Ballistic Annihilation with Continuous Isotropic Initial Velocity Distribution

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Ballistic annihilation with continuous initial velocity distributions is investigated in the framework of the Boltzmann equation. The particle density and the rms velocity decay as $c \sim t^{-\alpha}$ and $\bar{v} \sim t^{-\beta}$, with the exponents depending on the initial velocity distribution and the spatial dimension d. For instance, in one dimension for the uniform initial velocity distribution $\beta = 0.230\,472\ldots$. In the opposite extreme $d \to \infty$, the dynamics is universal and $\beta \to (1 - 2^{-1/2})d^{-1}$. We also solve the Boltzmann equation for Maxwell particles and very hard particles in arbitrary spatial dimension. These solvable cases provide bounds for the decay exponents of the hard sphere gas.

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Ballistic annihilation (BA) is the kinetic process which involves particles undergoing ballistic motion and annihilating upon colliding. Ballistic annihilation underlies numerous apparently unrelated phenomena, e.g., growth and coarsening processes and traffic flows [1–4]. In coarsening processes, for instance, domain walls have a natural particle interpretation. Yet little is known on irreversible processes where the reactants move ballistically while the contrasting situation of diffusion-controlled processes is well understood [5].

There exist few theoretical results on BA in one dimension with *discrete* velocity distributions. For the simplest binary velocity distribution, the BA process has been solved by Elskens and Frisch [6] (see also Refs. [1,7–9]). Some analytical results are also available for the ternary velocity distribution [10,11]. No solutions have been found for *continuous* initial velocity distributions, although the decay exponents have been determined numerically [12,13]. This lack of analytical results is especially striking given that one-dimensional ballistic aggregation processes admit exact solutions for arbitrary initial velocity distribution [14,15].

In this work, we consider BA with continuous isotropic initial velocity distributions in arbitrary dimension. Our analysis is performed in the framework of the Boltzmann equation approach which is *a priori* an uncontrolled approximation. However, the decay exponents obtained by this approach are in excellent agreement with simulation results [12,13]. Additionally, closely related ballistic aggregation processes [14,16] are solvable in one dimension [14,15], and these exact results also support the conjecture that the Boltzmann equation approach gives exact decay exponents for ballistically controlled processes with continuous velocity distributions.

For clarity, we start with the one-dimensional BA process. Initially, the system consists of identical point particles whose velocities are distributed according to a distribution $P(v, t = 0)$ and whose initial positions are uncorrelated. Particles move according to their initial velocities and every collision leads to the removal of both colliding particles. In the framework of the Boltzmann equation, the velocity distribution $P(v,t)$ obeys [12]

$$\frac{\partial P(v,t)}{\partial t} = -P(v,t)\int_{-\infty}^{\infty} dv'\,|v - v'|P(v',t). \quad (1)$$

In the long time limit, the velocity distribution approaches a scaling form

$$P(v,t) = t^{\beta-\alpha}F(x), \quad \text{with } x = vt^{\beta}. \quad (2)$$

The exponents α and β determine the decay of the particle density c and the rms velocity $\bar{v} = \sqrt{\langle v^2\rangle}$:

$$c \sim t^{-\alpha}, \qquad \bar{v} \sim t^{-\beta} \quad \text{as } t \to \infty. \quad (3)$$

Formally, c and $\langle v^2\rangle$ are defined via

$$c(t) = \int_{-\infty}^{\infty} dv\,P(v,t) = t^{-\alpha}\int_{-\infty}^{\infty} dx\,F(x), \quad (4)$$

and

$$\langle v^2\rangle = c^{-1}\int_{-\infty}^{\infty} dv\,v^2P(v,t) = t^{-2\beta}\,\frac{\int_{-\infty}^{\infty} dx\,x^2F(x)}{\int_{-\infty}^{\infty} dx\,F(x)}. \quad (5)$$

By inserting the scaling form (2) into Eq. (1) one finds $\alpha + \beta = 1$ and an equation for the scaling function

$$2\beta - 1 + \beta x\,\frac{F'(x)}{F(x)} = -\int_{-\infty}^{\infty} dx'\,|x - x'|F(x'). \quad (6)$$

In the following, we always consider isotropic initial velocity distributions. In one dimension, this requirement reads $P_0(v) = P_0(-v)$ and it implies the symmetry for later times $P(v,t) = P(-v,t)$ and the symmetry of the scaling function $F(x) = F(-x)$.

We should solve Eq. (6) subject to the boundary condition at $x = 0$ which is implied by the initial velocity distribution. For instance, $F'(0) = 0$ if the initial velocity distribution is flat near the origin. This problem has no solution except for a special value of β which is called an eigenvalue. Finding such an eigenvalue numerically gives $\beta_{\rm flat} = 0.230\,472\ldots$, to be compared with $\beta_{\rm flat} \approx 0.22$ [12] and $\beta_{\rm flat} \approx 0.19$ [13] found from Monte Carlo

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simulations. Similarly, for the initial velocity distribution satisfying $P_0(v) \sim |v|$ we find $\beta_{\rm linear} = 0.166\,649\ldots$.

In Fig. 1 we plot $F(x)$ for the flat initial velocity distribution. The exponential behavior of $F(x)$ for large x is confirmed analytically by simplifying the integral on the right-hand side of Eq. (6) to Cx, where $C = \int_{-\infty}^{\infty} dx\, F(x)$, and then solving the resulting differential equation to give

$$F(x) \sim x^{(1-2\beta)/\beta} e^{-Cx/\beta} \quad \text{when } x \to \infty. \tag{7}$$

We now outline an approximate computation of the exponent β. First we note that Eq. (6) reduces to an ordinary differential equation after a double differentiation. This equation is further simplified by the transformation $F(x) = F_*(x_*)$, $x = x_*\sqrt{\beta}$ that eliminates the β factor. The governing equation then reads

$$xf''' + 2f'' = 2\exp(-f), \tag{8}$$

where $f(x) = -\ln F(x)$ and the subscripts are omitted. We also use the relation

$$\int_0^\infty dx\, xF(x) = (2\beta)^{-1} - 1, \tag{9}$$

which plays the role of a normalization condition. Equation (9) is just Eq. (6) at $x = 0$.

Although Eq. (8) cannot be solved exactly, an approximate solution f_1 can be found by replacing the right-hand side by e^{-f_0}, where f_0 is a reasonable approximation for f. In principle, this approximation scheme can be repeated again starting from f_1 and should finally lead to the exact form for f. As the starting point, we choose

$$F_0(x) = e^{-f_0(x)} = \left(1 + \frac{x}{\nu} + \frac{1-\nu^2}{2\nu^2}x^2\right)e^{-x/\nu}. \tag{10}$$

The function $F_0(x)$ is constructed in such a way that the small x expansion is correct up to the second order. Additionally, $F_0(x)$ exhibits an exponential decay for large x, in agreement with the exact asymptotic behavior (7). The parameter ν is yet to be determined. Replacing now the right-hand side of Eq. (8) by $F_0(x)$ from Eq. (10) and solving the resulting *linear* differential equation gives

$$\begin{aligned} f_1(x) = {} & 2\nu(3-\nu^2)x + \nu^2(7-5\nu^2)(1-e^{-x/\nu}) \\ & - \nu(1-\nu^2)xe^{-x/\nu} \\ & - 6\nu^2(2-\nu^2)\int_0^{x/\nu} d\xi\, \frac{1-e^{-\xi}}{\xi}. \end{aligned} \tag{11}$$

The constant ν can now be calculated self-consistently by imposing the constraint

$$\nu^{-1} = 2\int_0^\infty F(x)\,dx, \tag{12}$$

which can be obtained by integrating Eq. (8). Plugging $F_{\rm appr} = e^{-f_1(x)}$ into Eq. (12) gives $\nu_{\rm appr}^{-1} \cong 2.671\,56$ to be compared to the exact numerical value $\nu^{-1} = 2.658\,26\ldots$. Finally, Eq. (9) leads to $\beta_{\rm appr} \cong 0.228\,98$, in good agreement with the exact value obtained above. The approximate scaled velocity distribution $F_{\rm appr} = e^{-f_1(x)}$ is extremely good as the relative error between f and f_1 is always less than 1.5%, and $|F(x) - F_{\rm appr}(x)| < 0.0036$.

In higher dimensions, the Boltzmann equation reads

$$\frac{\partial P(\mathbf{v},t)}{\partial t} = -P(\mathbf{v},t)\int d\mathbf{w}\,|\mathbf{v}-\mathbf{w}|P(\mathbf{w},t). \tag{13}$$

For isotropic initial velocity distributions, the appropriate scaling variable is $x = vt^\beta$ with $v = |\mathbf{v}|$, and the scaling form is $P(\mathbf{v},t) = t^{d\beta-\alpha}F(x)$. Plugging this scaling form into Eq. (13) and using the spherical coordinates to simplify the collision integral we obtain

$$\begin{aligned} (d+1)\beta - 1 + \beta x\frac{F'(x)}{F(x)} &= \\ & - \Omega_{d-1}\int_0^\infty dy\, y^{d-1}E_d(x,y)F(y). \end{aligned} \tag{14}$$

Here Ω_{d-1} is the surface area of the unit sphere in $d-1$ dimensions and

$$E_d(x,y) = \int_0^\pi d\phi\,(\sin\phi)^{d-2}\sqrt{x^2+y^2-2xy\cos\phi}. \tag{15}$$

In two dimensions, $E_2(x,y) = 2(x+y)E(k)$, where $E(k)$ is the complete elliptic integral of the second kind with modulus $k = \frac{2\sqrt{xy}}{x+y}$. Similarly, in an arbitrary even dimension $E_d(x,y)$ can be expressed via elliptic integrals. In odd dimensions, $E_d(x,y)$ can be expressed in terms of elementary functions. In the most interesting three-dimensional case, one reduces Eq. (14) to

$$\begin{aligned} 4\beta - 1 + \beta x\frac{F'(x)}{F(x)} \\ = -\frac{2\pi}{3}\int_0^\infty dy\, y\,\frac{(x+y)^3 - |x-y|^3}{x}F(y). \end{aligned} \tag{16}$$

One should solve Eq. (14) subject to an appropriate boundary condition at the origin. Overall, the task reduces again to the eigenvalue problem. Note that the approximation

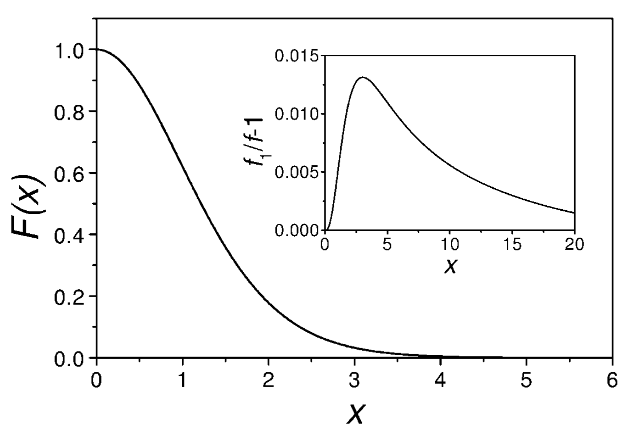

FIG. 1. Plot of the scaled velocity distribution F when the initial velocity distribution is flat. The inset shows the relative difference between $f = -\ln F$ and f_1, with f_1 given by Eq. (11).

scheme explicitly presented above in one dimension equally applies in higher dimensions.

Thus, the decay exponents can be determined with arbitrarily high precision in arbitrary spatial dimension for arbitrary isotropic initial velocity distribution. This method, however, does not solve the Boltzmann equation (1). We now provide two *approximate* solutions to the Boltzmann equation for arbitrary initial conditions and arbitrary spatial dimension d. These solutions are in fact *exact* solutions of Boltzmann equations with collision kernels $\sigma(g)$ different from $\sigma(g) = g$ characterizing the hard sphere gas (here $g \equiv |\mathbf{v} - \mathbf{w}|$ is the relative velocity). Note that such collision kernels naturally arise in kinetic theory of interacting particles [17]. In the present context, the generalized Boltzmann equation reads

$$\frac{\partial P(\mathbf{v},t)}{\partial t} = -P(\mathbf{v},t)\int d\mathbf{w}\,\sigma(g)P(\mathbf{w},t). \tag{17}$$

Let us compare dimensions of the left- and right-hand sides of Eq. (17). The velocity distribution has dimension $[P] = T/L^{d+1}$ which implies $[\sigma] = L^d/T$; therefore $[\sigma(g)/g] = L^{d-1}$. The remaining quantity with dimension of length, the "interaction" radius, should be extracted from the collision process. For hard sphere gas, the relevant interaction radius is simply the geometrical radius a of the spheres, so $\sigma(g) = ga^{d-1}$. The constant factor a^{d-1} can be absorbed into the time variable—this is what we have done in Eq. (13). For particles interacting through a two-body power-law potential, $U(r) \propto r^{-n}$, the energy conservation implies $g^2 \sim r^{-n}$. Thus, $\sigma(g) \sim gr^{d-1} \sim g^{\lambda}$ with $\lambda = 1 - \frac{2(d-1)}{n}$. The hard sphere gas ($\lambda = 1$) is recovered for $n = \infty$. The velocity independent kernel ($\lambda = 0$), the so-called "Maxwell" gas, arises when particles interact through the power-law potential with the exponent $n = 2(d-1)$. When $\lambda > 1$, the interaction is "harder" than in the hard sphere gas (though such behavior does not arise from a simple power-law interaction potential). One particularly tractable model corresponds to $\sigma(g) = g^2$, the so-called gas of very hard particles [18].

To provide a faithful analog of the original hard sphere gas, we replace $\sigma(g) = g$ by $\bar{v}$ in the Maxwell case and by the factor $g^2/\bar{v}$ for the very hard particles. Hence for the Maxwell gas the Boltzmann equation (17) becomes

$$\frac{\partial P(\mathbf{v},t)}{\partial t} = -P(\mathbf{v},t)c(t)\bar{v}(t). \tag{18}$$

Thus we get effectively noninteracting particles as different velocities remain uncoupled. Solving (18) yields

$$P(\mathbf{v},t) = c(t)P_0(\mathbf{v}), \qquad c(t) = \frac{1}{1+\bar{v}t}. \tag{19}$$

The moments of the velocity do not change with time, e.g., $\bar{v}(t) = \bar{v}(0)$, and thence $\alpha = 1$ and $\beta = 0$.

More interesting results are found for the very hard particles. The corresponding Boltzmann equation, i.e., Eq. (17) with $\sigma(g) = g^2/\bar{v}$, can be simplified by absorbing the $\bar{v}^{-1}$ factor into the time variable,

$$\tau = \int_0^t \frac{dt'}{\bar{v}(t')}, \tag{20}$$

and reducing the collision integral into a combination of the moments $M_j(\tau) = \int d\mathbf{w}\, w^j P(\mathbf{w},\tau)$ of the velocity distribution. The Boltzmann equation becomes

$$\frac{\partial P(\mathbf{v},\tau)}{\partial \tau} = -P(\mathbf{v},\tau)[v^2 M_0(\tau) + M_2(\tau)], \tag{21}$$

and easily solved to give

$$P(\mathbf{v},\tau) = P_0(\mathbf{v})e^{-v^2 L_0(\tau) - L_2(\tau)}. \tag{22}$$

Here $L_j(\tau) = \int_0^\tau d\tau'\, M_j(\tau')$.

To derive explicit results, it is natural to consider initial velocity distributions algebraic near the origin. To simplify algebra, we specifically choose

$$P_0(\mathbf{v}) = \frac{2v^{\mu}e^{-v^2}}{\Omega_d \Gamma[(\mu+d)/2]}, \tag{23}$$

where $\Omega_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the unit sphere in d dimensions, and the prefactor in (23) is chosen to set the initial density to unity. Combining (22) and (23) we can explicitly compute M_0 and M_2. Using then $M_j = \frac{dL_j}{d\tau}$ closes the problem. We find

$$\frac{dL_0}{d\tau} = \frac{e^{-L_2}}{(1+L_0)^{\frac{\mu+d}{2}}}, \tag{24}$$

and

$$\frac{dL_2}{d\tau} = \frac{\mu+d}{2}\,\frac{e^{-L_2}}{(1+L_0)^{\frac{\mu+d+2}{2}}}. \tag{25}$$

Solving Eqs. (24) and (25) subject to $L_0(0) = L_2(0) = 0$ yields $L_0(\tau)$ and $L_2(\tau)$ from which we compute the density $c = M_0$,

$$c = [1 + (\mu+d+1)\tau]^{-\frac{\mu+d}{\mu+d+1}}, \tag{26}$$

and the rms velocity $\bar{v} = \sqrt{M_2/M_0}$,

$$\bar{v} = \sqrt{\frac{\mu+d}{2}}\,[1 + (\mu+d+1)\tau]^{-\frac{1}{2(\mu+d+1)}}. \tag{27}$$

By inserting Eq. (27) into Eq. (20) we can express τ via the original time variable t. Finally, we arrive at

$$\alpha = \frac{2(\mu+d)}{2(\mu+d)+1}, \qquad \beta = \frac{1}{2(\mu+d)+1}. \tag{28}$$

Note that the exponent relation $\alpha + \beta = 1$ which is valid for any μ and d. This sum rule immediately follows from an elementary mean-free path argument: $a^{d-1}c\bar{v}t \sim 1$.

The above exact values of the exponents in the two solvable limits appear to provide the strict bounds for the hard sphere case:

$$\frac{2(\mu+d)}{2(\mu+d)+1} < \alpha < 1, \qquad 0 < \beta < \frac{1}{2(\mu+d)+1}. \tag{29}$$

These bounds are fair for small d and they get more and more stringent as the spatial dimension increases.

The above method of solving the Boltzmann equation can be adapted to the more general collision kernels $\sigma(\mathbf{v}, \mathbf{v}') = \bar{v}^{1-\kappa}(|\mathbf{v}|^\kappa + |\mathbf{v}'|^\kappa)$ (the most natural case corresponds to $\kappa = 1$). For $\kappa > 0$, the decay exponents are given by Eqs. (28). The collision kernels $\sigma(g) = g^{2n}$ constitute a more perspective generalization. One could try to solve the Boltzmann equation when n is an integer and then perform an analytic continuation to $n = 1/2$ corresponding to the hard sphere gas. Besides the cases of Maxwell and very hard particles ($n = 0$ and 1, respectively), it is possible to work out the case of $n = 2$. Unfortunately, we have not succeeded beyond that.

Finally, we consider the limit of infinite spatial dimension. In this case, the Boltzmann equation approach should become exact. Fortunately, the limit $d \to \infty$ also leads to considerable simplifications in many-body problems [19], and, for instance, the Boltzmann equation (13) becomes tractable. First of all, the collision kernel simplifies to $\sqrt{v^2 + w^2}$, which reflects an obvious fact that different vectors are orthogonal with probability one in infinite dimensions. The scaled Boltzmann equation reads

$$1 - B - \beta x \frac{F'}{F} = \Omega_d \int_0^\infty dy\, y^{d-1} \sqrt{x^2 + y^2}\, F(y), \tag{30}$$

with $B = (d + 1)\beta$. The bounds of Eq. (29) lead to $\beta \to 0$ but remain nontrivial for B, $0 < B < 1/2$. The right-hand side of Eq. (30) is computed by the saddle point technique to find $C\sqrt{x^2 + y_*^2}$, where C is the normalization factor $C = \Omega_d \int_0^\infty dy\, y^{d-1} F(y)$, and y_* is the saddle point which is found from $y_* F'(y_*)/F(y_*) + d - 1 = 0$. Hence, at the saddle point Eq. (30) gives $1 = C y_* \sqrt{2}$. Near the origin, $F(x) \sim x^\mu$ and thus $\beta x F'/F = \beta\mu \to 0$, so we find $1 - B = C y_* = 1/\sqrt{2}$. Therefore, we arrive at

$$\beta \simeq \left(1 - \frac{1}{\sqrt{2}}\right) d^{-1} \quad \text{when } d \to \infty. \tag{31}$$

Thus when the spatial dimension increases, the decay exponents become *universal,* i.e., independent on the initial velocity distribution.

In summary, we reduced the determination of the decay exponents for ballistic annihilation to an eigenvalue problem. We found that the exponents have very nontrivial values even for the simplest initial velocity distributions. Our approach manifestly demonstrates that the decay exponents are affected only by the spatial dimension d and by the exponent μ characterizing the initial velocity distribution in the $|\mathbf{v}| \to 0$ limit: $P_0(\mathbf{v}) \sim |\mathbf{v}|^\mu$. We also solved the Boltzmann equation for the Maxwell particles and very hard particles in arbitrary dimension. For the hard sphere gas, we found the exact asymptotic behavior of the exponents in the $d \to \infty$ limit.

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