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Efficient bipartite quantum state purification in arbitrary dimensional Hilbert spaces

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Abstract

A new purification scheme is proposed which applies to arbitrary dimensional bipartite quantum systems. It is based on the repeated application of a special class of nonlinear quantum maps and a single, local unitary operation. This special class of nonlinear quantum maps is generated in a natural way by a Hermitian generalized XOR-gate. The proposed purification scheme offers two major advantages, namely it does not require local depolarization operations at each step of the purification procedure and it purifies more efficiently than other known purification schemes.

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1. Introduction

Quantum information processing was proven to be superior to classical information processing in several respects. The essence of quantum information is its ability to employ the linearity of quantum mechanics on composite systems for practical purposes. Many of the already demonstrated procedures [1, 2] rely on the use of highly entangled quantum states. Entangled states are never generated in ideal form. Typically, either the source producing entangled quantum states or the communication channel with which entanglement is transferred to remote parties adds noise. In order to be able to exploit entanglement efficiently it is desirable to remove as much of this additional noise as possible. This goal can be achieved by purifying or concentrating entanglement.

The manipulation of quantum states is realized using quantum networks. Usually, they are constructed out of simple elements, so-called quantum gates. Among them the two-particle quantum XOR-gate [3] plays a fundamental role. In this two-qubit gate, the first qubit controls the target qubit: if the control is in state $|0\rangle$, the target is left unchanged, but if the control qubit is in state $|1\rangle$ the target's basis states are flipped. Together with one-qubit operations this gate forms a universal set of quantum gates allowing the implementation of arbitrary unitary operations acting on qubits [4]. It has been demonstrated that the quantum XOR-gate can be used for many practical tasks of quantum information processing, such as quantum state swapping [5], entangling quantum states [6], performing Bell measurements [7], dense coding [8], and teleportation [9]. Furthermore, in combination with selective measurements it can be used for implementing nonlinear quantum transformations which may be used for optimal state identification and for state purification [10, 11].

For many practical tasks of quantum information processing it is desirable to extend the basic notion of such a quantum XOR-operation to higher-dimensional Hilbert spaces. Indeed, most of the physical systems that have been proposed to hold qubits, such as multilevel atoms or ions [12] and multipath-interferometers [13], could equally well encode larger alphabets. However, there is a considerable degree of freedom involved in such a generalization.

In this paper we use a Hermitian generalization of the quantum XOR-gate which applies to arbitrary dimensional Hilbert spaces and which allows to implement a special class of nonlinear quantum transformations in a natural way. These nonlinear transformations can be used for the preparation of quantum states and for efficient quantum state purification. This will be exemplified by discussing state purification of generalized Bell states. These latter quantum states are of considerable interest in quantum information processing in higher-dimensional Hilbert spaces. Compared with the other known purification scheme which is valid in arbitrary dimensional Hilbert spaces (developed by Horodecki and Horodecki [14]) our proposed purification procedure offers two advantages. Firstly, it does not involve a depolarization operation at each step of the iteration procedure. Typically, it is not easy to implement such depolarization operations repeatedly. Secondly, it will be demonstrated that our newly proposed method is more efficient.

The paper is organized as follows. In section 2 the Hermitian generalized XOR-gate is introduced which allows one to implement the special class of nonlinear quantum transformations needed for our proposed purification scheme. The resulting class of nonlinear quantum maps is discussed in section 3. The new quantum state purification scheme, its basic properties and its efficiency are exemplified in section 4.

2. A Hermitian generalized GXOR-gate

Let us start by summarizing characteristic properties of the XOR-gate as they are known for qubit systems. For qubits the action of the quantum XOR-gate onto a chosen set of basis states $\{|i\rangle\}$ with $i \in \{0, 1\}$ of the Hilbert space of each qubit is defined by

$$\mathrm{XOR}_{12}\,|i\rangle_1\,|j\rangle_2 = |i\rangle_1\,|i \oplus j\rangle_2\,. \tag{1}$$

This transformation has the following characteristic properties: (i) it is unitary and thus reversible, (ii) it is Hermitian and (iii) $i \oplus j = 0$ if and only if $i = j$. The first (second) index denotes the state of the control (target) qubit and $\oplus$ denotes addition modulo (2).

Let us now consider the problem of generalizing the quantum XOR-gate to higher-dimensional Hilbert spaces. The desired generalized quantum XOR-gate (GXOR-gate) should act on two D-dimensional quantum systems. In analogy with qubits we will call these two systems qudits. The basis states $|i\rangle$ of each qudit are labelled by elements in the ring Z_D which we denote by the numbers $i = 0, \ldots, D-1$ with the usual rules for addition and multiplication modulo (D). In principle, the GXOR gate could be defined in a straightforward way by using equation (1) and by performing $i \oplus j$ modulo (D), i.e.

$$\mathrm{GXOR}_{12}\,|i\rangle_1\,|j\rangle_2 = |i\rangle_1\,|i \oplus j\rangle_2\,. \tag{2}$$

The GXOR-gate defined in (2) is unitary but not Hermitian for $D > 2$. Therefore it is no longer its own inverse. Thus, the inverse GXOR-gate has to be obtained from the GXOR-gate of equation (2) by iteration, i.e. $\text{GXOR}_{12}^{-1} = (\text{GXOR}_{12})^{D-1} = \text{GXOR}_{12}^{\dagger} \neq \text{GXOR}_{12}$. All these inconvenient properties of this preliminary definition (2) can be removed by the alternative definition

$$\text{GXOR}_{12}\,|i\rangle_1\,|j\rangle_2 = |i\rangle_1\,|i \ominus j\rangle_2\,. \tag{3}$$

In equation (3) $i \ominus j$ denotes the difference $i - j$ modulo (D). In the special case of qubits the definition of equation (3) reduces to equation (1) as $i \ominus j \equiv i \oplus j$ modulo (2). Furthermore, this definition preserves all the properties of equation (1) also for arbitrary values of D, namely it is unitary, Hermitian and $i \ominus j = 0$ modulo (D) if and only if $i = j$.

The GXOR-gate of equation (3) admits a natural extension to control and target systems with continuous spectra. In this case the basis states $|i\rangle$ are replaced by the basis states $\{|x\rangle\}$ with the continuous variable $x \in \boldsymbol{R}$. These new basis states are assumed to satisfy the orthogonality condition $\langle x|y\rangle = \delta(x - y)$. Furthermore, as the dimension D tends to infinity the modulo operation entering equation (3) can be omitted. Thus, for continuous variables the action of the GXOR-gate becomes

$$\text{GXOR}_{12}\,|x\rangle_1\,|y\rangle_2 = |x\rangle_1\,|x - y\rangle_2\,. \tag{4}$$

Let us note that this definition for the case of continuous variables is different from the generalized XOR-gate proposed in [15]. This latter gate is not Hermitian whereas the GXOR-gate of equation (4) is both unitary and Hermitian. The GXOR-gate of equation (4) can be represented in terms of a translation and a space inversion, namely

$$\text{GXOR}_{12}\,|x\rangle_1\,|y\rangle_2 = \hat{\Pi}_2 \mathrm{e}^{\mathrm{i}\hat{P}_y^{(2)}\hat{x}^{(1)}/\hbar}\,|x\rangle_1\,|y\rangle_2\,. \tag{5}$$

Thereby $\hat{P}_y^{(2)}$ denotes the canonical momentum operator which is conjugate to the position operator $\hat{y}^{(2)}$ acting on quantum system 2 and $\hat{\Pi}_2$ is the corresponding operator of space inversion.

Let us discuss a possible physical realization of the GXOR-gate defined by equation (3) which is based on nonlinear optical elements. For this purpose we assume that the two quantum systems which are going to be entangled are two modes of the radiation field. The basis states $|i\rangle_1$ $(i = 0, \ldots, D - 1)$ of the first quantum system are formed by n-photon states of mode one with $0 \leqslant n \leqslant D - 1$. The basis states of the second quantum system $|k\rangle_2$ $(k = 0, \ldots, D - 1)$ are formed by Fourier transformed n-photon states of this latter mode, i.e. $|k\rangle_2 = 1/\sqrt{D}\sum_{n=0}^{D-1}\exp(\mathrm{i}2\pi kn/D)|n\rangle_2$. Let us further assume that the dynamics of these two modes of the electromagnetic field are governed by the Kerr-effect [16]. Thus, in the interaction picture their Hamiltonian is given by $H = \hbar\chi a_1^{\dagger}a_1a_2^{\dagger}a_2$ with the creation and annihilation operators $a_{1,2}^{\dagger}$ and $a_{1,2}$ of modes 1 and 2, respectively. For the sake of simplicity the nonlinear susceptibility χ is assumed to be real-valued and positive. Initially preparing both quantum systems in state $|i\rangle_1|k\rangle_2$ after an interaction time of magnitude $t = 2\pi/(D\chi)$ this two-mode system ends up in state $|\psi\rangle_{12} = |i\rangle_1|k - i\rangle_2$. Applying to this latter state a time reversal transformation which may be implemented by the process of phase conjugation [16] we finally arrive at the desired state $|i\rangle_1|i - k\rangle_2$. Thus this combination of a Kerr-interaction with a time reversal transformation is capable of realizing the GXOR-gate of equation (3).

As the GXOR-gate of equation (3) differs from the alternative definition of equation (2) by local unitary operations both are only expected to exhibit similar properties as far as entanglement operations are concerned. However, with the help of this GXOR-gate alone, already a variety of interesting quantum operations can be implemented without having to use additional local unitary transformations. In addition, as will be shown in sections 3 and 4, this entanglement operation is particularly useful in the context of quantum state purification for implementing nonlinear quantum state transformations.

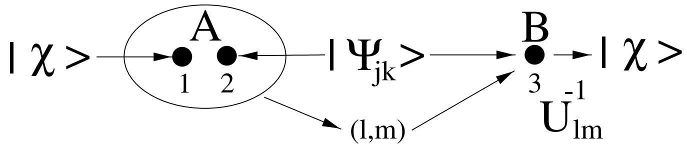

Figure 1. Schematic representation of the teleportation scheme involving Bell measurements onto the generalized Bell states of equation (6).

As a first application let us consider the preparation of a basis of entangled states from separable ones. If $|l\rangle|m\rangle$ with $l, m, = 0, \ldots, D-1$ denotes an orthonormal basis of factorized states an associated basis of entangled two-particle states is given by

$$|\psi_{lm}\rangle = \mathrm{GXOR}_{12}[(F|l\rangle)_1|m\rangle_2]. \tag{6}$$

Thereby F denotes the discrete Fourier transformation, i.e. $F|l\rangle = (1/\sqrt{D})\sum_{k=0}^{D-1} \exp(\mathrm{i}2\pi lk/D)|k\rangle$. For qubits this unitary quantum transformation leads to the well known basis of four Bell states. In the simplest higher-dimensional case of $D = 3$, for example, the first four states of this entangled generalized Bell basis are given by

$$\begin{aligned}
|\psi_{00}\rangle &= \frac{1}{\sqrt{3}}[|00\rangle + |11\rangle + |22\rangle] \\
|\psi_{10}\rangle &= \frac{1}{\sqrt{3}}[|00\rangle + \mathrm{e}^{\mathrm{i}2\pi/3}|11\rangle + \mathrm{e}^{-\mathrm{i}2\pi/3}|22\rangle] \\
|\psi_{20}\rangle &= \frac{1}{\sqrt{3}}[|00\rangle + \mathrm{e}^{-\mathrm{i}2\pi/3}|11\rangle + \mathrm{e}^{\mathrm{i}2\pi/3}|22\rangle] \\
|\psi_{01}\rangle &= \frac{1}{\sqrt{3}}[|02\rangle + |10\rangle + |21\rangle].
\end{aligned} \tag{7}$$

As the GXOR-gate is Hermitian it can also be used to disentangle this basis of generalized Bell states again by inverting equation (6). This basic disentanglement property is of practical significance. It enables one to reduce Bell measurements to measurements of factorized states. Examples where these latter types of measurements are of central interest are dense coding [8] and quantum teleportation schemes [9].

The basis of entangled Bell states resulting from equation (6) can be used for teleporting an arbitrary D-dimensional quantum state from A (Alice) to B (Bob). For this purpose let us assume that A and B share an entangled pair of particles prepared in state $|\psi_{lm}\rangle$ as defined by equation (6). If A wants to teleport an unknown quantum state $|\chi\rangle = \sum_{n=0}^{D-1} \alpha_n|n\rangle$ to B she has to perform a Bell measurement which yields one of the entangled basis states of equation (6) as an output state (compare with figure1). Conditioned on the measurement result of Alice, Bob has to perform an appropriate unitary transformation onto his particle which prepares this latter particle in state $|\chi\rangle$. This arbitrary dimensional teleportation scheme rests on the identity

$$\begin{aligned}
|\chi\rangle|\psi_{jk}\rangle_{23} &= \sum_{l,m=0}^{D-1} |\psi_{lm}\rangle_{12} \frac{\mathrm{e}^{-\mathrm{i}2\pi jm/D}}{D} U_{lm}|\chi\rangle \\
U_{lm}|n\rangle &= \mathrm{e}^{-\mathrm{i}2\pi n(l-j)/D}|n-k-m\rangle.
\end{aligned} \tag{8}$$

This basic relation for teleportation for an arbitrary dimensional state $|\chi\rangle$ can be derived in a straightforward way from equations (3) and (6). The classical communication requires $2\log_2(D)$ bits, which is the minimum necessary in all quantum teleportation schemes.

3. Nonlinear quantum maps on density matrices

With the help of the Hermitian GXOR-gate of equation (3) an interesting class of nonlinear quantum maps can be implemented in a natural way. Together with filtering measurements acting on a target quantum system t the GXOR-gate of equation (3) induces nonlinear transformations of quantum states of a control system c. This can be demonstrated most easily by considering the case of two qudits which are prepared in the quantum states σ^t and σ^c initially. Let us perform the quantum operation

$$T(\sigma^c, \sigma^t) = \frac{A\left(\sigma^c \otimes \sigma^t\right) A^\dagger}{\mathrm{Tr}[A\left(\sigma^c \otimes \sigma^t\right) A^\dagger]} \tag{9}$$

on these two qudits with

$$A = (\mathbf{1_c} \otimes P)\,\mathrm{GXOR}_{ct}. \tag{10}$$

Thereby $\mathbf{1_c}$ denotes the identity operator acting in the Hilbert space of the control system and $P = |p\rangle_{tt}\langle p|$ is the projector onto the state $|p\rangle_t$ of the target qudit. With the decomposition

$$\begin{aligned} \sigma^c &= \sum_{ij}^{D-1} \sigma^c_{ij} |i\rangle_{cc}\,\langle j| \\ \sigma^t &= \sum_{ij}^{D-1} \sigma^t_{ij} |i\rangle_{tt}\,\langle j|\,. \end{aligned} \tag{11}$$

Equations (9) and (10) may be rewritten in the form

$$T(\sigma^c, \sigma^t) = \frac{\sum_{ijkl}^{D-1} \sigma^c_{ij}\sigma^t_{kl}\,|i\rangle_{cc}\,\langle j| \otimes P\,|i \ominus k\rangle_{tt}\,\langle j \ominus l|\,P}{\sum_{ikl}^{D-1} \sigma^c_{ii}\sigma^t_{kl}\,\langle p|i \ominus k\rangle_{tt}\,\langle i \ominus l|p\rangle}. \tag{12}$$

Assuming that both control and target qudit are prepared in the same state initially, i.e. $\sigma^c \equiv \sigma^t$, it turns out that equation (9) is equivalent to the relations

$$\begin{aligned} T(\sigma^c, \sigma^t \equiv \sigma^c) &= \sigma^c_{\mathrm{output}} \otimes P \\ \sigma^c_{\mathrm{output}} &= \frac{\sum_{ij}^{D-1} \sigma^c_{i,j}\sigma^c_{i-p,j-p}\,|i\rangle_{cc}\,\langle j|}{\sum_{i}^{D-1} \sigma^c_{ii}\sigma^c_{i-p,i-p}}. \end{aligned} \tag{13}$$

As a result of the quantum operation (9) the combined system formed by the control and the target qudit forms a factorizable state with the target qudit being in state $|p\rangle\langle p|$. According to equation (13) the density matrix elements of σ^c with respect to the computational basis $|i\rangle$ $(i = 0, \ldots, D-1)$ have been multiplied with each other. The final state of the control qudit is prepared with probability $p_c = \sum_i^{D-1} \sigma^c_{i,i}\sigma^c_{i-p,i-p}$. In the case of qubits, for instance, the nonlinear transform of the (unnormalized) control density matrix is given by

$$\sigma^c_{\mathrm{output}} = \begin{pmatrix} \left(\sigma^c_{00}\right)^2 & \left(\sigma^c_{01}\right)^2 & \left(\sigma^c_{02}\right)^2 & \left(\sigma^c_{03}\right)^2 \\ \left(\sigma^c_{10}\right)^2 & \left(\sigma^c_{11}\right)^2 & \left(\sigma^c_{12}\right)^2 & \left(\sigma^c_{13}\right)^2 \\ \left(\sigma^c_{20}\right)^2 & \left(\sigma^c_{21}\right)^2 & \left(\sigma^c_{22}\right)^2 & \left(\sigma^c_{23}\right)^2 \\ \left(\sigma^c_{30}\right)^2 & \left(\sigma^c_{31}\right)^2 & \left(\sigma^c_{32}\right)^2 & \left(\sigma^c_{33}\right)^2 \end{pmatrix} \tag{14}$$

if projected onto the state $|0\rangle_t$. The elements of the original density matrix have been squared. If one projects onto the state $|1\rangle_t$, for example, the original density matrix elements are mixed in a more complicated way and one obtains the (unnormalized) density matrix

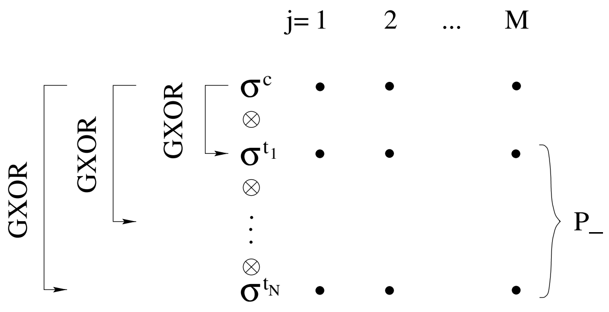

Figure 2. Schematic representation of the GXOR-gates and projections involved in the nonlinear quantum transformation of equation (18). The qudits are represented by dots. The dots of the first line represent the M qudits of the control system. The dots of the following lines represent the $M \times N$ qudits of the N target systems $t_1, t_2, \ldots, t_N$. The GXOR-gate $\mathrm{GXOR}_{ct_i}^{(j)}$ acts on the jth qudit of the control and target system t_i with $j \in \{1, 2, \ldots, M\}$ and $i \in \{1, 2, \ldots, N\}$. The operator P_- projects the state of the whole systems onto state $|\mathbf{0}\rangle\langle\mathbf{0}|$ with $|\mathbf{0}\rangle = |0\rangle_1|0\rangle_2 \ldots |0\rangle_{MN}$.

$$\sigma^c_{\text{output}} = \begin{pmatrix} \sigma^c_{00}\sigma^c_{33} & \sigma^c_{01}\sigma^c_{32} & \sigma^c_{02}\sigma^c_{31} & \sigma^c_{03}\sigma^c_{30} \\ \sigma^c_{10}\sigma^c_{23} & \sigma^c_{11}\sigma^c_{22} & \sigma^c_{12}\sigma^c_{21} & \sigma^c_{13}\sigma^c_{20} \\ \sigma^c_{20}\sigma^c_{13} & \sigma^c_{21}\sigma^c_{12} & \sigma^c_{22}\sigma^c_{11} & \sigma^c_{23}\sigma^c_{10} \\ \sigma^c_{30}\sigma^c_{03} & \sigma^c_{31}\sigma^c_{02} & \sigma^c_{32}\sigma^c_{01} & \sigma^c_{33}\sigma^c_{00} \end{pmatrix}. \tag{15}$$

From equation (13) it is easy to verify that the quantum operation (9) has the following basic properties: (i) it maps density matrices onto density matrices, (ii) it is not injective and nonlinear, and (iii) there are states invariant under this transformation. It is also possible to extended the quantum operation of equation (9) to cases in which there is more than one control system and in which both the control and the target systems are composite quantum systems each of which consists of M qudits. In this case σ^c describes a general M-qudit state of the form

$$\sigma^c = \sum_{ij} \sigma^c_{ij} \, |\boldsymbol{i}\rangle_{cc} \, \langle \boldsymbol{j}| \tag{16}$$

with $\boldsymbol{i} = (i_1, \ldots, i_M)$ and $\boldsymbol{j} = (j_1, \ldots, j_M)$. In equation (9) the operator A has to be replaced by

$$A = (\mathbf{1}_c \otimes P) \prod_{j=1}^{M} \prod_{i=1}^{N} \mathrm{GXOR}_{ct_i}^{(j)} \tag{17}$$

with the projection operators $P = \prod_{i=1}^{N} \otimes P_{t_i}$ and $P_{t_i} = |\boldsymbol{p_i}\rangle_{t_i t_i} \langle \boldsymbol{p_i}|$ onto state $|\boldsymbol{p_i}\rangle_{t_i}$ of the M-qudit target system t_i. Thereby the GXOR-gate $\mathrm{GXOR}_{ct_i}^{(j)}$ operates on the jth qudit of the control and of the ith target system. For the special case of $p_i = 0$ for all t_i, for example, the resulting final state of the control system is given by

$$\sigma^c_{\text{output}} = \frac{\sum_{ij} (\sigma^c_{ij})^{1+N} \, |\boldsymbol{i}\rangle_{cc} \, \langle \boldsymbol{j}|}{\sum_{i} (\sigma^c_{ii})^{1+N}} \tag{18}$$

and is prepared with probability $p_c = \sum_{i} (\sigma^c_{ii})^{1+N}$ (compare with figure 2).

4. Bipartite purification in higher-dimensional spaces

In general, for $N = 1$ the nonlinear quantum transformation of equation (18) has states which are not only invariant with respect to $P_{t_1} = |0\rangle_{t_1 t_1}\langle 0|$ but also with respect to other projectors P_{t_1}. This suggests to use this nonlinear quantum transformation for the purification of quantum states of a two-qudit system. For the special case of a control system consisting of two-qubits such a purification scheme which is based on the nonlinear quantum transformation of equation (18) has already been proposed previously [11]. In order to discuss an analogous purification scheme in arbitrary dimensional Hilbert spaces we start from the observation that for $M = 2$ the entangled basis state $|\psi_{00}\rangle$ of equation (6) is a fixed point of the nonlinear two-particle quantum map of equation (18). Inspection of the two 'squared' matrices (14) and (15) shows that this is true not only for the projector $P_{t_1} = |0\rangle_{tt}\langle 0|$ but also for $P_{t_1} = |1\rangle_{tt}\langle 1|$. Therefore, this nonlinear quantum transformation may be used for purifying quantum states towards the entangled state $|\psi_{00}\rangle$. Thereby the possibility to use both projectors in the purification process increases its efficiency considerably.

Purification in higher-dimensional Hilbert spaces has been considered previously by Horodecki and Horodecki [14]. These authors generalized the approach of Bennett *et al* [10] to arbitrary dimensional Hilbert spaces. The purpose of their protocol is to distill Bell states from a noisy channel. Their protocol combines two basic steps, namely a nonlinear quantum map which 'squares' the density matrix elements and a depolarizing channel converting the resulting output state into a Werner state. The depolarizing channel guarantees that at each step of the purification protocol an initially prepared Werner state is mapped again onto a Werner state with a higher admixture of the Bell state. This protocol is capable of purifying all non-separable Werner states in arbitrary dimensional Hilbert spaces. The depolarization operation involved in this purification protocol requires that suitably chosen local unitary operations have to be applied to a sufficiently large number of two-qudit systems. For the case of qubits it has been shown that such a depolarization may be achieved with a set of 12 suitably chosen unitary operations [17]. For qudits with $D > 2$ suitable minimal numbers of unitary operations are not known at present. In view of these inconveniences in implementing a depolarizing operation it appears desirable to develop alternative purification strategies which do not involve such a depolarization procedure. For the case of qubits such a procedure has already been developed by Deutsch *et al* [18]. In the following we propose such a method which applies to arbitrary dimensional Hilbert spaces. It is based on the nonlinear quantum transformation of equation (18) (with $N = 1$) followed by a single local unitary operation acting on both qudits.

In order to exemplify basic properties of our purification scheme let us consider the purification of a Werner state of the form

$$\sigma^c = \lambda|\psi_{00}\rangle\langle\psi_{00}| + (1-\lambda)\mathbf{1}/D^2 \tag{19}$$

where the parameter λ is related to the fidelity $F = \langle\psi_{00}|\sigma^c|\psi_{00}\rangle$ through the expression

$$F = \lambda + \frac{1-\lambda}{D^2}. \tag{20}$$

The state (19) may result from a physical situation where two spatially separated parties, say A(lice) and B(ob), want to share the entangled basis state $|\psi_{00}\rangle$ but with a probability of $(1-\lambda)$ the transmission of this entangled pair through a quantum channel leads to unwanted noise represented by the chaotic state $\mathbf{1}/D^2$. This initial quantum state σ^c is non-separable if and only if $\lambda > \lambda_D = (1+D)^{-1}$ [19] so that a purification scheme can succeed only for those values of λ.

We propose a purification scheme which is based on a two-step iteration procedure. In the first part of each iteration step we apply the nonlinear quantum map of equation (18) (with $N = 1$) by projecting onto an arbitrary two-qudit target state $|ii\rangle\langle ii|$ ($i = 0, \ldots, D-1$). Correspondingly, independent of the target state $|ii\rangle$, the initially prepared Werner state (19) is converted into the quantum state

$$\sigma^{c(1)}_{\text{output}} = \frac{\lambda_1}{N_D}|\psi_{00}\rangle\langle\psi_{00}| + \frac{\lambda_2}{N_D}\mathbf{1}/D^2 + \frac{\lambda_3}{N_D}\sum_{k=0}^{D-1}\frac{|kk\rangle\langle kk|}{D} \tag{21}$$

with the coefficients λ_i depending on the initial choice of λ only, i.e.

$$\begin{aligned} \lambda_1 &= \lambda^2 \\ \lambda_2 &= \frac{(1-\lambda)^2}{D} \\ \lambda_3 &= \frac{2\lambda(1-\lambda)}{D} \end{aligned} \tag{22}$$

and

$$N_D = \lambda_1 + \lambda_2 + \lambda_3. \tag{23}$$

However, it turns out that both the pure state $|\psi_{00}\rangle\langle\psi_{00}|$ and the mixed state $\sum_{k=0}^{D-1}|kk\rangle\langle kk|/D$ are fixed points of the nonlinear quantum transformation of equation (18). Therefore, an additional local unitary transformation is required in order to guarantee convergence of the iteration procedure towards the desired final state $|\psi_{00}\rangle\langle\psi_{00}|$. Thus, in the second part of each iteration step parties A and B perform a local twirling operation [14] $U \otimes U^*$, i.e.

$$\sigma^{c(1)}_{\text{output}} \to \sigma^{c(2)}_{\text{output}} \equiv U \otimes U^* \sigma^{c(1)}_{\text{output}} U^\dagger \otimes U^{*\dagger}. \tag{24}$$

Ultimately this transformation achieves convergence towards our desired final state $|\psi_{00}\rangle\langle\psi_{00}|$ by altering the mixed state $\sum_{k=0}^{D-1}|kk\rangle\langle kk|$ but still leaving state $|\psi_{00}\rangle\langle\psi_{00}|$ invariant. Therefore, the depolarization operation of the protocol of Horodecki *et al* [14] is replaced by a single twirling operation. For the unitary transformation involved in this twirling operation we propose to choose a discrete Fourier transform, i.e.

$$U|k\rangle = \frac{1}{\sqrt{D}}\sum_{n=0}^{D-1}\exp^{(\mathrm{i}2\pi kn/D)}|n\rangle. \tag{25}$$

This choice is motivated by the desire to increase the success probability of the purification process and to maximize the radius of convergence of the iteration procedure. As both the intermediate output state of equation (21) and the unitary transformation of equation (25) are invariant under transformations of the basis states of the form $|i\rangle \to |i+1\rangle$ ($i = 0, \ldots, D-1$), all the projections $P_{t_1} = |ii\rangle\langle ii|$ yield the same success probability thus increasing efficiency. Iterating the two-step procedure based on the nonlinear quantum transformation of equation (18) and the local twirling operation of equation (24) yields our proposed purification procedure. Thus, after the nth iteration the two-particle state $\sigma^{c(2n)}_{\text{output}}$ is given by

$$\begin{aligned} \sigma^{c(2n)}_{\text{output}} = {} & \frac{\tilde{\lambda}^{(n)}_1}{N_D}|\psi_{00}\rangle\langle\psi_{00}| + \frac{\tilde{\lambda}^{(n)}_2}{N_D}\mathbf{1}/D^2 + \frac{\tilde{\lambda}^{(n)}_3}{N_D}\sum_{k=0}^{D-1}\frac{|kk\rangle\langle kk|}{D} \\ & + \frac{\tilde{\lambda}^{(n)}_4}{N_D}\sum_{k=0}^{D-1} U \otimes U^* \frac{|kk\rangle\langle kk|}{D} U^\dagger \otimes U^{*\dagger} \end{aligned} \tag{26}$$

with

$$\begin{aligned}
\tilde{\lambda}_1^{(n)} &= \left[\left(\tilde{\lambda}_1^{(n-1)}\right)^2 + \frac{2\tilde{\lambda}_1^{(n-1)}\tilde{\lambda}_3^{(n-1)}}{D} \right] \\
\tilde{\lambda}_2^{(n)} &= \left[\frac{\left(\tilde{\lambda}_2^{(n-1)}\right)^2 + 2\tilde{\lambda}_2^{(n-1)}\tilde{\lambda}_3^{(n-1)}}{D} \right] \\
\tilde{\lambda}_3^{(n)} &= \left[\left(\tilde{\lambda}_4^{(n-1)}\right)^2 + \frac{2\tilde{\lambda}_1^{(n-1)}\tilde{\lambda}_2^{(n-1)}}{D} + \frac{2\tilde{\lambda}_1^{(n-1)}\tilde{\lambda}_4^{(n-1)}}{D} + 2\tilde{\lambda}_2^{(n-1)}\tilde{\lambda}_4^{(n-1)} + \frac{2\tilde{\lambda}_3^{(n-1)}\tilde{\lambda}_4^{(n-1)}}{D} \right] \\
\tilde{\lambda}_4^{(n)} &= \frac{\left(\tilde{\lambda}_3^{(n-1)}\right)^2}{D}
\end{aligned} \tag{27}$$

and

$$N_D = \tilde{\lambda}_1^{(n)} + \tilde{\lambda}_2^{(n)} + \tilde{\lambda}_3^{(n)} + \tilde{\lambda}_4^{(n)}. \tag{28}$$

The initial condition corresponding to the Werner state of equation (19) is given by $\tilde{\lambda}_1^{(0)} = \lambda,\ \tilde{\lambda}_2^{(0)} = \tilde{\lambda}_3^{(0)} = \tilde{\lambda}_4^{(0)} = 0$.

Numerical simulations performed for dimensions $2 \leqslant D \leqslant 20$ demonstrate that this purification procedure is capable of purifying almost all non-separable Werner states of the form of equation (19). Thereby, a depolarization operation is needed only once, namely in the possible preparation of this Werner state. The dependence of the range of convergence of this purification scheme on the dimension D of the Hilbert space is apparent from figure 3. The dashed curve indicates the minimum values $F_c = 1/D$ of the fidelity for which the Werner state of equation (19) is still non-separable. The purification protocol of Horodecki *et al* [14] converges for all initial values $F > F_c$. The solid curve represents the minimal initial value of

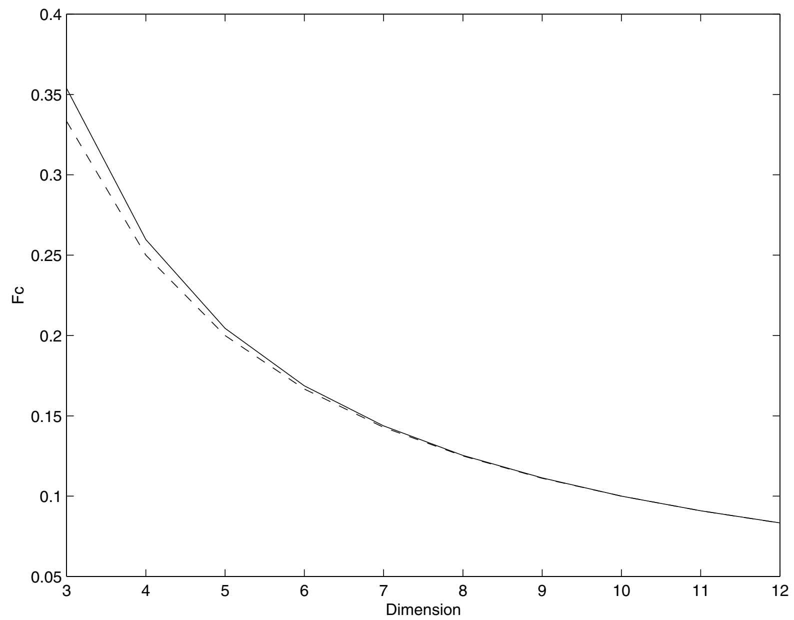

Figure 3. Dependence of the minimal initial fidelity F_c needed to purify a Werner state (compare with equation (19)) as a function of the dimension D: our protocol (full curve), Horodecki's protocol (dashed curve).

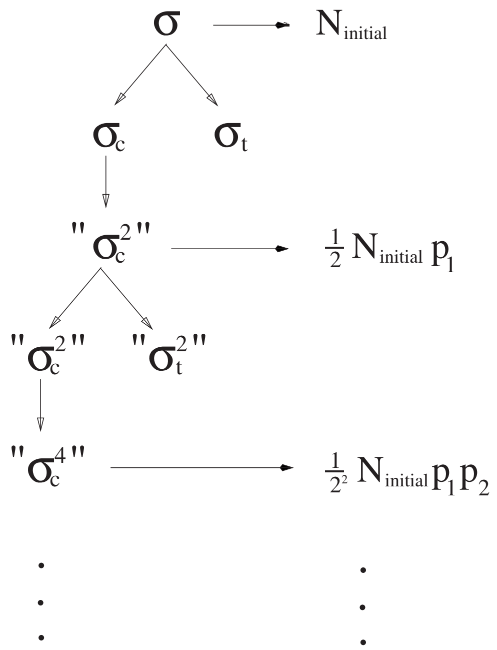

Figure 4. This scheme illustrates the calculation of the efficiency η. The initial collection of qudits is split into two equal parts. One part forms the control, the other the target qudits. The success probability of the process after the first step is p_1, the total number of 'squared' qudits is given by $(1/2)N_{\text{initial}}p_1$. The procedure is repeated until the required fidelity F_{final} for the resulting state is reached after n steps.

the fidelity for which our protocol purifies. It is clear from figure 3 that the range of convergence of our purification scheme is slightly smaller but approaches the ideal limit F_c with increasing dimension D of the Hilbert space. Already for moderately large dimensions D our range of convergence approaches the ideal range closely.

Let us now compare the efficiency of our purification protocol with the one proposed by Horodecki *et al* [14]. In particular, we are interested in answering the question, how many iterations are needed to obtain state $|\psi_{00}\rangle$ with a prescribed final fidelity $F_{\text{final}} = \langle\psi_{00}|\sigma^c_{\text{final}}|\psi_{00}\rangle$ for a given dimension D? In order to clarify the calculation of this efficiency let us briefly reconsider the basic steps involved in a purification protocol. They are represented schematically in figure 4. Initially the purification process starts with an ensemble of N_{initial} Werner states each of which is described by equation (19). In each step of the iteration procedure the ensemble of two-qudit states is divided into two equal parts which serve as control and target systems. The nonlinear quantum transformation is performed by projecting onto one of the target states $P_{t_1} = |ii\rangle\langle ii|\,(i = 0, \ldots, D-1)$. As our initial state and our unitary transformation of equation (25) are invariant under the transformation $|i\rangle \to |i+1\rangle$ of the basis states, all these projections are equally probable despite the fact that our purification procedure does not yield a Werner state at each step of the iteration procedure. This nonlinear quantum transformation is followed by a local twirling transformation (compare with equation (24)) which is based on the discrete Fourier transformation of equation (25). Thus, after the first step of our purification procedure we are left with $[p_1 \times N_{\text{initial}}/2]$ purified two-qudit systems. Thereby p_1 denotes the probability of obtaining the target qudit in one of its basis states $|i\rangle$

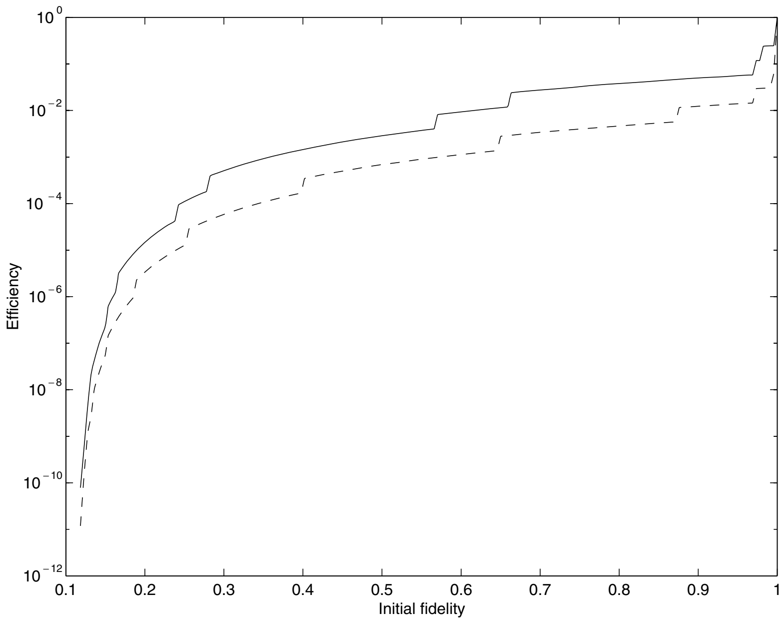

Figure 5. Dependence of the efficiency η (compare with equation (29)) on the initial fidelity F for a fixed final fidelity $F_{\text{final}} = 1 - 10^{-5}$ and for dimension $D = 6$. The solid curve gives the results of the proposed method and the dashed curve the results of Horodecki's protocol.

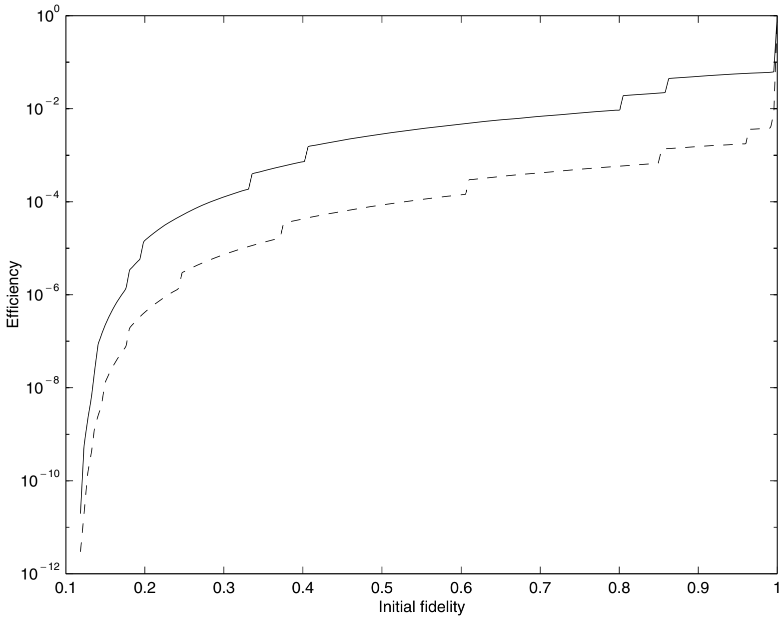

Figure 6. Dependence of the efficiency η (compare with equation (29)) on the initial fidelity F for a fixed final fidelity $F_{\text{final}} = 1 - 10^{-7}$ and for dimension $D = 9$. The solid curve gives the results of the proposed method and the dashed curve the results of Horodecki's protocol.

($i = 0, \ldots, D-1$). Continuing this iteration procedure after n iterations the number of remaining purified two-qudit systems is given by

$$N_{\text{purified}} = \frac{N_{\text{initial}}}{2^n} \prod_{l=1}^{n} p_l. \tag{29}$$

Accordingly, the efficiency η of this purification process is given by $\eta = N_{\text{purified}}/N_{\text{initial}}$.

In figures 5 and 6 the dependence of the efficiency η on the initial fidelity of the Werner state F (compare with equation (20)) is depicted for dimensions $D = 6$ and 9 and for different values of the final fidelity $F_{\text{final}} = \langle\psi_{00}|\sigma^c_{\text{final}}|\psi_{00}\rangle$. From these figures it is apparent that our protocol requires fewer steps than the protocol of Horodecki *et al* [14]. Furthermore, numerical studies also indicate that in both purification protocols the success probabilities p_l entering equation (29) are comparable in magnitude. Thus, the overall better efficiency of our purification protocol which is apparent from figures 5 and 6 reflects the fewer number of steps n which are required for achieving a given final accuracy. With increasing accuracy of the final purified state this difference in efficiencies between both purification protocols becomes larger and larger. From figures 5 and 6 one also notices a second characteristic feature which has been found also in other numerical simulations. For a given value of the final fidelity the differences between the efficiencies of both protocols becomes smaller with increasing dimension D of the Hilbert space involved.

5. Conclusions

A novel purification scheme has been proposed. It is based on the iterative application of a special class of nonlinear quantum maps and a single, local unitary transformation. The required nonlinear quantum map can be implemented conveniently by a Hermitian generalized quantum XOR-gate. The proposed purification scheme has several attractive features. Firstly, it applies to arbitrary dimensional bipartite quantum systems. Secondly, it does not require a depolarization operation at each step of the iteration procedure. In general, such a depolarization operation is required only for the preparation of the initial Werner state. The proposed scheme rests on a single twirling operation which is performed at each step of the iterative purification scheme. Thirdly, the proposed procedure achieves purification in a very efficient way. In particular, it has been demonstrated that it achieves purification of Werner states in a more efficient way than the other known purification protocol which has been introduced by Horodecki *et al* [14]. Furthermore, its almost maximal range of convergence indicates that the employed local twirling operation which is based on a discrete Fourier transform is a good choice. Such a transformation can be implemented easily in many quantum systems.

The proposed purification method may also be generalized to multi-partite quantum systems. In this context it would be particularly interesting to develop efficient purification protocols for GHZ-like quantum states.

Acknowledgments

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References

[1] Bouwmeester D, Ekert A and Zeilinger A 2000 *The Physics of Quantum Information* (Berlin: Springer)
[2] Alber G, Beth T, Horodecki M, Horodecki P, Horodecki R, Weinfurter H, Werner R and Zeilinger A *Quantum Information—An Introduction to Basic Theoretical Concepts and Experiments* (Berlin: Springer)
[3] Monroe C, Meekhof D M, King B E, Itano W M and Wineland D J 1995 *Phys. Rev. Lett.* **75** 4714
[4] Barenco A, Deutsch D, Ekert A and Jozsa R 1994 *Phys. Rev. Lett.* **74** 4083
DiVincenzo D 1995 *Phys. Rev.* A **51** 1015
Barenco A, Bennett C H, Cleve R, DiVincenzo D P, Margolus N, Shor P, Sleator T, Smolin J A and Weinfurter H 1995 *Phys. Rev.* A **52** 3457
[5] Feynman R 1985 *Opt. News* **11** 11
[6] Barenco A, Berthiaume A, Deutsch D, Ekert A, Josza R and Macchiavello C 1997 *SIAM J. Comput.* **26** 1541
[7] Braunstein S L, Mann A and Revzen M 1992 *Phys. Rev. Lett.* **68** 3259
[8] Bennett C H and Wiesner S J 1992 *Phys. Rev. Lett.* **69** 2881
[9] Bennett C H, Brassard G, Crépeau C, Jozsa R, Peres A and Wootters W K 1993 *Phys. Rev. Lett.* **70** 1895
[10] Bennett C H, Brassard G, Popescu S, Schumacher B, Smolin J and Wootters W K 1996 *Phys. Rev. Lett.* **76** 722
[11] Bechmann-Pasquinucci H, Huttner B and Gisin N 1998 *Phys. Lett.* A **242** 198
[12] Roos Ch, Zeiger Th, Rohde H, Nägerl H C, Eschner J, Leibfried D, Schmidt-Kaler F and Blatt R 1999 *Phys. Rev. Lett.* **83** 4713
[13] Brendel J, Gisin N, Tittel W and Zbinden H 1999 *Phys. Rev. Lett.* **82** 2594
[14] Horodecki M and Horodecki P 1999 *Phys. Rev.* A **59** 4206
[15] Braunstein S L 1998 *Phys. Rev. Lett.* **80** 4084
[16] Mandel L and Wolf E 1995 *Optical Coherence and Quantum Optics* (Cambridge: Cambridge University Press)
[17] Bennett C H, Di Vincenzo D P, Smolin J A and Wootters W K 1996 *Phys. Rev.* A **54** 3824
[18] Deutsch D, Ekert A, Josza R, Macchiavello Ch, Popescu S and Sanpera A 1996 *Phys. Rev. Lett.* **77** 2818
[19] Pittenger A O and Ruben M H 2000 *Opt. Commun.* **179** 447