sum_{k=1}^{\infty} \frac{k}{m+k-1} P_k[f](\omega) - Q_{k-1}[f](\underline{\omega}) = \sum_{k=0}^{\infty} \frac{k}{m+k-1} P_k[f](\omega) - Q_k[f](\omega)$$

In the above, $P_k[f]$ and $Q_k[f]$ are to be interpreted as the projections of the function f onto the vector spaces $\mathscr{M}^+(k)$ and $\mathscr{M}^-(k)$ of inner and outer spherical monogenics, respectively.

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In an integral form, they are given by

$$P_k : L_2(S^m) \mapsto \mathscr{M}^+(k)$$
$$f \mapsto P_k[f](\omega) = -\frac{1}{a_{m+1}}\omega \int_{S^m} \{C_k^{(m+1)/2}(t)\omega - C_{k-1}^{(m+1)/2}(t)\eta\} f(\eta)\, \mathrm{d}S(\eta)$$
$$Q_k : L_2(S^m) \mapsto \mathscr{M}^-(k)$$
$$f \mapsto Q_k[f](\omega) = -\frac{1}{a_{m+1}} \int_{S^m} \{C_k^{(m+1)/2}(t)\omega - C_{k-1}^{(m+1)/2}(t)\eta\}\eta f(\eta)\, \mathrm{d}S(\eta)$$

where $t = \langle \omega, \eta \rangle$ and $C_k^{(m+1)/2}(t)$ denotes the Gegenbauer polynomials in $m+1$ dimensions.

These formal calculations eventually lead to the following definition.

Definition 5.4

The interior $\mathscr{D}$-transform $\mathscr{D}^{\mathrm{int}}[f]$ of an L_2-function f on the unit sphere S^m is given in terms of the spherical monogenic decomposition of f by

$$\mathscr{D}^{\mathrm{int}}[f](\omega) = \sum_{k=0}^{\infty} \frac{k}{m+k-1} P_k[f](\omega) - Q_k[f](\omega)$$

The main properties of $\mathscr{D}^{\mathrm{int}}$ are summarized in the following proposition.

Proposition 5.5

The interior $\mathscr{D}$-transform, $\mathscr{D}^{\mathrm{int}} : L_2(S^m) \mapsto L_2(S^m)$, is a self-adjoint bounded linear operator.

Example 5.6 (*the exterior $\mathscr{D}$-transform*)

The definition and properties of the exterior $\mathscr{D}$-transform are obtained in a similar way as in previous section, now starting from

$$\mathscr{P}^{\mathrm{ext}}[f](x) = \sum_{k=0}^{\infty} \mathscr{K}(S_k[f](x)) = \sum_{k=0}^{\infty} r^{1-k-m} S_k[f](\omega)$$

Definition 5.7

The exterior $\mathscr{D}$-transform $\mathscr{D}^{\mathrm{ext}}[f]$ of an L_2-function f on the unit sphere S^m is given in terms of the spherical monogenic decomposition of f by

$$\mathscr{D}^{\mathrm{ext}}[f](\omega) = \sum_{k=1}^{\infty} -P_k[f](\omega) + \frac{m+k-1}{k} Q_{k-1}[f](\omega)$$

Proposition 5.8

The exterior $\mathscr{D}$-transform, $\mathscr{D}^{\mathrm{ext}} : L_2(S^m) \mapsto L_2(S^m)$, is a self-adjoint bounded linear operator.

Still note that, unlike the Hilbert transform, the $\mathscr{D}$-transforms do *not* square to the unity operator and that in particular the $\mathscr{D}$-transforms of a constant are zero. Moreover, only in the case where $m = 1$, the interior and exterior $\mathscr{D}$-transforms coincide and reduce to the so-called circular Hilbert transform (see e.g. Reference [9]). For other dimensions, one has the formula

$$\mathscr{D}^{\mathrm{int}} + \mathscr{D}^{\mathrm{ext}} = (m-1) \sum_{k=1}^{\infty} \left[-\frac{1}{m+k-1} P_k[f](\omega) + \frac{1}{k} Q_{k-1}[f](\omega) \right]$$

6. CONCLUSION

It may be clear that the notions of conjugate harmonicity in an open, bounded and simply connected domain Ω in Euclidean space, and of Hilbert transform on the smooth boundary $\partial\Omega$ of such a domain, both are not uniquely determined. However, they remain strongly connected to each other since each appropriate notion of conjugate harmonicity in Ω induces a Hilbert-like transform on $\partial\Omega$ and vice versa.

This situation is essentially different from the one in the complex plane. Indeed, given a real-valued harmonic function $u(x, y)$ in an open, simply connected region G of the complex plane, there is—up to a constant—a unique conjugate harmonic $v(x, y)$, such that $u + iv$ is holomorphic in G. As opposed to that, given a real-valued harmonic function $U(x)$ in an open, simply connected region Ω of Euclidean space, there exists a wealth of harmonic functions $V(x)$ such that $U(x) + V(x)$ is left-monogenic in Ω, which at the same time explains the absence of a standard notion of conjugate harmonicity.

In order to illustrate the interplay between the two protagonists under consideration, we summarize the results obtained for the unit ball, which offers the advantage of an explicitely known Poisson kernel,

$$P(x, \xi) = \frac{1}{a_{m+1}} \frac{1 - |x|^2}{|x - \xi|^{m+1}}, \quad x \in B_{m+1}(O, 1), \quad \xi \in S^m$$

Starting from the Hilbert transform of Reference [8] for $f \in L_2(S^m)$, which is given by

$$H[f](\xi) = \frac{2}{a_{m+1}} \operatorname{Pv} \int_{\partial\Omega} \frac{\xi - \eta}{|\eta - \xi|^{m+1}} \eta f(\eta) \, \mathrm{d}S(\eta), \quad \xi \in \partial\Omega$$

since $N(\eta) = \eta$, for all $\eta \in S^m$, the H-conjugate Poisson kernel has been shown to be

$$Q(x, \xi) = \frac{1}{a_{m+1}} \frac{1 + |x|^2 + 2x\xi}{|x - \xi|^{m+1}}, \quad x \in B_{m+1}(O, 1), \quad \xi \in S^m$$

Note that, if the function f is decomposed into spherical harmonics, and subsequently in spherical monogenics as well, viz

$$f(\omega) = \sum_{k=0}^{\infty} S_k[f](\omega) = \sum_{k=0}^{\infty} P_k[f](\omega) + Q_{k-1}[f](\omega), \quad \omega \in S^m$$

then the Hilbert transform of f takes the form

$$H[f](\omega) = \sum_{k=0}^{\infty} P_k[f](\omega) - Q_{k-1}[f](\omega), \quad \omega \in S^m$$

on account of the fact that $P_k[f]$ and $Q_{k-1}[f]$ are eigenfunctions of H with respective eigenvalues $+1$ and -1, see Reference [5]. Meanwhile, this also nicely illustrates the properties $H^2 = 1$ and $H[1] = 1$.

On the other hand, if we start from the notion of angular conjugate harmonics, a Hilbert-like integral transform $\mathscr{D}^{\text{int}}$ on S^m is obtained for which, again in terms of the decomposition of $f \in L_2(S^m)$ in spherical monogenics, we have found

$$\mathscr{D}^{\text{int}}[f](\omega) = \sum_{k=0}^{\infty} \frac{k}{m + k - 1} P_k[f](\omega) - Q_k[f](\omega), \quad \omega \in S^m$$

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As mentioned above, here we have that $(\mathscr{D}^{\text{int}})^2 \neq 1$ and that $\mathscr{D}^{\text{int}}[1] = 0$. So clearly, H and $\mathscr{D}^{\text{int}}$ are different operators. Moreover, the $\mathscr{D}^{\text{int}}$-conjugate Poisson kernel reads

$$\widetilde{Q}(x, \eta) = \frac{1}{a_{m+1}} \left(\frac{2}{|x - \eta|^{m+1}} - \frac{m-1}{r^m} F(r, \langle \omega, \eta \rangle) \right) x \wedge \eta$$

where the function F has been specified in Example 5.3. With P, the kernel $\widetilde{Q}$ adds up to a monogenic kernel, which obviously is not the Cauchy kernel. Note in particular that the H-conjugate Poisson kernel Q is para-bivector valued (i.e. a scalar plus a bivector), while the new $\widetilde{Q}$ is a pure bivector by construction.

ACKNOWLEDGEMENTS

David Eelbode was supported by FWO-Vlaanderen.

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