A CONVERSE TO THE MEAN VALUE THEOREM FOR HARMONIC FUNCTIONS.

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1. Introduction. In the present paper we shall confirm, in a special case, a conjecture concerning the "converse" to the mean value theorem for harmonic functions. We hope the ideas employed here will eventually be of use in settling the full conjecture.

Some notations are necessary in order to state our conjecture. Let Ω be a region in $\mathbf{R}^N, N \geqslant 1$, and for each $x \in \Omega$ let $d(x)$ be the distance from x to $\partial\Omega$ (the boundary of Ω. $d(x) < \infty$ unless $\Omega = \mathbf{R}^N$). Fixed for each $x \in \Omega$ is a ball, $B(x)$, with center x and radius $\delta(x) < \infty$, such that $B(x) \subseteq \Omega$. Define a kernel, $P = P_\delta$, on $\Omega \times \Omega$ by

$$P_\delta(x,y) = \begin{cases} \dfrac{1}{|B(x)|}\chi_{B(x)}(y) & (y \in B(x)) \\ 0 & (y \notin B(x)) \end{cases}$$

Here, as in later formulas, $|\cdot|$ stands for the Lebesgue measure (volume) of a set, and χ stands for characteristic function. While $P_\delta(\cdot,\cdot)$ may not be measurable (δ is not required to be measurable), it is still meaningful to speak of measurable functions f such that $P_\delta f = f$. Accordingly, we define $\mathcal{H}_\delta$ to be the set of all finite, nonnegative, Lebesgue measurable functions $f \geqslant 0$ such that

$$f(x) = \int_\Omega P_\delta(x,y) f(y)\, dy \tag{1.1}$$

for all $x \in \Omega$.

We denote by $\mathcal{H}$ the cone of nonnegative harmonic functions on Ω. By the mean value theorem for harmonic functions $\mathcal{H}_\delta \supseteq \mathcal{H}$ for any δ, and by a *converse* to the mean value theorem we mean a statement that $\mathcal{H}_\delta = \mathcal{H}$. Since there are well-known elementary examples to show $\mathcal{H}_\delta = \mathcal{H}$ is false in general for any region, any such converse must involve additional assumptions on $\delta(\cdot)$,

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at least. Now it is our conjecture that if $\mathcal{H}_\delta = \mathcal{H}$ whenever Ω is bounded and $\delta(\cdot)$ satisfies a mild growth restriction:

CONJECTURE 1.2. *With notations as above, if Ω is bounded, and if $\delta(\cdot)$ is locally bounded bounded away from 0 on Ω, then $\mathcal{H}_\delta = \mathcal{H}$.*

Many results in the recent literature lend support to our conjecture ([10], [1], [3], [11], [17], [12], [2]), however in each case in order to be able to conclude that a given $f \in \mathcal{H}_\delta$ belongs to $\mathcal{H}$, it is necessary to assume f satisfies an additional growth restriction (which, with the exception of [17], is that f be bounded). The main result of the present paper makes rather strong assumptions on Ω and $\delta(\cdot)$, but it is apparently the first instance in which $\mathcal{H}_\delta = \mathcal{H}$ has been verified.

THEOREM 1.3. *Assume of $\delta(\cdot)$ that there exist a constant $\alpha > 0$ and a function $0 < r(\cdot) \leqslant d(\cdot)$ on Ω such that for all $x, x' \in \Omega$ $|r(x) - r(x')| \leqslant \|x - x'\|$ and $\alpha r(x) \leqslant \delta(x) \leqslant (1-\alpha) r(x)$. If Ω is a bounded Lipschitz domain, then $\mathcal{H}_\delta = \mathcal{H}$.*

Remark. Heath has pointed out to us that some condition on Ω is necessary for $\mathcal{H}_\delta = \mathcal{H}$ to obtain in general. (Our conjecture was originally stated for arbitrary Ω.) He shows there are "admissible" δ's on $\Omega = \mathbf{R}^N$ such that $\mathcal{H}_\delta$ contains nonconstant bounded functions. (The examples will appear in [12].)

The last step in the proof of our Theorem 1.3 is to quote Theorem 1 of [17], and it is here that the "Lipschitz domain" assumption is used. The analysis in [17] required a careful study of individual points on the Martin boundary of Ω, a study which we have thus far been unable to carry out or replace with no smoothness assumption on $\partial\Omega$. It would be interesting to determine whether Heath's techniques could be combined with those of the present paper to obtain a closer approximation to our conjecture because Heath makes no assumption on $\partial\Omega$.

In outline, the proof of Theorem 1.3 is as follows: In Section 2 we use elementary Choquet theory to obtain an integral representation for the elements of $\mathcal{H}_\delta$ over the extremals of (the cone) $\mathcal{H}_\delta$. In this way we are reduced to proving an element of $\mathcal{H}_\delta$ is harmonic if it is extremal. After obtaining a probabilistic interpretation of the extremals in Section 3, we make in Section 4 the minor modifications in the Doob boundary theory ([5], [6], [14], [9]) which enable us to use this theory in our setting to obtain an analytic expression (more or less) for the extremals of $\mathcal{H}_\delta$. This analytic expression is used in Section 5 to prove each extremal of $\mathcal{H}_\delta$ is dominated by a positive harmonic function on Ω (i.e., an element of $\mathcal{H}$). The proof is completed by referring to our earlier

THEOREM. ([17], Theorem 1). *Let Ω be a bounded Lipschitz domain in* $\mathbf{R}^N$, $N \geqslant 1$, *and assume* $\delta(\cdot)$ *is locally bounded away from* 0 *on* Ω. *Then* $\mathcal{H}$ *is an extremal subcone of* $\mathcal{H}_\delta$. *That is, if* $f \in \mathcal{H}_\delta$ *and* $g \in \mathcal{H}$ *satisfy* $f \leqslant g$, *then also* $f \in \mathcal{H}$.

In Section 6 we combine Theorem 1.3 with the results in R. Hunt-Wheeden [15] to obtain an "elementary" computation of the Poisson kernel for a bounded Lipschitz domain:

THEOREM 1.4. *Let Ω and $\delta(\cdot)$ be as in the statement of Theorem* 1.3, *and assume in addition that $\delta(\cdot)$ is Borel. Form the kernel*

$$G(x,y) = \sum_{m=1}^{\infty} P_\delta^{(m)}(x,y), \tag{1.5}$$

the summands being the iterates of P_δ as an integral kernel. Then $G(\cdot,\cdot)$ is finite on Ω and moreover if $x_0 \in \Omega$ is a fixed point, and if $x \in \Omega$, $y \in \partial\Omega$, then

$$\lim_{z \to y} \frac{G(x,z)}{G(x_0,z)} = K_{x_0}(x,y) \tag{1.6}$$

exists and is the "Poisson kernel" for Ω. That is, K_{x_0} is continuous on $\Omega \times \partial\Omega$, $K_{x_0}(\cdot,y)$ is harmonic for each y, and $K_{x_0}(\cdot,y)$ vanishes continuously at each $y' \neq y$ in $\partial\Omega$.

2. **The Cone $\mathcal{H}_\delta$.** We begin by recalling Proposition 2.1 of [17]: Given $f \in \mathcal{H}_\delta$ there exist Borel functions f_0 and δ_0 with $f_0 \in \mathcal{H}_{\delta_0}$, $f = f_0$ a.e., and $\delta_0 \geqslant \delta$. If f_0 is proven to be harmonic, then $f_0 \equiv f$ because $f \equiv P_\delta f \equiv P_\delta f_0 \equiv f_0$. (It is an easy matter to modify the proof of this proposition so that if $\delta(\cdot)$ satisfies an inequality as in the statement of Theorem 1.3, then $\delta_0(\cdot)$ will satisfy the same inequality.) Therefore, it will be enough for us to prove Theorem 1.3 under the additional assumption that $\delta(\cdot)$ be Borel. This is assumed in all that follows.

Since $\delta(\cdot)$ is now a Borel function, we can form the iterates of the integral kernel P_δ, setting $P_\delta^{(1)} = P_\delta$, and

$$P_\delta^{(n+1)}(x,y) = \int_\Omega P_\delta^{(n)}(x,z) P_\delta(z,y)\, dz \qquad (n \geqslant 1).$$

LEMMA 2.1. *Assume $\delta(\cdot)$ is locally bounded away from* 0 *on Ω. If A and C are compact sets in Ω, there exist an integer m and a number $\lambda > 0$ such that $P_\delta^{(m)}(x,y) \geqslant \lambda$, $(x,y) \in A \times C$.*

Proof. By the compactness of $A \times C$ it is enough to prove for every pair $x, y \in \Omega$ the existence of fixed neighborhoods U, V $(x \in U, y \in V)$ such that for all sufficiently large $n \inf_{x' \in U, y' \in V} P_\delta^{(n)}(x', y') > 0$. If E and F are Borel sets in Ω, define

$$m_1(E,F) = \inf_{x \in E} P_\delta(x,F)$$

$$m_2(E,F) = \inf_{\substack{x \in E \\ y \in F}} P_\delta(x,y)$$

If $E_0, \ldots, E_n$ are Borel sets in Ω, then it is easily checked that if $x' \in E_0, y' \in E_n$

$$P_\delta^{(n)}(x',y') \geqslant \left(\prod_{j=0}^{n-2} m_1(E_j, E_{j+1}) \right) m_2(E_{n-1}, E_n) \tag{2.2}$$

Using the fact $\delta(\cdot)$ is locally bounded away from 0 on Ω it is possible to choose an integer t and balls $E_0, \ldots, E_t$ such that $x \in E_0$, $y \in E_t$, and $m_1(E_j, E_{j+1}) > 0$, $0 \leqslant j < t$, $m_2(E_{t-1}, E_t) > 0$ and $m_i(E_t, E_t) > 0$, $i = 1, 2$. Taking $U = E_0$ and $V = E_t$, (2.2) implies $P_\delta^{(n)}$ is bounded away from 0 on $U \times V$ for $n \geqslant t$.

LEMMA 2.3. *Assume both $\delta(\cdot)$ and $d(\cdot) - \delta(\cdot)$ are locally bounded away from 0 on Ω. There exists a locally bounded function $M(x_0, x)$ on $\Omega \times \Omega$ such that for all $f \in \mathcal{H}_\delta$ and $(x_0, x) \in \Omega \times \Omega$*

$$f(x) \leqslant M(x_0, x) f(x_0). \tag{2.4}$$

Proof. Let A and B be compact sets in Ω, and let C be the closure of the union $\cup_{z \in B} B(z)$. C is compact by our assumption on $d(\cdot) - \delta(\cdot)$. There exists an m and a number $\lambda = \lambda_n > 0$ such that $P_\delta^{(n)}(x,y) \geqslant \lambda_n$, $(x,y) \in A \times C$, by Lemma 2.1. Define $M = \sup_{z \in B_{w \in \Omega}} P_\delta(z, w)$. We have $P_\delta^{(n)}(x, \cdot) \geqslant (\lambda_n / M) P_\delta(z, \cdot)$ for $x \in A, z \in B$. It follows that $f(x) \geqslant (\lambda_n / M) f(z)$ for $f \in \mathcal{H}_\delta$ and the lemma is proved with $M(x_0, x) = M / \lambda_n$ for $x_0 \in A, x \in B$.

Let $\mathcal{L}_c^1$ be the space of equivalence classes of compactly supported Lebesgue integrable functions on Ω. If $\mathcal{L}_{\text{loc}}^\infty$, the space of equivalence classes of locally bounded measurable function on Ω, is given the $\mathcal{L}_c^1$ topology, then by Aloaglu's theorem any set $\Gamma \subseteq \mathcal{L}_{\text{loc}}^\infty$ whose elements are jointly locally uniformly bounded is relatively compact. Moreover because $\mathcal{L}_c^1$ is separable, such a set as Γ is *metrizable*. Let $x_0 \in \Omega$ be fixed, and define $\mathfrak{M}_{x_0} = \{ f \in \mathcal{H}_\delta \mid f(x_0) = 1 \}$. By Lemma 2.3 the elements of $\mathfrak{M}_{x_0}$ possess a local uniform bound $(M(x_0, x))$, and therefore $\mathfrak{M}_{x_0}$ is relatively compact in $\mathcal{L}_{\text{loc}}^\infty$. We claim (i) the canonical mapping $\mathfrak{M}_{x_0} \to \mathcal{L}_{\text{loc}}^\infty$ is an injection, and (ii) the range is closed. To see (i) simply note for each $x \in \Omega$ that $P_\delta(x, \cdot) \in \mathcal{L}_c$, and therefore if $f_1, f_2 \in \mathfrak{M}_{x_0}$ correspond to the same element in $\mathcal{L}_{\text{loc}}^\infty$, then $f_1 \equiv f_2$. Secondly, suppose f_n is a sequence in $\mathfrak{M}_{x_0}$

convergent to $f \in \mathcal{L}^{\infty}_{\text{loc}}$. We define $f_0(x) = (f, P_\delta(x, \cdot))$, $x \in \Omega$. If $\varphi \in \mathcal{L}^1_c$, then $\varphi P_\delta \in \mathcal{L}^1_c$, where $\varphi P_\delta(y) = {}_\Omega\int \varphi(x) P_\delta(x, y)\,dx$. Thus, $(\varphi, f_0) = (\varphi, P_\delta f) = (\varphi P_\delta, f) = \lim_n(\varphi P_\delta, f_n) = \lim_n(\varphi, P_\delta f_n) = \lim_n(\varphi, f_n) = (\varphi, f)$, and so $f = f_0$ a.e. It follows $P_\delta f_0 \equiv f_0$, and therefore the class of f is the image of f_0 under the map from $\mathfrak{M}_{x_0}$ to $\mathcal{L}^{\infty}_{\text{loc}}$. For one final remark we note that a sequence f_n in $\mathfrak{M}_{x_0}$ is convergent in the $\mathcal{L}^1_c$ topology if and only if $\lim_{n\to\infty} f_n(x)$ exists for every $x \in \Omega$. We have proved

PROPOSITION 2.5. *Assume both $\delta(\cdot)$ and $d(\cdot) - \delta(\cdot)$ are locally bounded away from 0 on Ω. For any $x_0 \in \Omega$ the space $\mathfrak{M}_{x_0} = \{ f \in \mathcal{H}_\delta \mid f(x_0) = 1\}$ is a compact metrizable space in the topology of pointwise everywhere convergence on Ω.*

$\mathfrak{M}_{x_0}$ is a compact convex set of functions on Ω, and we use $\mathcal{E}_{x_0}$ to denote the set of extreme points. Because $\mathfrak{M}_{x_0}$ is metrizable, $\mathcal{E}_{x_0}$ is a Borel set (in fact a G_δ) [16]. By Choquet's theorem there exists for every $f \in \mathcal{H}_\delta$ a measure ω_f on $\mathcal{E}_{x_0}$ such that

$$f = \int_{\mathcal{E}_{x_0}} e\,\omega_f(de)$$

in the sense that if $\varphi \in \mathcal{L}^1_c$, then

$$(\varphi, f) = \int_{\mathcal{E}_{x_0}} (\varphi, e)\,\omega_f(de)$$

Specializing to $\varphi(\cdot) = P_\delta(x, \cdot)$, we have in particular that

$$f(x) = \int_{\mathcal{E}_{x_0}} e(x)\,\omega_f(de) \qquad (x \in \Omega)$$

From this it is clear we have proved

PROPOSITION 2.6. *Assume both $\delta(\cdot)$ and $d(\cdot) - \delta(\cdot)$ are locally bounded away from 0 on Ω. If every extremal of $\mathcal{H}_\delta$ is harmonic, then $\mathcal{H}_\delta = \mathcal{H}$.*

Remark. If $f, g \in \mathcal{H}_\delta$, set $h_0 = f \wedge g$. Then $P_\delta h_0 \leqslant h_0$, and $P_\delta^{(n)} h_0$ decreases to a function $h \in \mathcal{H}_\delta$: If $h_1 \in \mathcal{H}_\delta$ and $h_1 \leqslant h_0$, then obviously $h_1 \leqslant h$. Therefore, $\mathcal{H}_\delta$ is a *lattice*. By the uniqueness portion of Choquet theory, the measure ω_f above is unique [16]. We shall have no need of this fact in what follows.

3. Probabilistic Interpretation of the Extremals. We assume for this section only that $\delta(\cdot)$ is (Borel and) locally bounded away from 0 on Ω. The consequence of this assumption which is used in what follows is that if $f \in \mathcal{H}_\delta$, then either $f \equiv 0$ or else $f(x) > 0$, all x.

Here and in later sections Λ will denote the infinite product space, $\Lambda=\Omega\times\Omega\times\cdots$. We use $x_j(\cdot)$, $j=0,1,\ldots$, for the coordinate functions and $\mathfrak{B}=\mathfrak{B}(x_0,x_1,\ldots)$ for the σ-field generated by these functions. The left shift, $T,(x_j(T\omega)=x_{j+1}(\omega))$, which is $\mathfrak{B}$ measurable, is used to define the *tail σ-field*, $\mathfrak{B}_\infty=\cap_{n=0}^{\infty}T^{-n}\mathfrak{B}=\cap_{n=k}^{\infty}T^{-n}\mathfrak{B}$ (for any $k\geqslant 0$). Finally, the *invariant σ-field*, $\mathfrak{B}_I$, is defined to be $\mathfrak{B}_I=\{X\in\mathfrak{B}\,|\,T^{-1}X=X\}$. $\mathfrak{B}_I$ is a sub-σ-field of $\mathfrak{B}_\infty$.

If $f\in\mathcal{H}_\delta$ is not 0, set up a kernel Q_f on $\Omega\times\Omega$ as

$$Q_f(x,y)=\frac{1}{f(x)}P_\delta(x,y)f(y).$$

Q_f is well defined because f never vanishes. Notice that $Q_f(x,\cdot)$ is a probability density for every x, and also

$$Q_f^{(n)}(x,y)=\frac{1}{f(x)}P_\delta^{(n)}(x,y)f(y)\qquad(n\geqslant 1).\tag{3.1}$$

If $X_0,X_1,\ldots,X_n$ are Borel sets in Ω, define for $x\in\Omega$

$$\mu_x^f\{\omega\in\Lambda|x_j(\omega)\in X_j,0\leqslant j\leqslant n\}$$
$$=\int_{X_0}\int_{X_1}\cdots\int_{X_n}\left(\prod_{j=0}^{n-1}Q_f(x_j,x_{j+1})\right)\delta_x(dx_0)dx_1\ldots dx_n,$$

where δ_x is the point mass at x. μ_x^f extends to be a probability measure on $\mathfrak{B}$. It is the realization of the random walk starting at x and governed by Q_f. The easily verified formula ([17], (2.7))

$$\mu_x^f(T^{-n}X)=\int_\Omega Q_f^{(n)}(x,y)\,\mu_y^f(X)\,dy\tag{3.2}$$

$X\in\mathfrak{B}$, $n\geqslant 1$ may be used to prove $\mu_x^f\prec\;\succ\mu_{x'}^f$ on $\mathfrak{B}_I$ for all $x,x'\in\Omega$([17], Prop. 2.3). Of course $\mu_x^f\perp\mu_{x'}^f,x\neq x'$ on B. If $X\in\mathfrak{B}_I$,and if $\alpha(x)=\mu_x^f(X)$, then by (3.2) $Q_f\alpha=\alpha$. This implies $P_\delta(\alpha f)=\alpha f$ so that $\alpha f\in\mathcal{H}_\delta$. Since $0\leqslant\alpha\leqslant 1$, if f is an extremal of $\mathcal{H}_\delta$, then αf is a constant multiple of f. Now to compute the constant we use the fact that $E(X|x_0\ldots,x_n)=\mu_{x_n}^f(X)=\alpha(x_n)$ ([17], Sec. 7) converges μ_x^f a.e. to χ_X. Thus, $\alpha\equiv 0$ or $\alpha\equiv 1$ if f is an extremal.

LEMMA 3.3. *If $\delta(\cdot)$ is locally bounded away from 0, and if $f\in\mathcal{H}_\delta$ is an extremal, then for every $X\in\mathfrak{B}_I\,\mu_x^f(X)\equiv 0$ or $\mu_x^f(X)\equiv 1$.*

Remark. Of course the lemma implies $\mathfrak{B}_I$ is μ_x^f trivial for every extremal f and $x\in\Omega$. On the other hand the mutual absolute continuity on $\mathfrak{B}_I$ (and a martingale argument) imply that if $\mathfrak{B}_I$ is μ_x^f trivial for some x, then f is an extremal. It was this fact which played an important role in the proof of Theorem 1 of [17].

4. The Doob Exit Boundary. In this section both $\delta(\cdot)$ and $d(\cdot)-\delta(\cdot)$ are assumed to be locally bounded away from 0.

Let $A\subseteq\Omega$ be a Borel set. The function $N_A(\omega)=\sum_{n=1}^{\infty}\chi_A(x_n(\omega)), \omega\in\Lambda$, is measurable and counts the number of terms in the sequence $x_j(\omega)$, $j\geqslant 1$ which belong to A. If $f\in\mathcal{H}_\delta$ and $x\in\Omega$, then

$$\int_\Lambda N_A(\omega)\,\mu_x^f(d\omega)=\sum_{n=1}^{\infty}\int_A Q_f^n\,(x,y)\,dy \tag{4.1}$$

because the nth summand on the right is just the integral of $\chi_A(x_n(\omega))$ with respect to μ_x^f. We will need a bound on the right hand side of (4.1), but for this bound it is most convenient to work with the left hand side. Because each $f\in\mathcal{H}_\delta$ is locally bounded away from both 0 and ∞ on Ω (use Lemma 2.1 for lower bounds), it will suffice to work with $P_\delta=Q_1$ (i.e., $f\equiv 1$). Let $\mu_x=\mu_x^1$ below.

LEMMA 4.2. *Assume $\delta(\cdot)$ is locally bounded away from 0 on Ω. For every $z\in\Omega$ there exists a ball $A=A(z)$ about z and a number $\lambda<1$ such that if $y\in A$, then $\mu_y\{\omega\in\Lambda|N_A(\omega)\geqslant 1\}\leqslant\lambda$.*

Proof. Fix $z\in\Omega$ and choose A to be a ball of radius $r/2>0$ centered at z such that if $y\in A$, then $\delta(y)>4r$. Define $\Omega_{r,z}=\{z'\in\Omega|\,\|z'-z\|>2r\}$. Define $\lambda(z'),z'\in\Omega$, by $\lambda(z')=\mu_{z'}\{\omega|N_A(\omega)>1\}$. If $y\in A$, then

$$\lambda(y)\leqslant P_\delta\,(y,\Omega_{r,z}^c)+\int_{\Omega_{r,z}}P_\delta\,(y,z')\lambda(z')\,dz'.$$

Let $C=\sup\{\lambda(z')|z'\in\Omega_{r,z}\}$. If we obtain an upper bound on C which is less than 1, then we will be done because $P_\delta(y,\Omega_{r,z}^c)$ is bounded away from 1 by a bound depending only upon N (the dimension).

Fix $z'\in\Omega_{r,z}$, and let $R=\|z'-y\|(>2r)$. Below M denotes $\sup\{\|x\|\,|x\in\Omega\}$. Let φ be the linear functional on $\mathbf{R}^N$ defined by $\varphi(w)=(w,R^{-1}(z'-y))$, and H the half-space $\{z'':\varphi(z'')\leqslant\varphi(y)+R/2\}$. Note $A\subseteq H$. The process $\varphi(x_n(\cdot))$ is a bounded $\mu_{z'}$ martingale and a simple martingale argument shows

$$\varphi(y)+R=\varphi(z')\leqslant\lambda\Big(\varphi(y)+\frac{R}{2}\Big)+(1-\lambda)M \tag{4.3}$$

where $\lambda=\mu_{z'}\{\omega|N_H(\omega)\geqslant 1\}\geqslant\lambda(z')$. Now $M>\|z'\|\geqslant\varphi(z')=\varphi(y)+R>\varphi(y)+$

$R/2+r$, and therefore (4.3) tells us

$$\begin{aligned}\lambda(z') &\leqslant \lambda \\ &\leqslant \frac{M-\varphi(y)-R}{M-\varphi(y)-R/2} \\ &\leqslant 1-\frac{R}{2M} \\ &\leqslant 1-\frac{r}{M}\end{aligned}$$

The lemma is proved.

Note we have proved there exists a number $c>0$, depending only upon dimension (a lower bound for $P_\delta(y',\Omega_{z,r}), y'\in A$) such that

$$\mu_{y'}\{\omega | N_A(\omega)\geqslant 1\}\leqslant 1-\frac{cr}{M} \qquad (y'\in A). \tag{4.4}$$

By (4.4) and the strong Markov property we have that

$$\mu_{y'}\{\omega | N_A(\omega)\geqslant k\}\leqslant\left(1-\frac{cr}{M}\right)^k \qquad (y'\in A) \tag{4.5}$$

and therefore

$$\int_\Lambda N_A(\omega)\,\mu_{y'}(d\omega)\leqslant\left(\frac{M}{cr}\right)\left(1-\frac{cr}{M}\right) \tag{4.6}$$

for $y'\in A$.

Now if $f\in\mathcal{H}_\delta$, and if A is as in (4.6) define $U(z)$ by

$$U(z)=\sup_{x,y\in A(z)}\frac{f(x)}{f(y)} \tag{4.7}$$

Since A has compact closure in Ω, $U(z)<\infty$. Since by (3.1) $Q_f^{(n)}(x,y) = \frac{1}{f(x)}P_\delta^{(n)}(x,y)f(y)\leqslant U(z)P_\delta^{(n)}(x,y)$, $x,y\in A(z)$, we see that (4.6) and (4.1) imply

$$\int_\Lambda N_A(\omega)\,\mu_{y'}^f(d\omega)\leqslant U(z)\left(\frac{M}{cr}\right)\left(1-\frac{cr}{M}\right) \tag{4.8}$$

for $y'\in A(z)$.

LEMMA 4.9. *If $f \in \mathcal{H}_\delta$, and if $A \subseteq \Omega$ is compact, there exists a constant $\Gamma = \Gamma(A,f) < \infty$ such that for all $x \in \Omega$*

$$\int_\Lambda N_A(\omega)\, \mu_x^f(d\omega) \leqslant \Gamma(A,f). \tag{4.10}$$

Proof. Define $\lambda_A^f(y) = \mu_y^f\{\omega | N_A(\omega) \geqslant 1\}$. By the strong Markov property

$$\int_\Lambda N_A(\omega)\, \mu_x^f(d\omega) \leqslant \lambda_A^f(x) + \sup_{z \in A} \int_\Lambda N_A(\omega)\, \mu_z^f(d\omega). \tag{4.11}$$

If $A = A(z)$ as in (4.8) this is

$$\leqslant 1 + U(z)\frac{M}{cr}\left(1 - \frac{cr}{M}\right). \tag{4.12}$$

The result for general compact A follows from (4.11)–(4.12), compactness, and the subadditivity of $A \to N_A(\omega)$ for every $\omega \in \Lambda$.

LEMMA 4.13. *Assume both $\delta(\cdot)$ and $d(\cdot) - \delta(\cdot)$ are locally bounded away from 0 on Ω. If $f \in \mathcal{H}_\delta$, define*

$$G_f(x,y) = \sum_{n=1}^{\infty} Q_f^{(n)}(x,y) \tag{4.14}$$

and

$$G(x,y) = \sum_{n=1}^{\infty} P_\delta^{(n)}(x,y) \quad (= G_1(x,y)) \tag{4.15}$$

For every compact set $A \subseteq \Omega$ and $f \in \mathcal{H}_\delta$ there exists a constant, $M = M(A,f) < \infty$, such that

$$\sup_{\substack{x \in \Omega \\ y \in A}} G_f(x,y) \leqslant M(A,f) \tag{4.16}$$

Proof. From Section 2 we know that if $A_1 = \text{closure}\{z \in \Omega | B(z) \cap A \neq \varnothing\}$, which is a compact subset of Ω, then

$$\sup_{\substack{z \in A_1 \\ y \in \Omega}} Q_f(z,y) = M_1 < \infty.$$

Therefore, if $x \in \Omega$

$$\begin{aligned}\sum_{n=1}^{\infty} Q_f^{(n)}(x,y) &\leqslant M_1 + \sum_{n=1}^{\infty} \int_{A_1} Q_f^{(n)}(x,z) Q_f(z,y)\,dz \\ &\leqslant M_1\left(1 + \sum_{n=1}^{\infty} Q_f^{(n)}(x,A_1)\right) \\ &\leqslant M_1(1+\Gamma(A_1,f))\end{aligned}$$

by Lemma 4.9.

Remark. In the opposite direction to (4.16) we have from Lemma 2.1 that if A, B are any compact sets in Ω, then

$$\inf_{\substack{x \in A \\ y \in B}} G_f(x,y) > 0. \tag{4.17}$$

LEMMA 4.18 *Assume both $\delta(\cdot)$ and $d(\cdot) - \delta(\cdot)$ are locally bounded away from 0 on Ω. There exists on $\Omega \times \Omega$ a function $\lambda(\cdot,\cdot)$ which is locally bounded away from both 0 and ∞, such that for all $f \in \mathcal{H}_\delta\ (\not\equiv 0)$ and $x, x', y \in \Omega$*

$$\frac{1}{\lambda(x,x')} \leqslant \frac{G_f(x,y)}{G_f(x',y)} \leqslant \lambda(x,x')$$

Proof. Since

$$\frac{G_f(x,y)}{G_f(x',y)} = \frac{f(x')}{f(x)} \frac{G(x,y)}{G(x',y)}$$

it will be enough to prove the lemma assuming $f \equiv 1$. (Local bounds for $\dfrac{f(x')}{f(x)}$ depend on δ, not f.)

By the monotone convergence theorem

$$\int_\Omega P_\delta^{(n)}(x,z) G(z,y)\,dz = G(x,y) - \sum_{k=1}^{n} P_\delta^{(k)}(x,y) \tag{4.19}$$

for all $n \geqslant 1$. If $A, B \subseteq \Omega$ are compact, there exists by Lemma 2.1 an integer n and $\lambda > 0$ such that

$$P_\delta^{(n)}(x,\cdot) \geqslant \lambda P_\delta(x',\cdot), x \in A, x' \in B.$$

Therefore, by (4.19) $G(x,\cdot) \geqslant \lambda(G(x',\cdot) - P_\delta(x',\cdot))$ for $x \in A, x' \in B$. Outside the closure of $\cup_{x' \in B} B(x') = C$, which is compact, we have $P_\delta(x',\cdot) = 0, x' \in B$, and then $G(x,\cdot) \geqslant \lambda G(x',\cdot)$. Define m to be the minimum of λ and the number

$$\lambda' = \inf_{\substack{x \in A \\ x' \in B \\ y \in C}} \frac{G(x,y)}{G(x',y)}$$

which is positive by (4.17). If $x \in A, x' \in B \dfrac{1}{\lambda(x,x')}$ can be m. This gives the desired local uniform bound, and the lemma is proved.

Lemma 4.20. *Assume both $\delta(\cdot)$ and $d(\cdot) - \delta(\cdot)$ are locally bounded away from 0 on Ω. Assume it is known for every $f \in \mathcal{H}_\delta$ and $x \in \Omega$ that*

$$\lim_{n \to \infty} \frac{G_f(x, x_n(\omega))}{G_f(x_0, x_n(\omega))} \tag{4.21}$$

exists for (any fixed x_0 and) μ_x^f almost all $\omega \in \Lambda$. Assume also that the integral of the terms in (4.21) converge to 1. Then if f is an extremal of $\mathcal{H}_\delta$, it is true for all $x \in \Omega$ that for μ_x^f almost all $\omega \in \Lambda$

$$\lim_{n \to \infty} \frac{G(z, x_n(\omega))}{G(x_0, x_n(\omega))} = \frac{f(z)}{f(x_0)} \tag{4.22}$$

holds for every $z \in \Omega$.

Proof. Let $\Lambda_x' \subseteq \Lambda$ be the set of all ω such that (4.21) exists. Then $\Lambda_x' \in \mathfrak{B}_I$ (Section 3), and by assumption $\mu_x^f(\Lambda_x') = 1$. Since $\mu_x^f \prec\ \succ \mu_z^f$ on $\mathfrak{B}_I$ for all $z, \mu_z^f(\Lambda_x') = 1$, $z \in \Omega$. If f is an extremal of $\mathcal{H}_\delta$, then by Lemma 3.3 the limit (4.21) is essentially constant $(\mu_z^f, z \in \Omega)$, and the mutual absolute continuity on $\mathfrak{B}_I$ tells us the constant is independent of z. By Lemma 4.18 the convergence in (4.21) is dominated, and by our assumption on the integrals, the limit has integral 1, meaning the constant above is 1.

Thus far, we have proved for every $x, z \in \Omega$ that

$$\lim_{n \to \infty} \frac{G(z, x_n(\omega))}{G(x_0, x_n(\omega))} = \frac{f(z)}{f(x_0)} \tag{4.23}$$

holds for μ_x^f almost all Ω. By the Fubini theorem it follows that for μ_x^f almost all ω (4.23) holds for almost all $z \in \Omega$. If $z_0 \in \Omega$ is fixed, then the dominated convergence theorem together with the fact, proved below, that $d(x_n(\omega)) \to 0$

a.e. μ_x^f, imply

$$\lim_{n\to\infty}\frac{G(z_0,x_n(\omega))}{G(x_0,x_n(\omega))}=\lim_{n\to\infty}\int_\Omega P_\delta(z_0,z)\frac{G(z,x_n(\omega))}{G(x_0,x_n(\omega))}\,dz$$

$$=\int_\Omega P_\delta(z_0,z)\frac{f(z)}{f(x_0)}\,dz$$

$$=\frac{f(z_0)}{f(x_0)}.$$

Therefore, for μ_x^f almost all $\omega\in\Lambda$, (4.22) holds for all $z\in\Omega$.

LEMMA 4.24. *With notations as above if $f\in\mathcal{H}_\delta$ and $x\in\Omega$, then $\lim_{n\to\infty}d(x_n(\omega))=0$ for μ_x^f almost all $\omega\in\Lambda$.*

Proof. The lemma follows from (4.10) which tells us for any compact set A that $N_A(\omega)<\infty\,\mu_x^f$ a.e.

The following lemma is a straightforward adaptation of a well known technique due to Doob [5], [6]. We follow closely the exposition in Dynkin [9].

LEMMA 4.25. *Assume both $\delta(\cdot)$ and $d(\cdot)-\delta(\cdot)$ are locally bounded away from 0 on Ω. If $f\in\mathcal{H}_\delta$ and $x\in\Omega$, then (4.21) exists for μ_x^f almost all $\omega\in\Lambda$. Moreover,*

$$\lim_{n\to\infty}\int_\Lambda\frac{G_f(x,x_n(\omega))}{G_f(x_0,x_n(\omega))}\,\mu_x^f(d\omega)=1.$$

Proof. Fix $x,x_0\in\Omega$, and let K be a compact subset of Ω which contains $B(x)\cup B(x_0)$. (Later on K will be allowed to increase to Ω.) Since $B(x)\subseteq K$ and $d(x_n)\to 0$ a.e. μ_x^f, the function $T(\omega)=\sup\{n\geqslant 1|x_n(\omega)\in K\}$ is well-defined and finite a.e. μ_x^f.

Next, define $\Lambda_0\in\mathcal{B}$ by $\Lambda_0=\{\omega|x_n(\omega)\not\in K,\ n=1,2,\dots\}$. We define $L(z)=\mu_z^f(\Lambda_0)$. A simple monotone class argument shows $L(\cdot)$ is a Borel function, and using $L(\cdot)$ we compute for $n\geqslant 1$

$$\mu_x^f\{\omega|T(\omega)=n\}=\int_K Q_f^{(n)}(x,z)L(z)\,dz. \tag{4.26}$$

Summing (4.26) over n and using the fact $1\leqslant T<\infty$ a.e. μ_x^f, we have

$$\int_K G_f(x,z)L(z)\,dz=1. \tag{4.27}$$

Note that $B(x_0) \subseteq K$ implies (4.27) with x_0 in place of x.

Define $x_j(\omega) \equiv x, j = -1, -2, \ldots$. Then on $\Lambda_1 = \{\omega | 1 \leqslant T(\omega) < \infty, x_0(\omega) = x\}$ define

$$y_n(\omega) = x_{T(\omega)-n}(\omega) \qquad (n = 0, 1, \ldots).$$

Thus $\{y_j(\cdot)\}$ has been defined a.e. μ_x^f. Given Borel sets $A_0, \ldots, A_k \subseteq \Omega - \{x\}$

$$\begin{aligned}
&\mu_x^f\{\omega | y_j(\omega) \in A_j, 0 \leqslant j \leqslant k\} \\
&= \sum_{i=1}^{\infty} \mu_x^f\{\omega | T(\omega) = k+i \quad \text{and} \quad x_{k+i-j}(\omega) \in A_j, 0 \leqslant j \leqslant k\} \\
&= \sum_{i=1}^{\infty} \int_{A_k} \cdots \int_{A_0} Q_f^{(i)}(x, z_k) \left\{ \prod_{j=0}^{k-1} Q_f(z_{k-j}, z_{k-j-1}) \right\} L(z_0)\, dz_k \ldots dz_0 \\
&= \int_{A_k} \cdots \int_{A_0} G_f(x, z_k) \left\{ \prod_{j=0}^{k-1} Q_f(z_{k-j}, z_{k-j-1}) \right\} L(z_0)\, dz_k \ldots dz_0.
\end{aligned}$$

Notice we have used the fact $x \notin A_j, 0 \leqslant j \leqslant k$, in the conclusion that $T(\omega) > k$ on the set whose measure we are computing. Define a kernel $H_f(u,v)$ by

$$H_f(u,v) = \frac{G_f(x,v)}{G_f(x,u)} Q_f(v,u).$$

Since

$$\begin{aligned}
\prod_{j=0}^{k-1} H_f(z_j, z_{j+1}) &= \prod_{j=0}^{k-1} \frac{G_f(x, z_{j+1})}{G_f(x, z_j)} Q_f(z_{j+1} z_j) \\
&= \frac{G_f(x, z_k)}{G_f(x, z_0)} \prod_{j=0}^{k-1} Q_f(z_{k-j}, z_{k-j-1})
\end{aligned}$$

we see that

$$\mu_x^f\{\omega | y_j(\omega) \in A_j, 0 \leqslant j \leqslant k\} = \int_{A_0} \cdots \int_{A_k} \left\{ \prod_{j=0}^{k-1} H_f(z_j, z_{j+1}) \right\} \times G_f(x, z_0) L(z_0)\, dz_0 \ldots dz_k.$$

Define $R(\cdot)$ on Ω by

$$R(y)=\begin{cases}0 & y=x\\ \dfrac{G_f(x_0,y)}{G_f(x,y)} & y\neq x\end{cases}.$$

We claim $R(y_j)$, $0\leqslant j<\infty$, is a (nonnegative) supermartingale. That is, we claim for all n that $E(R(y_{n+1})|\,y_0,\dots,y_n)\leqslant R(y_n)$. On the set $\{\omega|\,y_n(\omega)=x\}$ $E(R(y_{n+1})|\,y_0,\dots,y_n)$ is 0 for μ_x^f almost all ω. On the set $\{\omega|\,y_n\neq x\}$, the conditional expectation is computed as

$$\begin{aligned}E(R(y_{n+1})|\,y_0,\dots,y_n)&=0+\int H_f(y_n,y_{n+1})R(y_{n+1})\,dy_{n+1}\\&=R(y_n)-\frac{Q_f(x_0,y_n)}{G_f(x,y_n)}\leqslant R(y_n).\end{aligned}$$

For later reference we also compute

$$\begin{aligned}\int_\Lambda R(y_0(\omega))\,\mu_x^f(dw)&=\int_\Omega\frac{G_f(x_0,y_0)}{G_f(x,y_0)}G_f(x,y_0)L(y_0)\,dy_0\\&=\int_\Omega G_f(x_0,y_0)L(y_0)\,dy_0\\&=1.\end{aligned}\tag{4.28}$$

Let a and b be real numbers, $a<b$, and let $u=u_0,u_1,\dots$ be any numerical sequence. Define $D_{a,b}(u)$ to be the supremum of those integers l, if any, such that there exist subscripts $n_1<m_1<n_2<\cdots<n_l<m_l$ with $u_{n_j}>b$ and $u_{m_j}<a, 1\leqslant j\leqslant l$. $D_{a,b}(u)$ is called the number of *downcrossings* by $\{u_j\}$ of the interval (a,b). Given a nonnegative supermartingale $R(y_n)$, the "downcrossing inequality" is ([14], (1.17))

$$\begin{aligned}\int_\Lambda D_{a,b}(\{R(y_n(\omega))\})\,\mu_x^f(d\omega)&\leqslant\frac{1}{b-a}\int_\Lambda R(y_0(\omega))\,\mu_x^f(d\omega)\\&=\frac{1}{b-a}.\end{aligned}$$

Now if K increases to Ω through a sequence of compact sets, then for all ω $D_{a,b}(\{R(y_n(\omega))\})$ increases to $U_{a,b}(\{R(x_n(\omega))\})$ the number of "upcrossings" of (a,b) by $R(x_n(\omega))$, $n=0,1,\dots$. It follows that the number of upcrossings is finite a.e. μ_x^f, and letting a,b vary through all possible rational numbers, it follows that $\lim_{n\to\infty} R(y_n)$ exists a.e. Since

$$\int_{\Lambda} R(x_T)\,\mu_x^f(dw) = 1$$

by (4.28), and since $T \uparrow \infty$ as $K \uparrow \Omega$ the bounded convergence theorem implies the integrals in the statement of the lemma converge to 1 and the lemma is proved.

5. Domination of the Extremals. In this section we prove that if $\delta(\cdot)$ satisfies the hypotheses of Theorem 1.3, then each extremal of $\mathcal{H}_\delta$ is dominated by a positive harmonic function on Ω. As we have already remarked in Sec. 1 this, in conjunction with Theorem 1 of [17], implies Theorem 1.3.

LEMMA 5.1. *Let Ω be a bounded region, and suppose that $r(\cdot), \delta(\cdot)$, and $\alpha > 0$ satisfy the hypotheses of Theorem* 1.3. *There exist a constant $B > 0$ and an integer n such that* (*a*) $P_\delta^{(2)} \geqslant BP_\delta$, *and* (*b*) *if $z \in \Omega$ and $y \in B(z)$, then*

$$P_\delta^{(n)}(y, \cdot) \geqslant BP_\delta(z, \cdot), \tag{5.2}$$

and

$$P_\delta^{(n)}(z, \cdot) \geqslant BP_\delta(y, \cdot). \tag{5.3}$$

Proof. If $u, v \in \Omega$ and $v \in B(u)$, then $\|u - v\| < \delta(u) \leqslant (1-\alpha)r(u)$, and therefore $|r(u) - r(v)| \leqslant \|u - v\| < (1-\alpha)r(u)$. The inequalities $\alpha r(u) \leqslant r(v) \leqslant (2-\alpha)r(u)$ and $\alpha^2 r(u) \leqslant \delta(v) \leqslant (1-\alpha)(2-\alpha)r(u)$ then follow. If $v_1, v_2 \in B(u)$, we define a sequence $y_0, y_1, \ldots$ in $B(u)$, setting $y_0 = v_1$ and letting $y_{n+1} = v_2$ if $\|y_n - v_2\| < \frac{1}{4}\alpha^2 r(u)$ and otherwise letting y_{n+1} be the point closest to v_2 on the sphere of radius $\frac{1}{4}\alpha^2 r(u)$ about y_n. Since $\|v_1 - v_2\| < 2\delta(u) < 2(1-\alpha)r(u)$, it is the case that $y_n = v_2$ whenever $n \geqslant 8(1-\alpha)/\alpha^2$. Define $E_j = \{y \mid \|y - y_j\| < \frac{1}{4}\alpha^2 r(u)\}$, $j = 0, 1, \ldots$. Since $B(y) \supseteq E_{j+1}, y \in E_j$, and

$$\delta(y) \leqslant (1-\alpha)(2-\alpha)r(u) < \frac{4(1-\alpha)(2-\alpha)}{\alpha^2} \times \text{radius } E_{j+1},\ y \in E_j,$$

we have

$$m_1(E_j, E_{j+1}) \geqslant a$$

$j = 0, 1, \ldots$, where $a = \alpha^{2N}\{4(1-\alpha)(2-\alpha)\}^{-N}$ and $m_1(\cdot, \cdot)$ is as in the proof of

Lemma 2.1. If $m_2(\cdot,\cdot)$ is also as in the proof of Lemma 2.1, we have

$$m_2(E_j,E_{j+1}) \geqslant \alpha^N\{(1-\alpha)(2-\alpha)\}^{-N}P_\delta(u,v_2), \quad \text{for} \quad j=n-1 \quad \left(n \geqslant \frac{8(1-\alpha)}{\alpha^2}\right).$$

By (2.2) we have for any pair $y,z \in \Omega$ such that $y \in B(z)$

$$P_\delta^{(n)}(y,\cdot) > bP_\delta(z,\cdot) \tag{5.4}$$

where

$$b = a^{n-1}\alpha^N\{(1-\alpha)(2-\alpha)\}^{-N}, \qquad n \geqslant \frac{8(1-\alpha)}{\alpha^2}.$$

Thus (5.2) is established. To establish (5.3) it is necessary to connect z (with $y \in B(z)$) to points $z' \in B(y)$ by similar sequences of balls. One way to proceed is to first connect z to y and then to connect y to z'. This increases n to $\geqslant 16(1-\alpha)/\alpha^2$ but all other estimates are the same. We omit the details both of this argument and the argument that $P_\delta^{(2)} \geqslant BP_\delta$ (which is carried out in [19]).

In order to apply our lemma fix $y \in \Omega$ and let $A(y) = \{z \mid y \in B(z)\}$. ($A(y)$ is a Borel set.) If n and B are as in Lemma 5.1, then for all $z \in A(y)$

$$\begin{aligned} G(y,\cdot) &\geqslant P_\delta^{(n)}G(y,\cdot) \\ &\geqslant BP_\delta G(z,\cdot) \\ &= B\sum_{n=1}^{\infty} P_\delta^{(n+1)}(z,\cdot) \\ &\geqslant B^2 \sum_{n=1}^{\infty} P_\delta^{(n)}(z,\cdot) \\ &= B^2G(z,\cdot). \end{aligned}$$

Similarly, $G(z,\cdot) \geqslant B^2G(y,\cdot)$, $z \in A(y)$, and we have proved

LEMMA 5.5. *Let Ω be a bounded region, and let $r(\cdot), \delta(\cdot)$, and $\alpha > 0$ satisfy the hypotheses of Theorem 1.3. There exists a number $B = B(\alpha, N) > 0$ such that if $y \in \Omega$ and $z \in A(y)$, then*

$$\frac{1}{B^2}G(z,\cdot) \geqslant G(y,\cdot) \geqslant B^2G(z,\cdot) \tag{5.6}$$

LEMMA 5.7. *Let Ω be a bounded region and suppose $u \in \partial\Omega$ is such that there exists a positive harmonic function h on Ω which vanishes continuously at u. Then for all $y \in \Omega$ $\lim_{x \to u} G(x,y) = 0$.*

Proof. By (4.16) we have for any $y \in \Omega$ that

$$\sup_{x \in \Omega} \frac{G(x,y)}{f(x)} f(y) \leqslant M(\{y\}, f) < \infty.$$

Let u, h be as in the statement of the lemma and set $f = h$ in the above. Then $\lim_{x \to u} h(x) = 0$ implies $\lim_{x \to u} G(x,y) = 0$.

A point $u \in \partial\Omega$ has the property of Lemma 5.7 if and only if u is regular for the Dirichlet problem for Ω ([13], Chapter 8). If $Z \supseteq \partial\Omega$ is the set of *non-regular* points, there exists a ball $S \supseteq \Omega$ and a positive superharmonic function h on S ($h < \infty$ a.e.) such that $h = \infty$ on Z ([13], Theorems 7.3–7.4). Let $Z_0 = \{z \in S | h(z) = \infty\}$. If $x \in \Omega$ is such that $h(x) < \infty$, then $h(x_n(\omega)$, $n = 0, 1, \ldots,$ is a μ_x supermartingale. Therefore $H(\omega) = \lim_n h(x_n(\omega))$ exists a.e. μ_x, and

$$h(x) \geqslant {}_{\Lambda}\!\int H(\omega)\, \mu_x(d\omega). \tag{5.8}$$

Now $Z_k = \{z \in S | h(z) > k\}$ is an open set because h is l.s.c. It follows from this and (5.8) that $\lim_{n \to \infty} x_n(\omega) = x_\infty(\omega)$ (which exists a.e. μ_x) is in Z with probability 0. Therefore if $y \in \Omega$

$$\lim_{n \to \infty} G(x_n(\omega), y) = 0 \tag{5.9}$$

for μ_x almost all $\omega \in \Lambda$. Since (5.9) is a $\mathcal{B}_I$ event, it is true for all μ_x once it is true for one.

LEMMA 5.10. *Assume $r(\cdot), \delta(\cdot)$, and $\alpha > 0$ satisfy the hypotheses of Theorem 1.3. If $x, y \in \Omega$, define $\nu(x,y) = \mu_x\{\omega | x_n(\omega) \in A(y)$, some $n \geqslant 0\}$. Then there exists $B > 0$ (the same as in Lemma 5.5) such that for all $x, y \in \Omega$*

$$\frac{1}{B^2} G(y,y)\nu(x,y) \geqslant G(x,y) \geqslant B^2 G(y,y)\nu(x,y). \tag{5.11}$$

Proof. Define T on Λ by $T(\omega) = \min\{n \geqslant 0 | x_n(\omega) \in A(y)\}$ if any such n exist and $T = \infty$ otherwise. Set $T_k(\omega) = \min(k, T(\omega))$. Then $G(y_k, y)$, $y_k = x_{T_k}$, is a uniformly bounded martingale. Since for almost all ω either $y_k = y_{k+1} \in A(y)$ for large k or else $y_k = x_k \to x_\infty$ the martingale theorem and (5.9) tell us there is a measure, ν_x, on $A(y)$, $\nu_x(A | y) = \nu(x,y)$, such that

$$G(x,y) = \int_{A(y)} G(z,y)\nu_x(dz).$$

This and (5.6) combine to yield (5.11). The lemma is proved.

In preparation of our final estimate fix $r(\cdot), \delta(\cdot)$, and $\alpha > 0$ satisfying the hypotheses of Theorem 1.3. To each $y \in \Omega$ we associate balls U_y, V_y, and W_y centered at y of radii $\frac{1}{10}\alpha^2 r(y), \frac{1}{2}\alpha^2 r(y)$, and $\alpha^2 r(y)$, respectively. There exists an harmonic function, h_y, on $\Omega - U_y, 0 \leqslant h_y \leqslant 1$, which assumes boundary values 1 on ∂U_y and 0 at all regular points of $\partial\Omega$ (this is just the harmonic measure of ∂U_y in $\Omega - U_y$. See [13]).

Define $A_0(y) = \{z | B(z) \cap U_y \neq \varnothing\}$. A_0 is a Borel set and so $T(\omega) = \min\{n \geqslant 0 | x_n(\omega) \in A_0(y)\}$ if any such n exist, otherwise $T(\omega) = \infty$, is $\mathfrak{B}$-measurable. Setting $T_k = \min(k, T)$ the process $h_y(y_k), y_k = x_{T_k}$ is a bounded μ_x martingale for every x. As in the argument for Lemma 5.10, there exists a measure ν_x^0, supported on $A_0(y)$, such that if $x \not\in U_y$

$$h_y(x) = \int_{A_0(y)} h_y(z)\nu_x^0(dz). \tag{5.12}$$

LEMMA 5.13. *With notations and assumptions as above there exists a constant $c = c(\alpha, N) > 0$ such that if $x \not\in U_y$,*

$$c\nu^0(x,y) \leqslant h_y(x) \leqslant \nu^0(x,y) \tag{5.14}$$

where $\nu^0(x,y) = \nu_x(A_0(y))$.

Proof. The upper bound is obvious from (5.12). Since h_y dominates, on $W_y - U_y$, the harmonic measure of ∂U_y (in $W_y - U_y$), there is a constant $c_0 = c_0(\alpha, N) > 0$ such that $h_y \geqslant c_0$ on $V_y - U_y$. If $z \not\in V_y - U_y$, but $B(z) \cap U_y \neq \phi$, and if δ_0 denotes the distance from z to ∂V_y, we claim $\delta_0/\delta(z)$ is bounded away from 1 by a constant depending only upon α. Once this has been established, then Harnack's inequality and the fact that $h_y \geqslant c_0$ on ∂V_y will imply there exists a constant $c_1 = c_1(\alpha, N) > 0$ such that $h_y(z) \geqslant c_1$, and we may set $c = \min(c_0, c_1)$. To obtain the upper bound on $\delta_0/\delta(z)$, notice that $\delta_0 \leqslant \delta(z) - \frac{2}{5}\alpha^2 r(y)$ or $\delta_0/\delta(z) \leqslant 1 - \frac{2}{5}\alpha^2(r(y)/\delta(z))$. Now since $\|z - y\| \leqslant \delta(z) + \frac{1}{10}\alpha^2 r(y) \leqslant (1-\alpha)r(z) + \frac{1}{10}\alpha^2 r(y)$, we have $r(z) \leqslant (1/\alpha)(1 + \alpha^2/10)r(y)$, and therefore $r(y)/\delta(z) \geqslant \alpha(1 + \alpha^2/10)^{-1}$. This gives the desired upper bound on $\delta_0/\delta(z)$ and completes the proof of the lemma.

With notations and assumptions as above we notice that if $z \in A_0(y)$, then $P_\delta(z, V_y) \geqslant c_2 = c_2(\alpha, N) > 0$. We claim $V_y \subseteq A(y)$. For if $w \in V_y$, then $r(w) > (1 - \frac{1}{2}\alpha^2)r(y)$ and $\delta(w) > \alpha(1 - \frac{1}{2}\alpha^2)r(y) > (\alpha^2/2)r(y)$. Thus $y \in B(w)$. We conclude from these observations and the strong Markov property that $\nu(x,y) \geqslant c_2\nu^0(x,y), x \not\in U_y$. Putting this together with (5.14) and (5.11) we have proved

LEMMA 5.15. *There exists a constant* $a = a(\alpha, N) > 0$ *such that if* $x, y \in \Omega$ *and* $x \notin U_y$, *then*

$$\frac{1}{a} G(y,y) h_y(x) \geqslant G(x,y) \geqslant aG(y,y) h_y(x) \tag{5.16}$$

where $U_y = \{z \mid \|z - y\| < \frac{1}{10}\alpha^2 r(y)\}$ *and* h_y *is the harmonic measure of* ∂U_y *in* $\Omega - U_y$.

We apply (5.16) *for* $x, x_0 \notin U_y$ *to conclude*

$$\frac{G(x,y)}{G(x_0,y)} \leqslant \frac{1}{a^2} \frac{h_y(x)}{h_y(x_0)}. \tag{5.17}$$

PROPOSITION 5.18. *Let* Ω *be a bounded region in* $\mathbf{R}^N$, *and let* $r(\cdot), \delta(\cdot)$, *and* $\alpha > 0$ *satisfy the hypotheses of Theorem* 1.3. *If* f *is an extremal element of* $\mathcal{H}_\delta$, *there exists an harmonic function,* h, *on* Ω *such that* $f \leqslant h$.

Proof. The hypotheses imply both $\delta(\cdot)$ and $f(\cdot) - \delta(\cdot)$ are locally bounded away from 0 on Ω. Therefore by Lemmas 4.20 and 4.25 there exists for any extremal $f \in \mathcal{H}_\delta$ a sequence $\{y_n\}$ in Ω such that $\lim_{n\to\infty} d(y_n) = 0$ and for all $x \in \Omega$

$$\lim_{n\to\infty} \frac{G(x,y_n)}{G(x_0,y_n)} = \frac{f(x)}{f(x_0)}. \tag{5.19}$$

Define h_y as above and let $g_y(x) = (h_y(x)/h_y(x_0))$. By Harnack's principle and the fact $d(y_n) \to 0$, we may assume $h_n = g_{y_n}$ is locally uniformly convergent to some $h \in \mathcal{H}$. Then by (5.17) $f(x) \leqslant (f(x_0)/a^2) h(x), x \in \Omega$. The proposition is proved.

With Proposition 5.18 we have come to the end of the proof of Theorem 1.3, since all that is left is to quote Theorem 1 of [17]. (Sec. 1.)

6. Remark on the Poisson Kernel. If Ω is a bounded Lipschitz domain, then the results of R. Hunt and Wheeden [15] allow us to glean more information from (5.17). For if $y_0 \in \partial\Omega$ is fixed, their results imply that

$$\lim_{y\to y_0} \frac{h_y(x)}{h_y(x_0)} = K_{x_0}(x, y_0) \tag{6.1}$$

exists for all x, and moreover K_{x_0} has these properties: (1) $K_{x_0}(\cdot,\cdot)$ is continuous on $\Omega\times\partial\Omega$, (2) $K_{x_0}(\cdot,y_0)\in\mathcal{H}$ vanishes continuously at every $y'\neq y_0$ in $\partial\Omega(K_{x_0}(\cdot,y_0)$ is a "kernel function" at y_0), and (3) $K_{x_0}(\cdot,y_0)$ is the unique kernel function at y_0 which is 1 at x_0. Now (5.17) and (6.1) tell us that if $f\in\mathcal{H}_\delta$ has the form

$$\lim_{n\to\infty}\frac{G(x,y_n)}{G(x_0,y_n)}=\frac{f(x)}{f(x_0)}$$

and if $y_0\in\partial\Omega$ is a cluster point of $\{y_n\}$, then $f(x)\leqslant(f(x_0)/a^2)K_{x_0}(x,y_0)$. Since f is harmonic, f is therefore a kernel function at y_0. By (3) above $f(x)=f(x_0)$ $K_{x_0}(x,y_0)$. We have proved

THEOREM 1.4. *Let $\Omega\subseteq\mathbf{R}^N$ be a bounded Lipschitz domain, and let δ be any Borel function on Ω which satisfies the hypotheses of Theorem 1.3. If $G(x,y)=\sum_{n=1}^{\infty}P_\delta^{(n)}(x,y)$, then for every $(x,y_0)\in\Omega\times\partial\Omega$*

$$\lim_{y\to y_0}\frac{G(x,y)}{G(x_0,y)}=K_{x_0}(x,y_0). \tag{6.3}$$

Theorem 6.2 enables one in principle to compute the Poisson kernel for a bounded Lipschitz domain using the elementary kernels $P_\delta^{(n)}$. If Conjecture 1.2 is true, then (6.3) probably is also true provided we replace $\partial\Omega$ by the Martin boundary of Ω.

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