DEFINITION. *A set T is a tail if and only if there exists a positive integer t, such that T contains exactly those integers $\geqq t$.*

THEOREM. *If S is a set of positive integers containing no tail and if S is closed under finite sums, then* g.c.d. $(S) > 1$.

Proof. It is a well-known theorem (see [1]) that if x and y are positive integers with $(x, y) = 1$ and if A is the set of all integers of the form $ax + by$, where a and b range over the set of positive integers, then A contains every integer greater than xy. Thus, since S does not contain a tail, $(a, b) > 1$ for all $a, b \in S$. This implies that g.c.d.$(S) > 1$.

For completeness, we include a proof of the above quoted result.

Proof. We can clearly find nonzero integers c and d such that $cx + dy = n$, for any given positive integer n. If $d < 0$, let rx be the least positive multiple of x such that $d + rx > 0$. Then, $(c-(r-1)y)x + (d+rx)y = n + xy$ and $(c-(r-1)y)x + xy \geqq n + xy$. Thus, $c - (r-1)y \geqq n/x > 0$, and $c - (r-1) > 0$. Since A is closed under finite sums, $n + xy \in A$, for every positive integer n. Thus, A contains every integer $> xy$. If $c < 0$, a similar argument holds. Finally, if $c, d > 0$, it follows easily that $n + xy \in A$.

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Reference

1. I. Niven and H. S. Zuckerman, An Introduction to the Theory of Numbers, Wiley, New York and London, 1960, p. 97, problem 5.

A MEAN VALUE THEOREM—AN EXTENSION

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Let $f(x)$ be defined in the interval $[a, b]$ and differentiable in the interval. Further let $f'(a) = f'(b)$. Then there is a point ξ in (a, b) such that

$$\frac{f(\xi) - f(a)}{\xi - a} = f'(\xi).$$

The above result is a mean value theorem due to Flett ([1], p. 121; [2], p. 39).

In this note we obtain an extension of the above result.

THEOREM. *Let $f(x)$, defined in the interval $[a, b]$, be continuous in $[a, b]$ and differentiable at $x = a$ and $x = b$. Further let the four Dini derivates ([1]) $f^+(x)$, $f_+(x)$, $f^-(x)$, $f_-(x)$ exist and be finite in (a, b). Then if $f'(a) = f'(b)$ there exists a ξ_1, $a < \xi_1 < b$, such that*

$$f^+(\xi_1) \leqq \frac{f(\xi_1) - f(a)}{\xi_1 - a} \leqq f_-(\xi_1) \tag{1}$$

or a ξ_2, $a<\xi_2<b$, *such that*

$$f^-(\xi_2) \leqq \frac{f(\xi_2)-f(a)}{\xi_2-a} \leqq f_+(\xi_2). \tag{2}$$

Proof. We shall assume that $f'(a)=f'(b)=0$. For otherwise we have to consider only $f(x)-xf'(a)$. Consider the function $g(x)$ defined by

$$g(x) = \frac{f(x)-f(a)}{x-a}, \qquad a < x \leqq b, \; g(a) = f'(a) = 0.$$

The function $g(x)$ so defined is continuous in $[a, b]$. Further,

$$g'(b) = -g(b)/(b-a).$$

If $g(b)>0$ then $g'(b)<0$. Hence g is a decreasing function at b, while $g(a)=0$. Now $g(x)$ being continuous in $[a, b]$ attains its maximum at a point ξ_1 between a and b. Hence ([1], p. 117, Ex. 21.6),

$$g^+(\xi_1) \leqq 0 \quad \text{and} \quad g_-(\xi_1) \geqq 0.$$

But

$$g^+(\xi_1) = \frac{f^+(\xi_1)}{\xi_1-a} - \frac{f(\xi_1)-f(a)}{(\xi_1-a)^2}$$

and

$$g_-(\xi_1) = \frac{f_-(\xi_1)}{\xi_1-a} - \frac{f(\xi_1)-f(a)}{(\xi_1-a)^2}.$$

Inequalities (1) now follow from these.

If, on the other hand, $g(b)<0$, then $g'(b)>0$, so that g is increasing at b while $g(a)=0$. Hence, $g(x)$ attains a minimum at a point ξ_2 between a and b; so that

$$g_+(\xi_2) \geqq 0 \quad \text{and} \quad g^-(\xi_2) \leqq 0,$$

and we get inequalities (2).

Finally, if $g(b)=0$ then, since $g(x)$ is continuous in the closed interval $[a, b]$ and $g(a)=0$, $g(x)$ attains either a maximum at a point ξ_1 or a minimum at a point ξ_2 between a and b. Hence, as before, either $g^+(\xi_1)\leqq 0$ and $g_-(\xi_1)\geqq 0$ or $g_+(\xi_2)\geqq 0$ and $g^-(\xi_2)\leqq 0$. These inequalities are equivalent to (1) and (2) in the respective cases. Our theorem is thus completely established.

References

1. R. P. Boas, A primer of real functions, Carus Monograph No. 13, 1963.
2. T. M. Flett, A mean value theorem, Math. Gaz., 42 (1958) 38–39.