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SOME REMARKS ON THE MAXIMAL IDEAL STRUCTURE OF H^∞

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It has been conjectured [1] that the open disc, $|z| < 1$, is dense in the maximal ideal space of H^∞. (H^∞ is, as usual, the algebra of functions analytic and bounded in $|z| < 1$). This, the so-called "Corona problem", seems quite difficult and in this paper we content ourselves with proving the affirmative result only in a special case; namely, for maximal ideals which do not contain any Blaschke products.

We should mention at this point that such maximal ideals actually exist. Namely, consider H^∞ as a (closed) subalgebra of $L^\infty(|z| = 1)$. Let $\mathcal{M}$ be a maximal ideal in L^∞ and set $M = \mathcal{M} \cap H^\infty$. Clearly M is a maximal ideal in H^∞ and furthermore $\mathcal{M}$ (and hence M) does not contain a B-product. This follows from the fact on $|z| = 1$ a B-product has modulus 1 almost everywhere, so that it has an inverse in L^∞. Our Theorem 4 will show that, conversely, this is the only way to obtain such an M in H^∞.

It is of some interest to point out that the original "Corona" conjecture can be stated in purely analytical terms without any reference to algebraic concepts; namely, let $f_1(z), \cdots, f_n(z)$ be analytic and bounded in $|z| < 1$, and suppose that $|f_1(z)| + \cdots + |f_n(z)| \geqq 1$, throughout $|z| < 1$, then there exist $g_1(z), \cdots, g_n(z)$ which are analytic and bounded in $|z| < 1$, and such that $f_1 g_1 + \cdots + f_n g_n = 1$ throughout $|z| < 1$.

(Unfortunately the special case with which we deal can not be so *cleanly* interpreted).

We shall have need of the following definitions:

D1. If M is a maximal ideal in H^∞ and if $f(z) \in H^\infty$, we denote by $f(M)$ the number for which $f(z) - f(M) \in M$. (If $\exists \zeta$, $|\zeta| < 1$, such that $f(M) = f(\zeta)$ for all $f \in H^\infty$ then we identify M with ζ.)

D2. If $f(z) \in H^\infty$ we denote, for fixed θ, $f(e^{i\theta})$ the limit, when it exists, of $f(re^{i\theta})$ as $r \to 1^-$.

D3. $U(z)$ is said to be an *inner function* if $|U(e^{i\theta})| = 1$ for almost all θ.

D4. $B(z)$ is said to be a B-*product* if $B(z) = cz^k \prod_n ((z_n - z)/(1 - \bar{z}_n z)) \cdot (|z_n|)/(z_n)$ where $|c| = 1$, c constant, $0 < |z_n| < 1$ and $\sum_n 1 - |z_n| < \infty$.

D5. If $F(\theta)$ is a measurable function, a is said to be an *essential value* of F if for any $\varepsilon > 0$ the set of θ for which $|F(\theta) - a| < \varepsilon$ has positive measure.

D6. If $f(z) \in H^\infty$ then $\|f\| = \sup_{|z|<1} |f(z)|$

D7. The *Corona* is the set of all maximal ideals in H^∞ which are not in the closure of $|z|<1$; that is, maximal ideals M for which $\exists f_1, \cdots, f_n$ all in M for which $|f_1(z)| + \cdots + |f_n(z)| \geqq 1$ for all z in $|z|<1$.

We shall also assume the following fundamental facts [2]:

F1. If $f(z) \in H^\infty$, then $f(e^{i\theta})$ exists for almost all θ.

F2. All B-products are inner functions.

F3. If $z_1, z_2, \cdots$ are the zeros of $f(z) \in H^\infty$, then there is a B-product, $B(z)$, whose zeros are precisely these z_n.

F4. If $f(z) \in H^\infty$ with zeros z_n and $B(z)$ is a B-product with these zeros, then $\|f(z)\| = \|f(z)/B(z)\|$.

F5. If $U(z)$ is an inner function, then $\|U(z)\| = 1$.

F6. $|f(M)| \leqq \|f\|$.

Our theorem will be proved in two stages. First we will show that any maximal ideal which contains no Blaschke products contains no inner functions. Then it will be shown that any maximal ideal containing no inner functions cannot lie in the Corona.

First, however, we need to prove the following theorem which, in itself, is of some interest.

THEOREM 1. *Every inner function is the uniform limit of Blaschke products.*

PROOF. Let $U(z)$ be any inner function. We may assume $U(0) \neq 0$. Define:

$$\varphi(r, \alpha) = \frac{1}{2\pi r}\int_{|z|=r} \log\left|\frac{U(z)-\alpha}{1-\bar\alpha U(z)}\right| |dz| \quad \text{for } |\alpha|<1.$$

and note that, by Jensen's theorem, if $\alpha \neq U(0)$,

$$\varphi(r, a) = \log\frac{r^n\left|\dfrac{U(0)-\alpha}{1-\bar\alpha U(0)}\right|}{|z_1|\cdots|z_n|}$$

where the z_i are the points in $|z| \leqq r$ where $U(z)=\alpha$. Hence, $\varphi(r,\alpha) \leqq 0$, $\varphi(r,\alpha)$ is non-decreasing in r, and throughout the region $|\alpha - U(0)| \geqq \delta$, $\varphi(r,\alpha)$ is uniformly bounded.

Call $\varphi(\alpha) = \lim_{r\to1}\varphi(r,\alpha)$, $[\varphi(\alpha) \leqq 0]$, let $\rho < |U(0)|$, and consider $\int_0^{2\pi}\varphi(\rho e^{i\theta})\,d\theta$. By the uniform boundedness of $\varphi(r,a)$ along $|\alpha| = \rho$ this is equal to

$$\lim_{r\to1}\int_0^{2\pi}\varphi(r,\rho e^{i\theta})\,d\theta = \lim_{r\to1}\frac{1}{2\pi r}\int_0^{2\pi}\int_{|z|=1}\log\left|\frac{U(z)-\rho e^{i\theta}}{1-\rho e^{-i\theta}U(z)}\right| |dz|\,d\theta$$

$$= \lim_{r\to1}\int_{|z|=r}\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{U(z)-\rho e^{i\theta}}{1-\rho e^{-i\theta}U(z)}\right| d\theta\,|dz|\,.$$

Using Jensen's theorem again on the inner integral gives

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{U-\rho e^{i\theta}}{1-\rho e^{-i\theta}U}\right|d\theta = \frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{U-\rho e^{i\theta}}{e^{i\theta}-\rho U}\right|d\theta$$
$$= \max(\log|U(z)|, \log\rho)\ .$$

Now note that $F = \mathrm{Max}(\log|U(z)|, \log\rho)$ is bounded and since $\log|U(re^{i\theta})| \to 0$ as $r \to 1$ for almost all θ, we have

$$\lim_{r\to 1}\int F\,|dz| = 0$$

and so

$$\int_0^{2\pi}\varphi(\rho e^{i\theta})\,d\theta = 0$$

Since, however, $\varphi(\alpha) \leqq 0$, it follows that $\varphi(\rho e^{i\theta}) = 0$ for almost all θ.

Let $\alpha = \rho e^{i\theta}$ be chosen such that $\varphi(\alpha) = 0$. Now consider $(U(z) - \alpha)/(1 - \bar{\alpha}U(z))$ and let $B(z)$ be a B-product with the same zeros. Let us form $V(z) = (U(z) - \alpha)/((1 - \bar{\alpha}U(z))\cdot B(z))$. It follows readily that $V(z)$ is an inner function. We have, however,

$$0 = \varphi(\alpha) = \lim_{r\to 1}\varphi(r, \alpha) = \lim_{r\to 1}\log\frac{r^n\left|\dfrac{U(0)-\alpha}{1-\bar{\alpha}U(0)}\right|}{|z_1|\cdots|z_n|}$$

$$= \log\frac{\left|\dfrac{U(0)-\alpha}{1-\bar{\alpha}U(0)}\right|}{|z_1|\cdots|z_n|} = \log\frac{\left|\dfrac{U(0)-\alpha}{1-\bar{\alpha}U(0)}\right|}{|B(0)|} = \log|V(0)|\ .$$

Hence $|V(0)| = 1$. By using the maximum modulus theorem, since $V(z)$ is an inner function, $V(z) = \text{constant} = e^{it}$.

Hence, finally, with $B_1(z) = e^{it}B(z)$, $(U(z) - \alpha)/(1 - \bar{\alpha}U(z)) = B_1(z)$, a B-product. Also, $|U(z) - B_1(z)| = |\alpha||(1 - U(z))/(1 - \alpha U(z))| \leqq (2\rho)/(1-\rho)$ and the result follows since ρ can be made arbitrarily small.

THEOREM 2. *If M contains no B-product, then M contains no inner function.*

LEMMA. *If M contains no B-product and if $B(z)$ is a Blaschke product, then $|B(M)| = 1$.*

PROOF. Since $|B(z)| \leqq 1$ it follows that $|B(M)| \leqq 1$ and we need only show $|B(M)| \nless 1$.

CASE 1. $B(z) = z$. If $|B(M)| < 1$ then $(z - B(M))/(1 - \bar{B}(M)z)$ would be a Blaschke product in M contradicting the hypothesis.

CASE 2. $B(z)$ a finite Blaschke product $B(z) = \prod_{n=1}^{N} (z-z_n)/(1-\bar{z}_n z)$. By case 1 we know that $z \to e^{i\theta}$ for some θ hence

$$B(M) = \prod_{n=1}^{N} \frac{e^{i\theta} - z_n}{1 - \bar{z}_n e^{i\theta}}$$

and hence $| B(M) | = 1$.

CASE 3. $B(z) = \prod_{n=1}^{\infty} ((z_n - z)/(1 - \bar{z}_n z)) \cdot (| z_n |)/(z_n)$ where $\prod | z_n | > 0$. Here we can find a non-decreasing sequence of positive integers $a_n \to \infty$ such that $\prod | z_n |^{a_n} > 0$. Consider the Blaschke products

$$B_0(z) = \prod_{n=1}^{\infty} \left(\frac{z_n - z}{1 - \bar{z}_n z} \frac{| z_n |}{z_n} \right)^{a_n}$$

$$B_1(z) = \prod_{n=1}^{N} \left(\frac{z_n - z}{1 - \bar{z}_n z} \frac{| z_n |}{z_n} \right)^{a_N - a_n}$$

$$B_2(z) = \prod_{n=N+1}^{\infty} \left(\frac{z_n - z}{1 - \bar{z}_n z} \frac{| z_n |}{z_n} \right)^{a_n - a_N}$$

and note that

$$(B(z))^{a_N} B_2(z) = B_0(z) B_1(z)$$

hence

$$| B(M) |^{a_N} | B_2(M) | = | B_0(M) || B_1(M) |$$

but by case 2, $| B_1(M) | = 1$, and by hypothesis $B_0(M) \neq 0$, hence

$$| B(M) \geqq | B_0(M) |^{1/a_N} ,$$

and letting $N \to \infty$ gives the result.

Theorem 2 now follows readily. Let $U(z)$, an inner function, be given. By Theorem 1, there is a Blaschke product for which $\| U(z) - B(z) \| \leqq \frac{1}{2}$, $| U(M) | \geqq | B(M) | - \frac{1}{2}$ and the latter expression is equal to by $\frac{1}{2}$ the lemma. Hence, $U(M) \neq 0$ and the proof is complete.

THEOREM 3. *If $M \in C$ then M contains no B-product.*
We require several lemmas.

LEMMA 1. *If M contains no B-product and $f \in H^\infty$, then $f(M)$ is an essential value of $f(e^{i\theta})$.*

PROOF. Assume otherwise, that, for some $f \in H^\infty$, $\varepsilon > 0$, $| f(e^{i\theta}) - f(M) | \geqq \varepsilon$

for almost all θ. Hence there exists a real bounded harmonic function $u(r, \theta)$ with boundary values $\log |f(e^{i\theta}) - f(M)|$. If $v(r, \theta)$ is the conjugate function to $u(r, \theta)$, then $g(z) = e^{u+iv}$ is bounded analytic and has a reciprocal $1/(g(z)) = e^{-u-iv}$ which is bounded analytic. Therefore, $g(z) \notin M$, but $(f(z)) - f(M))/g(z)$ is an inner function and so, by Theorem 1, $(f(z) - f(M))/g(z) \notin M$. From these two results, we conclude the contradiction $f(z) - f(M) \notin M$ and the lemma is proved.

We now consider the real algebra, A, of all bounded measurable real functions on $|z| = 1$. For $f \in A$, let $\varphi = \varphi(f)$ denote the unique bounded harmonic function in $|z| < 1$ with $f(e^{i\theta})$ as boundary values. Let $\psi(f)$ be the analytic function whose real part is $\varphi(f)$. Let $F = F(f) = e^{\psi}$ and note that $F \in H^\infty$, also note that $1/F = e^{-\psi}$ is $\in H^\infty$ so that $F(M) \neq 0$. Finally, define $l(f) = \log |F(M)|$.

This l thus defined is then a real functional on the real algebra A.

LEMMA 2. *l is a bounded, linear, multiplicative functional on A.*

PROOF. We first show l is linear, so let $f_1, f_2 \in A$, α real. Suppose φ_1, φ_2 are harmonic, bounded, and have f_1, f_2 respectively as boundary values. Clearly $\alpha\varphi_1 + \varphi_2$ is harmonic, bounded, and has $\alpha f_1 + f_2$ as boundary values. Hence $\psi(\alpha f_1+f_2)=\alpha\psi(f_1)+\psi(f_2)$ and so $F(\alpha f_1+f_2)=F(f_1)^\alpha \cdot F(f_2)$. It is now easily shown that if F, $1/F \in H^\infty$, then $|F^\alpha(M)| = |F(M)|^\alpha$ and so we have, finally

$$l(\alpha f_1 + f_2) = \alpha l(f_1) + l(f_2) .$$

As for boundedness, if $|f| \leqq 1$ then, by the maximum modulus principle, $|\varphi| \leqq 1$ and so $1/e \leqq |F| \leqq e$ hence $1/e \leqq |F(M)| \leqq e$ and $|l(f)| = |\log |F(M)|| \leqq 1$.

The multiplicativity is of a less trivial nature and for this we need some (sub)lemmas.

LEMMA 3. *The algebra A is isometric to a real $C(X)$ where X is compact, Hausdorff, and totally disconnected. Furthermore, if $f \in A$ corresponds to $f^* \in C(X)$ then* Ess range *of $f \equiv$* range *of f^*.*

PROOF. In fact this is true of the complex algebra of bounded measurable functions, see [3].

LEMMA 4. *Let X be compact, Hausdorff, and totally disconnected. Let l be a bounded linear functional on real $C(X)$ such that for each $f \in C(X)$, $l(f) \in$* range *of f. Then l is multiplicative.*

PROOF. Since l is bounded linear it must have the form $l(f) = \int_X f \, d\mu$.

To show that l is multiplicative, it suffices to show that $d\mu$ is a point mass of mass 1.

Suppose that $d\mu$ were not a point mass, then there exist two points x_1, x_2 both in X such that all neighborhoods of x_1 or x_2 have positive measure. Since X is totally disconnected and Hausdorff, we may choose neighborhoods N_1 of x_1 and N_2 of x_2 which are disjoint and both open and closed. The function f defined as a on N_1, b on N_2 and 0 elsewhere would then be $C(X)$ and so

$$l(f) = \int f d\mu = a\mu(N_1) + b\mu(N_2)$$

with $a = 0$, $b = 1$, the condition $l(f) \in$ range of f insures that $\mu(N_2) = 1$. Similarly, $a = 1$, $b = 0$ gives $\mu(N_1) = 1$, hence $a = 1$, $b = 1$ gives $l(f) = 2$ which contradicts $l(f) \in$ range of f.

Hence $d\mu$ is a point mass. Choosing f as 1 in an open and closed neighborhood of this point and 0 elsewhere insures that mass must be 1 and the lemma is proven.

Proof of Lemma 2. To prove that our l is multiplicative it suffices, by Lemmas 3 and 4 to show that for each $f \in A$ $l(f)$ is an essential value of $f(e^{i\theta})$. But, by Lemma 1, $F(M)$ is an essential value of $F(e^{i\theta})$, and so $\log |F(M)|$ is an essential value of $f(e^{i\theta})$, and the proof is complete.

Lemma 5. *Let $f_1, \cdots, f_k$ be in A. Suppose $l(f_1) = l(f_2) = \cdots = l(f_k) = 0$, and let $\varepsilon > 0$. There exists a set S of positive measure on $|z| = 1$ such that $|f_1(e^{i\theta})| + \cdots + |f_k(e^{i\theta})| < \varepsilon$ for $e^{i\theta} \in S$.*

Proof. Suppose not, then $|f_1(e^{i\theta})| + \cdots + |f_k(e^{i\theta})| \geqq \varepsilon$ almost everywhere and so

$$f_1^2 + \cdots + f_k^2 \geqq \frac{\varepsilon^2}{k} \qquad \text{a. e.}$$

hence

$$\frac{1}{f_1^2 + \cdots + f_k^2} \in A$$

and so, l being multiplicative

$$l(1) = l\left(\frac{1}{f_1^2 + \cdots + f_k^2}\right)[l^2(f_1) + \cdots + l^2(f_k)] = 0$$

a contradiction since, directly from the definition of l, $l(1) = 1$.

Lemma 6. *Let $f_1, f_2, \cdots, f_n \in M$, M containing no B-products and let $\varepsilon > 0$. There exists a set S of positive measure on $|z| = 1$ wherein*

$$|f_j| < \varepsilon \qquad j = 1, \cdots, n.$$

PROOF. We may assume $\|f_j\| \leqq k$ for all $j = 1, \cdots, n$. Now consider the following $2n$ functions in A, namely

$$\log \left|\frac{1 + f_j(e^{i\theta})}{2k}\right| \quad \text{and} \quad \log \left|1 - \frac{f_j(e^{i\theta})}{2k}\right|$$

clearly

$$l\left(\left|1 + \frac{f_j(e^{i\theta})}{2k}\right|\right) = \log \left|1 + \frac{f_j(M)}{2k}\right| = 0$$

and

$$l\left(\left|1 - \frac{f_j(e^{i\theta})}{2k}\right|\right) = \log \left|1 - \frac{f_j(M)}{2k}\right| = 0 .$$

By lemma 5, then, it follows that

$$\left|1 + \frac{f_j(e^{i\theta})}{2k}\right| < 1 + \delta, \qquad \left|1 - \frac{f_j(e^{i\theta})}{2k}\right| < 1 + \delta$$

throughout some set S, of positive measure. By the parallelogram law

$$2\left(1 + \frac{|f_j|^2}{4k^2}\right) = \left|1 + \frac{f_j}{2k}\right|^2 + \left|1 - \frac{f_j}{2k}\right|^2 < 2(1 + \delta)^2$$

through S. Therefore (assuming $\delta \leqq 1$)

$$|f_j|^2 \leqq 4K^2(2\delta + \delta^2) \leqq 12K^2\delta \qquad \text{in } S.$$

The proof is completed upon choosing $\delta = \min(1, \varepsilon^2/12K^2)$.

PROOF OF THEOREM 3. Let $f_1, f_2, \cdots f_n \in M$. We must display a $\zeta, |\zeta| < 1$, where $|f_1| + \cdots + |f_n| < 1$. By Lemma 6. $|f_j(e^{i\theta})| < 1/2n$ throughout S of positive measure but $f_j(re^{i\theta}) \to f_j(e^{i\theta})$ almost everywhere, thus there exists an $e^{i\theta} \in S$ for which $f_j(re^{i\theta}) \to f_j(e^{i\theta})$, $j = 1, \cdots, n$ and so, for some $r < 1$,

$$|f_j(re^{i\theta})| < \frac{1}{n} \qquad j = 1, \cdots, n.$$

The choice $\zeta = re^{i\theta}$ completes the proof.

THEOREM 4. *If M is a maximal ideal in H^∞ then M can be extended to a maximal ideal in $L^\infty(|z| = 1)$ if and only if M contains no B-product.*

PROOF. The "only if" part has already been proven in our introductory remarks. Conversely, assume M contains no B-products, we must show that the ideal generated by M in L^∞ is proper, i. e., $F_1f_1 + \cdots + F_nf_n = 1$

a. e. is impossible with $F_i \in L^\infty$, $f_i \in M$. This, however, follows immediately from Lemma 6, and the proof is complete.

A COUNTER-EXAMPLE. It might be felt that what is left is the easy case; namely, if M is a maximal ideal in H^∞ and the Blaschke product $B(z)$ with zeros at z_n lies in M, then M is actually a limit point of these z_n. Such is not the case, however, as the following example illustrates:

Let $B(z)$ be a B-product with zeros at $1 - 1/n^2$ $n = 1, 2, \cdots$ and let $F(z) = e[(1 + z)/(1 - z)]^i - 1$. Then it can be shown that B and F both tend to zero along the sequence $(e^{2\pi n + i} - 1)/(e^{2\pi n + i} + 1)$ and so $\exists M_0$ containing both B and F. Consider the neighborhood of M_0 consisting of all $M \ni |F(M) - F(M_0)| < 1$. This neighborhood contains no point $1 - 1/n^2$ for $|F(1 - 1/n^2)| \geqq e - 1 > 1$ and $F(M_0) = 0$.

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