Lewis-Riesenfeld approach to the solutions of the Schrödinger equation in the presence of a time-dependent linear potential

Pi-Gang Luan[1] and Chi-Shung Tang[2]
[1]*Institute of Optical Sciences, National Central University, Chung-Li 32054, Taiwan, Republic of China*
[2]*Physics Division, National Center for Theoretical Sciences, P.O. Box 2-131, Hsinchu 30013, Taiwan, Republic of China*
(Received 23 September 2003; published 5 January 2005)

We reexamine the general solution of a Schrödinger equation in the presence of a time-dependent linear potential in configuration space based on the Lewis-Riesenfeld framework. For comparison, we also solve the problem in momentum space and then Fourier transform the solution to get the general wave function. Appropriately choosing the weight function in the latter method, we can obtain the same wave function as the former method. It is found that a non-Hermitian time-dependent linear invariant can be used to obtain Gaussian-type wave-packet solutions of the time-dependent system. This operator is a specific linear combination of the initial momentum and initial position operators. This fact indicates that the constants of integration such as the initial position and initial momentum that determine the classical motion play important roles in the time-dependent quantum system. The eigenfunction of the linear invariant is interpreted as a wave packet with a "center of mass" moving along the classical trajectory, while the ratio between the coefficients of the initial position and initial momentum determines the width of the wave packet.

DOI: 10.1103/PhysRevA.71.014101 PACS number(s): 03.65.Ge, 03.65.Fd

INTRODUCTION

The study of time-dependent systems has been a growing field not only for its fundamental physical perspective but also for its applicability, such as quantum transport [1–3], quantum optics [4–6], quantum information [7,8], and spintronics [9–11]. Recently, the analytical solutions of the one-dimensional Schrödinger equation with a time-dependent linear potential have attracted the attention of physicists [12–15]. First, Guedes found the wave function for a Schrödinger equation with a time-dependent linear potential [12], using the Lewis-Riesenfeld (LR) method [16,17]. Later on, Feng [13] found the plane-wave-type and the Airy-packet solutions using a space-time transformation method. However, Bekkar *et al.* pointed out that the Airy-packet solution is in fact only a superposition of the plane-wave-type solution [14]. Moreover, Bauer [15] showed that the solution proposed by Ref. [12] is only a special case of the so-called Volkov solution with a zero wave vector k. He also showed how to do the gauge transformation appropriately to the time-dependent Schrödinger equation to get the different expressions of the Hamiltonian and the corresponding Volkov solution.

Besides the solutions described above, we shall show that the linear invariant can be *non-Hermitian*. As such, a Gaussian-type wave-packet (GWP) solution is then derived naturally based on the LR approach. This solution was ruled out in previous studies because the linear LR invariant $\hat{I}(t)$ as a *Hermitian* operator had been assumed [12–15]. Although in Ref. [14] the authors pointed out the incorrectness of setting $B=0$ in Ref. [12], the Hermiticity assumption still led them to the conclusion of $B=0$, which they referred to as "a constraint that must be taken to get physical solutions." This assumption, however, is actually unnecessary.

The main results of this Brief Report are as follows. First, we show that the GWP solution is derived using a non-Hermitian linear LR invariant. Second, we solve the Schrödinger equation in momentum space and then transform back to the coordinate space to obtain a general wave-function solution. Third, we present a physical interpretation to the LR invariant for the realization of constructing an invariant and the selection of an appropriate space-time transformation to find the general solution.

The Schrödinger equation for describing the motion of a particle in the presence of a time-dependent linear potential is of the form

$$i\hbar\frac{\partial\psi}{\partial t}=\hat{H}\psi, \tag{1}$$

where the Hamiltonian $\hat{H}(t)$ is induced by an external time-dependent driving force $F(t)$, given by

$$\hat{H}(t)=\frac{\hat{p}^2}{2m}-F(t)\hat{x}. \tag{2}$$

This time-dependent dynamical problem could be solved in either configuration or momentum space.

CONFIGURATION SPACE

To utilize the LR method [17] solving the time-dependent system, one should first find an operator $\hat{I}(t)$ such that

$$i\hbar\frac{d\hat{I}}{dt}=i\hbar\frac{\partial\hat{I}}{\partial t}+[\hat{I},\hat{H}]=0, \tag{3}$$

and then find its eigenfunction $\varphi_\lambda(x,t)$ satisfying

$$\hat{I}(t)\varphi_\lambda(x,t)=\lambda\varphi_\lambda(x,t), \tag{4}$$

with λ being the corresponding eigenvalue. The general wave function $\psi(x,t)$ is then written as

$$\psi(x,t) = \int d\lambda g(\lambda)\psi_\lambda(x,t), \tag{5}$$

where $g(\lambda)$ is a weight function for λ.

The wave function $\psi_\lambda(x,t)$ in Eq. (5) is related to $\varphi_\lambda(x,t)$,

$$\psi_\lambda(x,t) = e^{i\alpha_\lambda(t)}\varphi_\lambda(x,t), \tag{6}$$

where $\alpha(t)$ is a function of time only, satisfying

$$\dot{\alpha}_\lambda = \varphi_\lambda^{-1}(i\partial/\partial t - \hat{H}/\hbar)\varphi_\lambda. \tag{7}$$

We note that the integration in Eq. (5) includes, in general, all possible degeneracies of λ. It turns out that the time-dependent invariant operator $\hat{I}(t)$ takes the linear form [12]

$$\hat{I}(t) = A(t)\hat{p} + B(t)\hat{x} + C(t), \tag{8}$$

in which $A(t)$, $B(t)$, and $C(t)$ are time-dependent c-number functions to be determined.

The operators describing the equations of motion are given by

$$\frac{d\hat{x}}{dt} = \frac{[\hat{x},\hat{H}]}{i\hbar} = \frac{\hat{p}}{m} \tag{9}$$

and

$$\frac{d\hat{p}}{dt} = \frac{[\hat{p},\hat{H}]}{i\hbar} = F(t). \tag{10}$$

By solving the above two equations, the space and momentum operators can be obtained in terms of initial conditions, given by

$$\hat{x}(t) = \hat{x}(0) + \frac{\hat{p}(0)t + G_1(t)}{m} \tag{11}$$

and

$$\hat{p}(t) = \hat{p}(0) + G(t), \tag{12}$$

where $G_1(t)$ and $G(t)$ are defined, respectively, as

$$G_1(t) \equiv \int_0^t G(\tau)d\tau \tag{13}$$

and

$$G(t) \equiv \int_0^t F(\tau)d\tau. \tag{14}$$

Substituting Eqs. (8)–(10) into Eq. (3), and solving these operator equations, we get

$$A(t) = A_0 - \frac{B_0}{m}t, \quad B(t) = B_0, \tag{15}$$

$$\begin{aligned} C(t) &= C_0 - A_0\int_0^t F(\tau)d\tau + \frac{B_0}{m}\int_0^t F(\tau)\tau d\tau \\ &= C_0 - A(t)G(t) - \frac{B_0}{m}G_1(t), \end{aligned} \tag{16}$$

where A_0, B_0, and C_0 are arbitrary complex constants. Furthermore, substituting Eqs. (11)–(16) into Eq. (8), we find

$$\hat{I}(t) = A_0\hat{p}(0) + B_0\hat{x}(0) + C_0 = \hat{I}(0). \tag{17}$$

In other words, the invariant $\hat{I}$ is precisely the linear combination of the initial momentum $\hat{p}(0)$ and the initial position $\hat{x}(0)$ with an arbitrary constant C_0.

For the convenience of later discussion, we define $x_c(t)$ and $p_c(t)$ as the expectation value of $\hat{x}(t)$ and $\hat{p}(t)$ with respect to the wave function $\psi_\lambda(x,t)$, i.e.,

$$x_c(t) \equiv \langle\hat{x}(t)\rangle_\lambda = x_0 + \frac{p_0 t + G_1(t)}{m}, \tag{18}$$

$$p_c(t) \equiv \langle\hat{p}(t)\rangle_\lambda = p_0 + G(t), \tag{19}$$

where $x_0 = x_c(0)$ and $p_0 = p_c(0)$ are, respectively, the initial position and initial momentum of the corresponding classical problem.

To find a solution of Eq. (1), we have to solve Eq. (4) first. Note that in Eq. (6) the phase factor $e^{i\alpha(t)}$ is a function of time only, thus $\psi_\lambda(x,t)$ is also an eigenfunction of $\hat{I}$ with the same eigenvalue λ. It turns out that

$$\lambda = A(t)p_c(t) + B(t)x_c(t) + C(t) = A_0 p_0 + B_0 x_0 + C_0. \tag{20}$$

By solving Eq. (4), after some algebra, we find

$$\varphi_\lambda(x,t) = \exp\left\{\frac{i}{\hbar}\left[\frac{2[\lambda - C(t)]x - B_0x^2}{2A(t)}\right]\right\}. \tag{21}$$

Substituting Eq. (21) into Eq. (7), we obtain

$$\alpha_\lambda(t) = \alpha_\lambda(0) - \int_0^t \left[\frac{[\lambda - C(\tau)]^2 + i\hbar B_0 A(\tau)}{2m\hbar A^2(\tau)}\right]d\tau. \tag{22}$$

Here we see that in general $\alpha(t)$ is a complex function.

Using the identities

$$\begin{aligned} \frac{\lambda - C(t)}{A(t)} &= p_c(t) + \frac{B_0}{A(t)}x_c(t), \\ \frac{d}{dt}[B_0/A(t)] &= B_0^2/[mA^2(t)], \\ \frac{d}{dt}x_c^2(t) &= \frac{2}{m}p_c(t)x_c(t), \end{aligned} \tag{23}$$

we finally obtain the general expression of the wave-packet solution

$$\psi_\lambda(x,t)=\frac{e^{i\alpha_\lambda(0)}}{\sqrt{A(t)/A_0}}\exp\left[-\frac{i}{\hbar}\int_0^t\frac{p_c^2(\tau)}{2m}d\tau\right]\times\exp\left[-\frac{iB_0[x-x_c(t)]^2}{2\hbar A(t)}+\frac{i}{\hbar}p_c(t)x\right]. \quad (24)$$

It is important to note that the probability density is of the form

$$|\psi_\lambda(x,t)|^2=\frac{e^{-2\ \mathrm{Im}[\alpha_\lambda(0)]+\mathrm{Im}(\mathcal{F}_0)\{[x-x_c(t)]^2/\hbar|A(t)/A_0|^2\}}}{|A(t)/A_0|}, \quad (25)$$

where the factor $\mathcal{F}_0$ is defined by $\mathcal{F}_0 \equiv B_0/A_0$. It is crucial to note that $\mathcal{F}_0$ must satisfy

$$\mathrm{Im}(\mathcal{F}_0)\leqslant 0 \quad (26)$$

to ensure Eq. (24) really provides a physically acceptable solution. This is a key result of this Brief Report.

In the case in which $\mathrm{Im}(\mathcal{F}_0)<0$, the ψ_λ obtained in Eq. (24) describes exactly a GWP with position uncertainty

$$\Delta x=\sqrt{\frac{\hbar}{2}}\left(\frac{|A(t)/A_0|}{\sqrt{-\mathrm{Im}(\mathcal{F}_0)}}\right), \quad (27)$$

and momentum uncertainty

$$\Delta p=\sqrt{\frac{\hbar}{2}}\left(\frac{|\mathcal{F}_0|}{\sqrt{-\mathrm{Im}(\mathcal{F}_0)}}\right), \quad (28)$$

which leads to the uncertainty relation

$$\Delta x\Delta p=\frac{\hbar}{2}\left(\frac{\left|\mathcal{F}_0\left(1-\mathcal{F}_0\frac{t}{m}\right)\right|}{-\mathrm{Im}(\mathcal{F}_0)}\right)\geqslant\frac{\hbar}{2}. \quad (29)$$

The equality holds at $t=\mathrm{Re}(m/\mathcal{F}_0)$. That is, at that time the position uncertainty of the particle goes to the minimum.

We can see that, in general, B_0 (or $\mathcal{F}_0$) is not zero, although it is taken to be zero in Refs. [12–15]. In the case in which $\mathrm{Im}(\mathcal{F}_0)=0$, the result obtained in Eq. (25) enforces $\mathcal{F}_0=0$. This means that the solution becomes plane-wave-like rather than GWP, otherwise $|\psi_\lambda|^2$ becomes divergent at the moment $t=m/\mathcal{F}_0$.

MOMENTUM SPACE

We now turn to solve the problem in the momentum space by denoting the wave function as $\phi(p,t)$ and using the substitution $x\rightarrow i\hbar\partial/\partial p$. We thus have

$$i\hbar\left(\frac{\partial}{\partial t}+F\frac{\partial}{\partial p}\right)\phi=\frac{p^2}{2m}\phi. \quad (30)$$

By changing the momentum and time variables from (p,t) to (p',t'),

$$p'\equiv p-G(t),\quad t'\equiv t, \quad (31)$$

the corresponding differential operators can be transformed to the form

$$\frac{\partial}{\partial p}=\frac{\partial}{\partial p'},\quad \frac{\partial}{\partial t}=\frac{\partial}{\partial t'}-F(t)\frac{\partial}{\partial p'}. \quad (32)$$

It turns out that Eq. (30) becomes

$$i\hbar\frac{\partial\phi}{\partial t'}=\frac{[p'+G(t')]^2}{2m}\phi, \quad (33)$$

which yields

$$\phi(p',t')=\phi_0(p')\exp\left\{-\frac{i}{\hbar}\int_0^{t'}\frac{[p'+G(\tau)]^2}{2m}d\tau\right\}, \quad (34)$$

or equivalently

$$\phi(p,t)=\phi_0[p-G(t)]\exp\left\{-\frac{i}{\hbar}\int_0^{t}\frac{[p-G(t)+G(\tau)]^2}{2m}d\tau\right\}. \quad (35)$$

Here ϕ_0 is an arbitrary single-variable function.

The general solution of the wave function $\psi(x,t)$ can be obtained by the Fourier transform,

$$\psi(x,t)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\phi(p,t)e^{ipx/\hbar}dp,$$
$$=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\phi_0(p')\exp\left\{\frac{i}{\hbar}[p'+G(t)]x-\frac{i}{\hbar}\int_0^t\frac{[p'+G(\tau)]^2}{2m}d\tau\right\}dp'. \quad (36)$$

COMPARISON

We now show that the GWP solution Eq. (24) obtained using the LR method can also be obtained from the general solution (36). We consider the time-dependent wave function $\phi_0(p')$ in momentum space, given by

$$\phi_0(p')=\left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{1/4}\exp\left[-\frac{\sigma^2(p'-p_0)^2}{\hbar^2}-i\frac{(p'-p_0)x_0}{\hbar}\right]. \quad (37)$$

After some algebra, we obtain the space-time wave function, given explicitly as

$$\phi(p,t)=\left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{1/4}\exp\left[-\frac{i}{\hbar}\int_0^t\frac{p_c^2(\tau)}{2m}d\tau\right]\times\exp\left[-\frac{\sigma^2(1+it/T)}{\hbar^2}[p-p_c(t)]^2\right]\times\exp\left[-\frac{i}{\hbar}[p-p_c(t)]x_c(t)\right]. \quad (38)$$

Here the parameter

$$T\equiv\frac{2m\sigma^2}{\hbar} \quad (39)$$

indicates a measure of the *spreading time* of the GWP.

Substituting Eqs. (37) and (38) into Eq. (36) and accomplishing the integration, we get

$$\psi(x,t) = \frac{1}{(2\pi)^{1/4}\sqrt{\sigma(1+it/T)}}\exp\left[-\frac{i}{\hbar}\int_0^t \frac{p_c^2(\tau)}{2m}d\tau\right]\times\exp\left[-\frac{[x-x_c(t)]^2}{4\sigma^2(1+it/T)}+\frac{i}{\hbar}p_c(t)x\right]. \tag{40}$$

By suitably choosing the initial condition parameters as

$$e^{i\alpha_\lambda(0)} = \frac{1}{(2\pi\sigma^2)^{1/4}}, \quad \frac{B_0}{A_0} = -\frac{im}{T}, \tag{41}$$

we can see that this wave function is exactly of the form given by Eq. (24).

SUMMARY

In this Brief Report, we have studied the Schrödinger equation with a time-dependent linear potential. We reexamine the linear invariant proposed by Guedes [12]. We have shown that if we assume this opterator to be a non-Hermitian one, then a GWP solution can be obtained. This GWP has a "center of mass" moving along the trajectory of the corresponding classical particle. The trajectory is determined by the classical initial position x_0 and initial momentum p_0. In the corresponding quantum problem, $\hat{x}(0)$ does not commute with $\hat{p}(0)$, thus the particle described by a wave function proportional to the eigenfunction of the linear invariant operator $\hat{I}=A_0\hat{p}(0)+B_0\hat{x}(0)+C_0$ acquires position and momentum uncertainties, as given by Eqs. (27) and (28), respectively. The size of the uncertainties is determined by the factor $\mathcal{F}_0=B_0/A_0$.

On the other hand, we have investigated the time-dependent system in momentum space. After performing a transformation of variables, the problem becomes exactly solvable. Moreover, we have presented a specific example for comparison between these two approaches. Our analysis has shown that the key to solving the time-dependent Schrödinger equation is to find a way to transform the problem to a standard form. For a linear time-dependent case, the standard form is simply a free-particle problem. It is interesting to note that if we treat the driving force as a time-dependent gravity, then an observer in the "free-fall frame" will not be able to feel the gravity. As a result, the frame effectively becomes an inertial frame. This provides a physical picture for the transformation Eq. (31) we have performed.

ACKNOWLEDGMENTS

We would like to thank D.H. Lin and Y.M. Kao for discussion of the results. This work was partly supported by the National Science Council, by the National Center for Theoretical Sciences, and by the National Central University in Taiwan.

[1] G. Burmeister and K. Maschke, Phys. Rev. B **57**, 13050 (1998).
[2] C. S. Tang and C. S. Chu, Phys. Rev. B **60**, 1830 (1999).
[3] W. Li and L. E. Riechl, Phys. Rev. B **62**, 8269 (2000).
[4] C. Figueira de M. Faria, M. Dorr, and W. Sandner, Phys. Rev. A **55**, 3961 (1996).
[5] H. Zeng, Phys. Rev. A **57**, 388 (1997).
[6] H. Maeda and T. F. Gallagher, Phys. Rev. Lett. **92**, 133004 (2004).
[7] For a general overview see, e.g., M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
[8] C. E. Creffield and G. Platero, Phys. Rev. B **65**, 113304 (2002).
[9] M. Governale, F. Taddei, and R. Fazio, Phys. Rev. B **68**, 155324 (2003).
[10] A. G. Mal'shukov, C. S. Tang, C. S. Chu, and K. A. Chao, Phys. Rev. B **68**, 233307 (2003).
[11] E. I. Rashba, e-print cond-mat/0408119.
[12] I. Guedes, Phys. Rev. A **63**, 034102 (2001).
[13] M. Feng, Phys. Rev. A **64**, 034101 (2002).
[14] H. Bekkar, F. Benamira, and M. Maamache, Phys. Rev. A **68**, 016101 (2003); I. Guedes, *ibid.* **68**, 016102 (2003).
[15] J. Bauer, Phys. Rev. A **65**, 036101 (2002).
[16] H. R. Lewis, Jr., Phys. Rev. Lett. **18**, 510 (1967); J. Math. Phys. **9**, 1967 (1968).
[17] H. R. Lewis, Jr. and W. B. Reisenfeld, J. Math. Phys. **10**, 1458 (1969).