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A Mean Value Theorem-An Extension

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DEFINITION. A set  $T$  is a tail if and only if there exists a positive integer  $t$ , such that  $T$  contains exactly those integers  $\geq t$ .

THEOREM. If  $S$  is a set of positive integers containing no tail and if  $S$  is closed under finite sums, then  $\text{g.c.d.}(S) > 1$ .

*Proof.* It is a well-known theorem (see [1]) that if  $x$  and  $y$  are positive integers with  $(x, y) = 1$  and if  $A$  is the set of all integers of the form  $ax + by$ , where  $a$  and  $b$  range over the set of positive integers, then  $A$  contains every integer greater than  $xy$ . Thus, since  $S$  does not contain a tail,  $(a, b) > 1$  for all  $a, b \in S$ . This implies that  $\text{g.c.d.}(S) > 1$ .

For completeness, we include a proof of the above quoted result.

*Proof.* We can clearly find nonzero integers  $c$  and  $d$  such that  $cx + dy = n$ , for any given positive integer  $n$ . If  $d < 0$ , let  $rx$  be the least positive multiple of  $x$  such that  $d + rx > 0$ . Then,  $(c - (r-1)y)x + (d + rx)y = n + xy$  and  $(c - (r-1)y)x + xy \geq n + xy$ . Thus,  $c - (r-1)y \geq n/x > 0$ , and  $c - (r-1)y > 0$ . Since  $A$  is closed under finite sums,  $n + xy \in A$ , for every positive integer  $n$ . Thus,  $A$  contains every integer  $> xy$ . If  $c < 0$ , a similar argument holds. Finally, if  $c, d > 0$ , it follows easily that  $n + xy \in A$ .

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#### Reference

1. I. Niven and H. S. Zuckerman, *An Introduction to the Theory of Numbers*, Wiley, New York and London, 1960, p. 97, problem 5.

#### A MEAN VALUE THEOREM—AN EXTENSION

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Let  $f(x)$  be defined in the interval  $[a, b]$  and differentiable in the interval. Further let  $f'(a) = f'(b)$ . Then there is a point  $\xi$  in  $(a, b)$  such that

$$\frac{f(\xi) - f(a)}{\xi - a} = f'(\xi).$$

The above result is a mean value theorem due to Flett ([1], p. 121; [2], p. 39).

In this note we obtain an extension of the above result.

THEOREM. Let  $f(x)$ , defined in the interval  $[a, b]$ , be continuous in  $[a, b]$  and differentiable at  $x = a$  and  $x = b$ . Further let the four Dini derivatives ([1])  $f^+(x)$ ,  $f_+(x)$ ,  $f^-(x)$ ,  $f_-(x)$  exist and be finite in  $(a, b)$ . Then if  $f'(a) = f'(b)$  there exists a  $\xi_1$ ,  $a < \xi_1 < b$ , such that

$$(1) \quad f^+(\xi_1) \leq \frac{f(\xi_1) - f(a)}{\xi_1 - a} \leq f_-(\xi_1)$$

or a  $\xi_2$ ,  $a < \xi_2 < b$ , such that

$$(2) \quad f^-(\xi_2) \leq \frac{f(\xi_2) - f(a)}{\xi_2 - a} \leq f_+(\xi_2).$$

*Proof.* We shall assume that  $f'(a) = f'(b) = 0$ . For otherwise we have to consider only  $f(x) - xf'(a)$ . Consider the function  $g(x)$  defined by

$$g(x) = \frac{f(x) - f(a)}{x - a}, \quad a < x \leq b, \quad g(a) = f'(a) = 0.$$

The function  $g(x)$  so defined is continuous in  $[a, b]$ . Further,

$$g'(b) = -g(b)/(b - a).$$

If  $g(b) > 0$  then  $g'(b) < 0$ . Hence  $g$  is a decreasing function at  $b$ , while  $g(a) = 0$ . Now  $g(x)$  being continuous in  $[a, b]$  attains its maximum at a point  $\xi_1$  between  $a$  and  $b$ . Hence ([1], p. 117, Ex. 21.6),

$$g^+(\xi_1) \leq 0 \quad \text{and} \quad g_-(\xi_1) \geq 0.$$

But

$$g^+(\xi_1) = \frac{f^+(\xi_1)}{\xi_1 - a} - \frac{f(\xi_1) - f(a)}{(\xi_1 - a)^2}$$

and

$$g_-(\xi_1) = \frac{f_-(\xi_1)}{\xi_1 - a} - \frac{f(\xi_1) - f(a)}{(\xi_1 - a)^2}.$$

Inequalities (1) now follow from these.

If, on the other hand,  $g(b) < 0$ , then  $g'(b) > 0$ , so that  $g$  is increasing at  $b$  while  $g(a) = 0$ . Hence,  $g(x)$  attains a minimum at a point  $\xi_2$  between  $a$  and  $b$ ; so that

$$g_+(\xi_2) \geq 0 \quad \text{and} \quad g^-(\xi_2) \leq 0,$$

and we get inequalities (2).

Finally, if  $g(b) = 0$  then, since  $g(x)$  is continuous in the closed interval  $[a, b]$  and  $g(a) = 0$ ,  $g(x)$  attains either a maximum at a point  $\xi_1$  or a minimum at a point  $\xi_2$  between  $a$  and  $b$ . Hence, as before, either  $g^+(\xi_1) \leq 0$  and  $g_-(\xi_1) \geq 0$  or  $g_+(\xi_2) \geq 0$  and  $g^-(\xi_2) \leq 0$ . These inequalities are equivalent to (1) and (2) in the respective cases. Our theorem is thus completely established.

#### References

1. R. P. Boas, A primer of real functions, Carus Monograph No. 13, 1963.
2. T. M. Flett, A mean value theorem, Math. Gaz., 42 (1958) 38-39.