Classification of scalar field potentials with cosmological scaling solutions

Andrew R. Liddle*
Astronomy Centre, University of Sussex, Falmer, Brighton BN1 9QJ, United Kingdom

Robert J. Scherrer
Department of Physics and Department of Astronomy, The Ohio State University, Columbus, Ohio 43210 and NASA/Fermilab Astrophysics Center, Fermi National Accelerator Laboratory, Batavia, Illinois 60510

(Received 23 September 1998; published 10 December 1998)

An attractive method of obtaining an effective cosmological constant at the present epoch is through the potential energy of a scalar field. Considering models with a perfect fluid and a scalar field, we classify all potentials for which the scalar field energy density scales as a power law of the scale factor when the perfect fluid density dominates. There are three possibilities. The first two are well known; the much-investigated exponential potentials have the scalar field mimicking the evolution of the perfect fluid, while for negative power laws, introduced by Ratra and Peebles, the scalar field density grows relative to that of the fluid. The third possibility is a new one, where the potential is a positive power law and the scalar field energy density decays relative to the perfect fluid. We provide a complete analysis of exact solutions and their stability properties, and investigate a range of possible cosmological applications. [S0556-2821(99)07102-7]

PACS number(s): 98.80.Cq

I. INTRODUCTION

The evidence in favor of a cosmological constant, or something very much like it, playing a significant dynamical role in our present Universe is becoming overwhelming. Most prominent have been the recent measurements of the apparent magnitude-redshift relationship using type Ia supernovae [1], but other factors such as the consistently low measurements of the matter density, including the baryon fraction in galaxy clusters [2], have also been pointing in that direction. While many of these latter measurements are insensitive to the presence of a cosmological constant, there is some observational motivation for a flat universe from the favored location of the first acoustic peak of the microwave background anisotropies and some theoretical motivation from a desire to utilize the simplest models of cosmological inflation as the source of density perturbations. In combination, these favor a present cosmological constant (in units of the critical density) of $\Omega_\Lambda \sim 0.7$.

Since a genuine cosmological constant requires extreme fine-tuning in order to have only begun to dominate recently, it is extremely tempting to model the cosmological constant as an effective one. As the supernova observations are requiring an accelerating universe, which is precisely the definition of inflation, the minimal approach is to assume that the same mechanism drives inflation now as is presumed in the early Universe, namely the potential energy of a scalar field. Three possibilities present themselves. The field could be at an absolute minimum of non-zero potential energy. It could be in a metastable false vacuum, tunneling at some later stage into the true vacuum and perhaps even reheating. Or it could be slowly rolling down a potential, as in the chaotic inflation models favored for the early Universe.

The first two of these possibilities are dynamically indistinguishable from a true cosmological constant, and so we shall concentrate on the third, which is often called ''quintessence.'' As stressed in a recent paper by Zlatev *et al.* [3], a rolling scalar field offers the opportunity to address another mystery, that of why the cosmological constant took so long to become dominant. If, for example, the scalar field behaves in such a way as to remain insignificant during the radiation domination era, perhaps it can be ''triggered'' in some way to begin to grow in the matter era and come to dominate only in the recent past. Solutions where the scalar field energy density follows that of radiation or matter have been called ''scaling solutions,'' and, more recently, ''trackers,'' and several examples have been described in the literature.

For the purposes of this paper, we will define a ''scaling solution'' as one in which the scalar field energy density ρ_ϕ scales exactly as a power of the scale factor, $\rho_\phi \propto R^{-n}$, when the dominant component has an energy density which scales as a (possibly different) power: $\rho \propto R^{-m}$. An equivalent, and perhaps more fundamental, definition is that the scalar field kinetic and potential energies maintain a fixed ratio. We will use the term ''tracker solution'' to refer to the special case $m=n$, i.e., where the scalar field energy density scales in the same way as the dominant component. The case where $m=n$ is produced by an exponential potential [4–8], while negative power-law potentials give $n<m$ [3,9].

In this paper we provide a comprehensive classification of all solutions of this type, when the energy density is dominated by the perfect fluid. We show that the *only* potentials which lead to this sort of behavior are the previously studied exponential and negative power-law potentials, and a new class of positive power-law potentials. We study the general properties of such solutions, including stability, and examine how well they might do in giving the desired cosmological behavior.

*Present address: Astrophysics Group, The Blackett Laboratory, Imperial College, London SW7 2BZ, United Kingdom

II. SCALING SOLUTIONS

A spatially flat homogeneous universe containing a perfect fluid with energy density ρ and pressure p, plus a scalar field ϕ with potential energy $V(\phi)$, satisfies the equations

$$H^2=\frac{8\pi G}{3}\left[V(\phi)+\frac{1}{2}\dot{\phi}^2+\rho\right], \tag{1}$$

$$\dot{\rho}=-3H(\rho+p), \tag{2}$$

where overdots are time derivatives. We will assume that the perfect fluid has an equation of state $p=(\gamma-1)\rho$, which immediately implies

$$\rho\propto\frac{1}{R^m}, \quad m=3\gamma. \tag{3}$$

The scalar field ϕ evolves according to

$$\ddot{\phi}=-3H\dot{\phi}-\frac{dV}{d\phi}. \tag{4}$$

The total scalar field energy is

$$\rho_\phi=V(\phi)+\frac{1}{2}\dot{\phi}^2, \tag{5}$$

and we are interested in solutions for which $\rho_\phi\propto R^{-n}$ when $\rho_\phi\ll\rho$ and $\rho\propto R^{-m}$.

Equation (5) allows the scalar field equation to be written in the useful form

$$\dot{\rho}_\phi=-3H\dot{\phi}^2. \tag{6}$$

If we divide Eq. (6) by ρ_ϕ and use $\dot{\rho}_\phi/\rho_\phi=-n(\dot{R}/R)$, then we obtain

$$\frac{\dot{\phi}^2/2}{\rho_\phi}=\frac{n}{6}. \tag{7}$$

Thus, power-law behavior for the scalar field energy density requires that the scalar field kinetic energy remain a fixed fraction of the total scalar field energy. The converse is true as well. This makes sense: if the kinetic energy evolves to become either dominant or negligible, then ρ_ϕ will scale as $1/a^6$ or remain constant, respectively. The former is not what we want, and the latter is no different from a genuine cosmological constant. These two extreme cases also delimit the possible scaling behavior for the scalar field energy density: $0\leq n\leq 6$, with the lower limit corresponding to potential energy domination and the upper limit to kinetic-energy domination.

A. Exact solutions

Our basic method of solution is to assume the desired behavior of ρ_ϕ and ρ and substitute into Eq. (4). A similar procedure was first undertaken by Ratra and Peebles [9], who confined their attention to the cases of matter and radiation domination, $m=3,4$, and were interested in certain classes of solutions. Our development parallels and extends their analysis.

When the perfect fluid with $\rho\propto R^{-m}$ dominates,

$$R\propto t^{2/m}, \tag{8}$$

and Eq. (4) becomes

$$\ddot{\phi}=-\frac{6}{m}\frac{1}{t}\dot{\phi}-\frac{dV}{d\phi}. \tag{9}$$

The desired scaling behavior for ρ_ϕ, substituted into Eq. (7), gives

$$\dot{\phi}\propto t^{-n/m}. \tag{10}$$

Consider first the case $m=n$. Then Eq. (10) can be integrated to give $\phi\propto\ln(t)$. Substituting this into Eq. (9) and solving for $V(\phi)$, we obtain

$$V(\phi)=\frac{2}{\lambda^2}\left(\frac{6}{m}-1\right)\exp(-\lambda\phi). \tag{11}$$

This is the well-investigated exponential potential [4] for the limiting case where $\rho_\phi\ll\rho$. Although λ can be positive or negative, those cases are physically identical, simply corresponding to a reflection of the ϕ trajectory about the vertical axis.

Provided $\lambda^2>m$, the unique late-time attractor is a tracker solution with $\rho_\phi=(m/\lambda^2)\rho_{\rm total}$ [5,9]. For example, the scalar field will redshift as $1/a^4$ during radiation domination, and then switch to $1/a^3$ once matter domination commences. Although we derived it assuming $\rho_\phi\ll\rho$, in fact this solution exists for any fractional scalar field density Ω_ϕ, through the appropriate choice of λ.

While mathematically intriguing, such solutions seem uninteresting as candidates for a cosmological constant. During nucleosynthesis they behave as radiation and hence act like extra neutrino species, and are limited to $\Omega_\phi<0.2$ during radiation domination and hence $\Omega_\phi<0.15$ during matter domination, well below the desired density [6,7]. A similar constraint arises from suppression of density perturbation growth [7]. Anyway, such a scalar field is presently evolving like matter and so will not explain the supernova measurements even if it were permitted with a more substantial density.

These bounds can be evaded if the field does not enter the scaling regime until late in the cosmological evolution, e.g. after nucleosynthesis for the first bound, and after structure formation has been initiated for the second. However, this requires that the scalar field begin with more or less its present density, and so provides no answer to the original fine-tuning problem.

Now consider the case $m\neq n$. In this case, integrating Eq. (10) yields

$$\phi=At^{1-n/m}. \tag{12}$$

The second integration constant has been absorbed by horizontal translation of ϕ. Substituting the required behavior into the scalar field equation leads to the potential

$$V(\phi)=A^2\left(1-\frac{n}{m}\right)^2\left(\frac{6-n}{2n}\right)\left(\frac{\phi}{A}\right)^{\alpha}, \tag{13}$$

where

$$\alpha=\frac{2n}{n-m}. \tag{14}$$

The constant of integration, which would otherwise appear in $V(\phi)$, vanishes because for scaling we need the kinetic energy to be a fraction $n/6$ of the scalar field energy density.

Scaling behavior can therefore occur provided the potential has a power-law form. If the exponent α is negative, then $m>n$ and the scalar field energy density grows compared to the matter, whereas if it is positive the opposite is true. We can rewrite Eq. (14) as

$$n=\left(\frac{\alpha}{\alpha-2}\right)m. \tag{15}$$

Since m and n are positive, Eq. (15) shows that scaling solutions exist for positive α only when $\alpha>2$ (in Sec. III below, we consider what happens for $\alpha\leqslant 2$).

We have thus determined *all* potentials which give power-law scaling of ρ_ϕ when the dominant density component also scales as a power of R. The negative power-law and exponential potentials have been studied in detail [3,9,4–8]; our new result is the existence of scaling solutions with the positive power-law potentials.

For most of these potentials, the differential equation governing the evolution of ϕ, Eq. (9), is nonlinear, and the solutions we have derived for $\phi(t)$ are particular rather than general solutions (in the study of nonlinear differential equations, these are known as ''singular solutions''). Hence, although there can be no other potentials which produce scaling behavior, there is as yet no guarantee that the potentials we have derived produce general solutions (as opposed to singular solutions) which display the desired scaling behavior. Put another way, we must show that the singular solutions we have derived in this section are attractors of the equations of motion.

B. Attractor structure

The attractor structure of the exponential potential has been analyzed in detail elsewhere [8], and so we will not concern ourselves with that potential here. The attractor structure of the negative power-law potentials has been discussed by Ratra and Peebles [9] for the cases $m=3,4$. We extend their discussion to the case of arbitrary m, and also consider the case of positive power laws.

We substitute a potential of the form $V(\phi)=V_0\phi^\alpha$ into Eq. (9). However, note that the multiplicative constant in front of $dV/d\phi$ can be absorbed into a rescaling of t. Henceforth, we assume such a rescaling and write

$$\ddot{\phi}=-\frac{6}{m}\frac{1}{t}\dot{\phi}-\phi^{\alpha-1}. \tag{16}$$

For this rescaled equation, the constant A in Eq. (12) is

$$A=\left[\left(\frac{2}{\alpha-2}\right)\left(\frac{6}{m}-\frac{\alpha}{\alpha-2}\right)\right]^{1/(\alpha-2)}. \tag{17}$$

Note that A is well defined, and the solution given by Eq. (12) valid, only for

$$\frac{6}{m}-\frac{\alpha}{\alpha-2}=\frac{6-n}{m}>0, \tag{18}$$

which is satisfied automatically as long as $n<6$.

Following Ratra and Peebles [9], we make the change of variables

$$t=e^{\tau}, \quad u(\tau)=\frac{\phi(\tau)}{\phi_{\rm e}(\tau)}, \tag{19}$$

where $\phi_{\rm e}(\tau)$ is the exact (singular) solution given by Eqs. (12) and (17). With these changes, Eq. (16) becomes

$$u''+\left(\frac{4}{2-\alpha}+\frac{6}{m}-1\right)u'+\frac{2}{\alpha-2}\left[\frac{\alpha}{\alpha-2}-\frac{6}{m}\right](u-u^{\alpha-1})=0, \tag{20}$$

where the prime denotes the derivative with respect to τ. This can be split into the autonomous system

$$u'=p,$$

$$p'=\left(1-\frac{6}{m}-\frac{4}{2-\alpha}\right)p+\frac{2}{\alpha-2}\left[\frac{6}{m}-\frac{\alpha}{\alpha-2}\right](u-u^{\alpha-1}). \tag{21}$$

For positive α, the interesting case is when α is an even integer, and then there are three critical points, all with $p=0$ and with $u=-1$, 0 and 1. All three of these represent solutions which asymptotically approach $\phi=0$, $\dot{\phi}=0$. The $u=+1$ and $u=-1$ critical points, when they are attractors, represent solutions which asymptotically approach the singular solution. They give mirror-image trajectories; the $+1$ attractor represents solutions which go to $\phi=0$ from the positive ϕ direction, while the -1 attractor gives solutions which approach $\phi=0$ from the negative ϕ direction. The $u=0$ critical point corresponds to solutions in which ϕ goes to zero faster than in the exact solution in Eq. (12).

Linearizing Eqs. (21) about the $u=1$, $p=0$ critical point and solving for the eigenvalues $\lambda_\pm$ of small perturbations about this point, we find

$$\lambda_\pm=\frac{1}{2}-\frac{3}{m}-\frac{2}{2-\alpha}\pm\sqrt{\left(\frac{1}{2}-\frac{3}{m}-\frac{2}{2-\alpha}\right)^2+2\left(\frac{\alpha}{\alpha-2}-\frac{6}{m}\right)}. \tag{22}$$

For the cases $m=3$ and $m=4$, this equation reduces to the Ratra-Peebles results [9]. The behavior of these eigenvalues is somewhat clearer if written in terms of m and n, using Eq. (14):

$$\lambda_{\pm}=\frac{2n-m-6\pm\sqrt{(2n-m-6)^2+8m(n-6)}}{2m}. \quad (23)$$

The necessary and sufficient condition for stability is that the real part of both λ_+ and λ_- be negative. If the quantity under the square root in Eq. (23) is negative, then this corresponds to the requirement that $2n-m-6<0$ and gives a stable spiral. Note, however, that because $n<6$, the second term under the square root is always negative. Hence, if the quantity under the square root is positive (so that both eigenvalues are real), then $2n-m-6+\sqrt{(2n-m-6)^2+8m(n-6)}<0$ whenever $2n-m-6<0$. Hence, the condition for stability is just $2n-m-6<0$, regardless of the value of the quantity under the square root (although that will determine whether the stable singular point is a stable spiral or a stable node). In terms of α, the stability condition is

$$\alpha<2\left(\frac{6+m}{6-m}\right), \quad \text{negative } \alpha, \quad (24)$$

$$\alpha>2\left(\frac{6+m}{6-m}\right), \quad \text{positive } \alpha. \quad (25)$$

The first of these is always satisfied, showing that the scaling solution for the Ratra-Peebles potentials ($\alpha<0$) is a stable attractor for all values of α (as noted by Ratra and Peebles for $m=3,4$). For positive α, however, the scaling solution is a stable attractor only for sufficiently large α. For example, in the matter-dominated era, attractor scaling solutions exist only for $\alpha>6$, while in the radiation-dominated era, this condition becomes $\alpha>10$. For $m=3$, $\alpha=6$, and $m=4$, $\alpha=10$, we have a vortex point at the singularity, which is neutrally stable but not an attractor.

C. Phase plane analysis

A complete analysis requires a numerical solution. Three phase planes are shown in Fig. 1 for the case $m=4$ (a radiation-dominated universe). From top to bottom, the choices are for the singular solution ($u=\pm1$, $p=0$) to be unstable ($\alpha=8$), to be marginally stable ($\alpha=10$), and to be stable ($\alpha=12$), respectively. To visualize the physical meaning of these trajectories, note that trajectories which cross the $u=0$ vertical axis correspond to solutions in which the ϕ field is oscillating about the minimum in the potential at $\phi=0$, while trajectories which are confined to the left or right side of the $u=0$ axis correspond to solutions in which ϕ goes to zero without oscillating.

In the top figure, any point in phase space spirals out to infinity. This corresponds to a solution which never stops oscillating. The amplitude of the ϕ oscillations is decreasing; the spirals move outwards because they lose amplitude more slowly than the exact solution $\phi_e(\tau)$. Although such a solu-

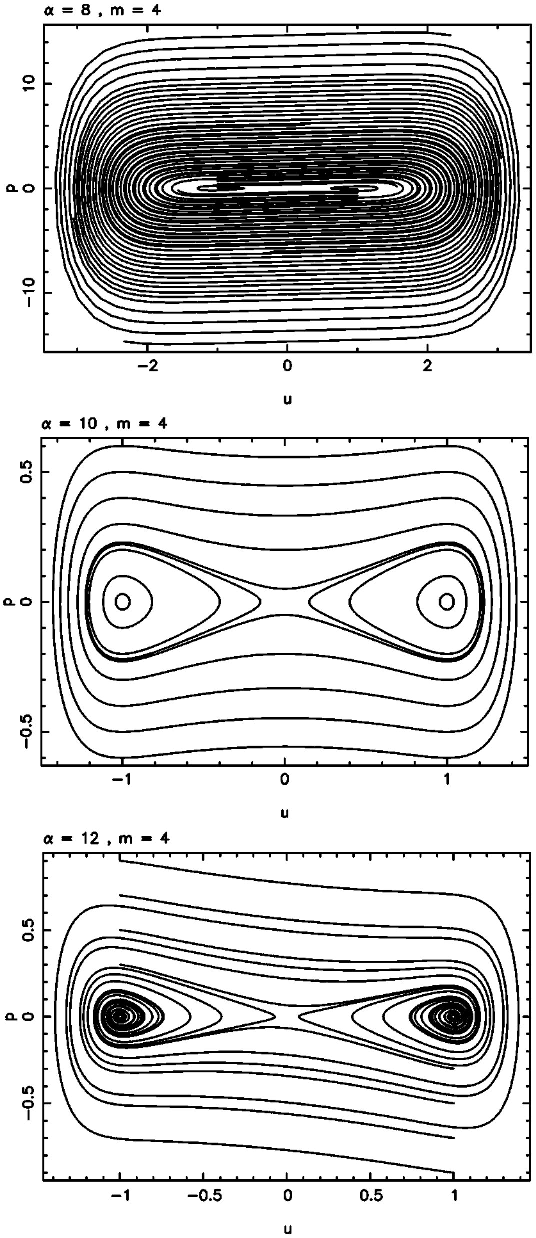

FIG. 1. Three phase planes for the radiation-dominated $m=4$ case. From top to bottom, $\alpha=8,10,12$. The top panel spirals outwards and the bottom one inwards.

tion exhibits a form of scaling behavior (see Sec. III B below), the scaling exponent is not given by Eq. (15). The middle figure corresponds to marginal stability and the phase trajectories are closed loops (which circulate clockwise), with the solution oscillating around the singular solution but not approaching it. The trajectories which cross the $\phi=0$ axis represent solutions in which ϕ oscillates forever. The trajectories closer to the singular point which do not cross the $\phi=0$ axis represent solutions in which ϕ does not oscillate about the minimum, but approaches zero as $t^{1-n/m}$ times an oscillatory function. The bottom figure shows the attractor situation; depending on the initial conditions the trajectory may circulate several times around the two critical points (which corresponds to the field oscillating about the minimum) before circulating only about a single point (in which case the field stops oscillating and falls steadily toward the minimum). In the latter case the exact solution given by Eq. (12) is multiplied by an oscillatory function which has a steadily decreasing amplitude.

We can repeat our stability analysis for the $u=0$, $p=0$ critical point. We find that the eigenvalues are both real, with $\lambda_-=(n-6)/m$, which is negative for $n<6$, and $\lambda_+=n/m-1$, which is positive (since $n>m$ for positive α). Thus, the (0,0) critical point is always an unstable saddle point. This may seem bizarre, since the (0,0) critical point corresponds to the field lying motionless at the bottom of the potential. However, remember that the singular solutions also asymptotically reach the minimum, and our result simply means that if the field is perturbed slightly from this minimum, it returns to the minimum via the singular solution.

III. APPLICATIONS AND SPECIAL CASES

A. Negative power laws

If α is negative, we have a decaying power-law potential, in which the field can roll forever. These are the potentials first investigated by Ratra and Peebles [9]. They were recently reexamined in some detail by Zlatev *et al.* [3], in the context of the current observational situation. Because the scalar field density grows relative to the fluid, eventually the approximation that the fluid energy density is dominant will break down. When that happens, the Universe enters an inflationary regime, which has in fact been investigated in the early Universe context under the name ''intermediate inflation'' [10]. The expansion rate asymptotically becomes

$$R\propto \exp[t^{4/(4-\alpha)}], \tag{26}$$

and the fluid becomes less and less relevant. The inflationary regime may be preceded by a period of non-inflationary scalar field domination, if the scalar field comes to dominate while ϕ is sufficiently small.

The scalar field density grows with respect to the fluid regardless of whether the Universe is radiation or matter dominated; so these solutions do not exhibit a ''triggered'' transition into the inflationary regime. Rather, the timing of that transition is governed by the initial conditions, and for the domination to be a recent event, one has to arrange for the initial scalar field density to be well below the radiation density. The tuning is not however as severe as with a pure cosmological constant, since the redshifting of the scalar field may be quite similar to that of the fluid [3]. A particularly interesting case arises for $\alpha=-6$; such a scalar field will scale as matter during the radiation-dominated era, and then grow relative to matter, as $\rho_\phi\propto R^{-9/4}$, once the matter-dominated era begins. If the field is generated initially with $\rho_\phi\approx\rho_{\rm matter}\ll\rho_{\rm rad}$, then it will continue to evolve with $\rho_\phi\approx\rho_{\rm matter}$ until matter domination. The onset of matter domination then triggers a change in the evolution of the scalar field energy density, and ρ_ϕ begins to evolve in a manner close to a curvature density until it comes to dominate.

An interesting question is whether it might be possible to find inflationary scenarios capable of providing suitable initial conditions. A possible objection to the above is that in standard cosmological scenarios the energy density which today is in non-relativistic particles (especially the baryons) starts out as highly relativistic, only later to change its equation of state on cooling, rather than already existing as a trace amount in the early Universe. However, the scenario just outlined bears some similarity to suggestions for the creation of cold dark matter at the end of the inflationary epoch [11].

B. Positive power laws

Although positive power-law potentials are more commonly associated with driving an inflationary expansion, provided they satisfy Eq. (25), we then have shown that they too permit stable scaling solutions. In this case $m>n$, and so the scalar field becomes progressively less important as the evolution proceeds, better and better justifying the neglect of the scalar field terms in the Friedmann equation. The scaling solution for ϕ goes smoothly to zero as $t\rightarrow\infty$, without oscillations.

Note, however, that these potentials (for even α) can also support oscillatory behavior, with [12]

$$\rho_\phi\propto R^{-6\alpha/(\alpha+2)}. \tag{27}$$

Despite the power-law behavior, these solutions are not encompassed in our definition of scaling, as the scaling law arises only after averaging over oscillations, while within each oscillation, energy is continually being converted between potential and kinetic. When the field is oscillating, and Eq. (27) applies, the scaling of ρ_ϕ with R is independent of the equation of state of the dominant component of the density; this differs from the attractor solution in which the scaling of ρ_ϕ with R depends on m. Furthermore, such solutions apply to oscillating fields even when the scalar field density itself is dominant. In our phase diagram, Fig. 1, this oscillating solution corresponds to the regime in which the phase space trajectory winds around both attractors.

Whether the oscillatory behavior perseveres depends on the stability condition of Eq. (25), which if satisfied implies that the scaling behavior found in Sec. II is the attractor. The oscillating solution has an amplitude $\phi_{\rm max}\sim R^{-6/(\alpha+2)}$, which matches the redshift dependence of the singular scaling solution if the stability condition is saturated, leading to the closed loops seen in the middle panel of Fig. 1.

If the stability condition is satisfied, this can lead to some interesting behavior. Consider the limit where α is very large and positive in the radiation-dominated era. If initially $\rho_\phi\gg\rho_{\rm rad}$, but ϕ is oscillating rapidly, then the ϕ energy density will scale roughly as $\rho_\phi\propto 1/R^6$ and eventually fall below the radiation density. When this happens, however, the scaling behavior will take over and the ϕ energy density will scale as $1/R^{4+\epsilon}$ with $\epsilon\ll1$. We show this evolution with $\alpha=30$ in Fig. 2. Initially we have inflation, and then the field undergoes oscillations which are heavily dominated by the kinetic energy; so its energy falls off at nearly $1/R^6$, similar to kination [13]. Finally the scalar field becomes subdominant and stops oscillating, instead entering the scaling solution with $\rho_\phi\propto 1/R^{(4+2/7)}$. This scenario provides yet another ''natural'' mechanism to give a scaling solution with ρ_ϕ

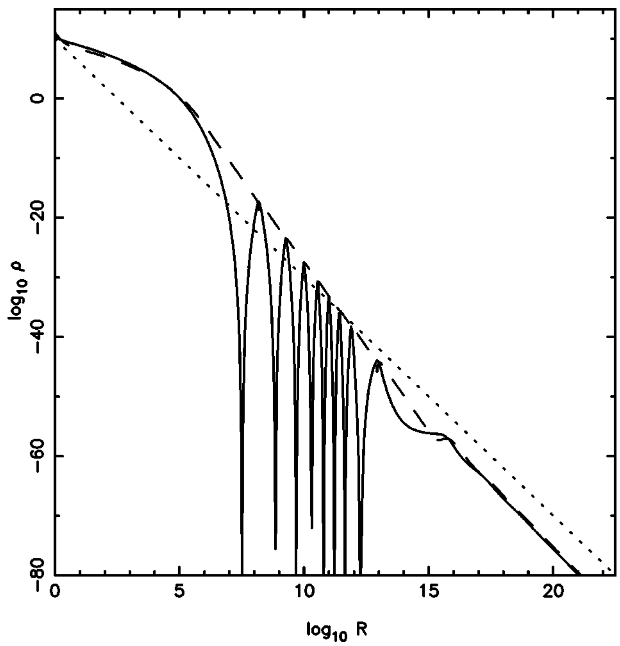

FIG. 2. The evolution of the energy densities in a radiation-dominated universe with $\alpha=30$. The vertical axis is in arbitrary units. The dotted line is the radiation energy density, and the solid line is the scalar field potential energy. The scalar field kinetic energy, shown as the dashed line, oscillates through zero out of phase with the potential energy, but does so too sharply for the plotting resolution.

roughly equal to the density of the dominant component, since the scalar field density drops rapidly relative to the radiation density until $\rho_\phi \approx \rho_{\rm rad}$, after which ρ_ϕ decreases at nearly the same rate as $\rho_{\rm rad}$. A similar behavior has also been noted for the case of negative power laws when the ϕ density is initially much larger than its attractor value [3]; the difference in the case discussed here is that the attractor is reached even when $\rho_\phi \gg \rho_{\rm rad}$ initially.

If the stability condition Eq. (25) is not satisfied, then the oscillations continue indefinitely as in the top panel of Fig. 1. Depending on parameters, the scalar field energy density may be either increasing or decreasing relative to the fluid energy density, e.g. for the choices in the figure, $\rho_{\rm rad} \propto 1/R^4$ while $\rho_\phi \propto 1/R^{4.8}$; so the scalar field becomes less and less important.

What happens for $0<\alpha \leqslant 2$? For $\alpha=1$ or 2 it is easy to find the exact solutions. First consider $\alpha=1$. Then Eq. (16) becomes

$$\ddot{\phi}+\frac{6}{m}\frac{1}{t}\dot{\phi}+1=0, \tag{28}$$

and the exact solution is

$$\phi=A+Bt^{1-6/m}-\frac{1}{2}\frac{m}{m+6}t^2, \tag{29}$$

where A and B are constants to be determined by initial conditions. This solution does not display scaling behavior, and, not surprisingly, $\phi \rightarrow -\infty$ as $t\rightarrow\infty$; so it is of little physical interest.

The $m=2$ case is more interesting. For this case, we get the linear equation

$$\ddot{\phi}+\frac{6}{m}\frac{1}{t}\dot{\phi}+\phi=0. \tag{30}$$

Taking $\phi=\theta t^{1/2-3/m}$, this equation reduces to

$$t^2\ddot{\theta}+t\dot{\theta}+t^2[1-(1/2-3/m)^2]\theta=0. \tag{31}$$

This is Bessel's equation of order $|1/2-3/m|$; so the general solution for ϕ is

$$\phi=t^{(1/2-3/m)}[AJ_\nu(t)+BN_\nu(t)], \tag{32}$$

where A and B are constants determined by the initial conditions, and J_ν and N_ν are Bessel functions of order ν, with $\nu=|1/2-3/m|$. In the limit of large t, the solutions in Eq. (32) all oscillate sinusoidally, with amplitude decaying as $t^{-3/m}$, and so $\rho_\phi \propto R^{-3}$. Thus, the solutions for this potential always oscillate, and the density scales as in Eq. (27).

Finally, some exact solutions exist for the case $m=3$ (matter domination), for which Eq. (16) reduces to the Lane-Emden equation. This equation can be solved exactly [14] for $\alpha=1$, 2 and 6, with the Lane-Emden boundary conditions corresponding to the initial condition $\dot{\phi}=0$ at $t=0$. The solutions for $\alpha=1$ and $\alpha=2$ are special cases of the solutions discussed above. The case $\alpha=6$ corresponds to the transition between attractor and non-attractor behavior and represents the analogue (for the matter-dominated case) of the $\alpha=10$ potential for the radiation-dominated universe shown in Fig. 1. The phase diagram in this case resembles the middle diagram in Fig. 1. However, the Lane-Emden solution is *not* the $u=\pm 1$ attractor solution; rather it corresponds to the unstable singular point at $u=0$. This arises because the Lane-Emden boundary conditions correspond to initial conditions which lie exactly on the singular point $u=0$, $p=0$, and the solution remains there as $t\rightarrow\infty$. The Lane-Emden boundary conditions are unphysical when applied to the scalar field evolution equation, since $t=0$ is undefined in the cosmological context.

C. ZWS potential

Zlatev *et al.* [3] made a detailed analysis of the rather unusual potential

$$V(\phi)\propto \exp\left(\frac{m_{\rm Pl}}{\phi}\right)-1. \tag{33}$$

This potential is introduced in recognition of the fact that simple power laws do not exhibit the ideal cosmological behavior, in that the scalar field density grows relative to matter during the matter-dominated era only if it also grows relative to radiation during the radiation-dominated era. While the fine-tuning problem of why the cosmological constant took so long to dominate is certainly less severe with these power-law potentials than with a pure Λ term, it still remains and one requires either an extremely low density in the scalar field at early times or, alternatively, to have n extremely close to m (i.e. very large positive or negative α) so that the scalar field requires a very long time to catch up with the conventional matter, say from an initial state of equipartition with a large number of fluid components. Further, the latter resolution, while superficially attractive, will fail for the same reasons that the exponential potential does, namely nucleosynthesis and structure formation; its dynamics are extremely close to the exponential case.

The purpose of a potential such as in Eq. (33) is to change the slope of the scalar field and, hence, alter the character of the scaling solutions with epoch. For $m_{\rm Pl}/\phi \ll 1$, the potential decreases more rapidly than any power law. This initial steepness guarantees $n \simeq m$, and the field is drawn to this approximate tracker behavior. Later, when $\phi \gg m_{\rm Pl}$, this potential asymptotically approaches the form $V(\phi) \propto 1/\phi$, with the scaling solution $n = m/3$, producing scalar field domination. This strategy certainly does yield the attractive observational consequences explored by Zlatev *et al.* [3]. The drawback is that the change of behavior is now governed by the form of the potential, and not primarily by the equation of state of the accompanying fluid. That the scalar field begins to change its behavior around the epoch of matter-radiation equality is because the feature of changing steepness in the potential has been placed in the appropriate place. This represents tuning of a different sort to the usual tuning of Λ models, but a tuning nonetheless.

IV. DISCUSSION

Our results indicate that exact solutions for the scalar field, which give scaling behavior when the expansion of the universe is driven by a dominant component with density $\rho_{\rm dominant}$, are possible for only three classes of potentials:

(1) Exponential potentials (ρ_ϕ scales as $\rho_{\rm dominant}$).
(2) Negative power-law potentials (ρ_ϕ decreases less rapidly than $\rho_{\rm dominant}$).
(3) Positive power-law potentials (ρ_ϕ decreases more rapidly than $\rho_{\rm dominant}$).

The first two cases have been extensively discussed elsewhere; the existence of the third class is our major new result. The negative power-law potentials $V \propto \phi^\alpha$ have attractor solutions for all values of the exponent α, while the positive power-law solutions require an exponent $\alpha > 2(6+m)/(6-m)$ for attractor behavior to occur.

Our results do have one practical limitation: we have confined our attention to exact solutions. It is certainly possible, for example, for approximate solutions to exist which are very close to scaling behavior, e.g., $\rho_\phi \propto R^{-n} f(R)$, where $f(R)$ is a slowly varying function of R. If $f(R)$ varies sufficiently slowly, then there may be no practical distinction between a solution of this type and our exact solutions. The particular potential of Zlatev *et al.* is of this type. It is not practical to systematically classify all such approximate scaling solutions, although it may be possible to provide conditions on $V(\phi)$ which allow for such solutions [15].

ACKNOWLEDGMENTS

A.R.L. was supported by the Royal Society, and R.J.S. by NASA (NAG 5-2788) at Fermilab and by the DOE at Fermilab and at Ohio State (DE-FG02-91ER40690). We thank Limin Wang for useful discussions. We thank the Fermilab astrophysics group for hospitality during the start of this work, and A.R.L. also thanks the Università di Padova. A.R.L acknowledges use of the Starlink computer system at the University of Sussex.

[1] S. Perlmutter *et al.*, Nature (London) **391**, 51 (1998); B. P. Schmidt *et al.*, Astrophys. J. (to be published), astro-ph/9805200.
[2] S. D. M. White, J. F. Navarro, A. E. Evrard and C. S. Frenk, Nature (London) **366**, 429 (1993).
[3] I. Zlatev, L. Wang and P. J. Steinhardt, astro-ph/9807002.
[4] F. Lucchin and S. Matarrese, Phys. Rev. D **32**, 1316 (1985); J. J. Halliwell, Phys. Lett. B **185**, 341 (1987); A. B. Burd and J. D. Barrow, Nucl. Phys. **B308**, 929 (1988).
[5] C. Wetterich, Nucl. Phys. **B302**, 668 (1988).
[6] D. Wands, E. J. Copeland and A. R. Liddle, Ann. (N.Y.) Acad. Sci. **688**, 647 (1993).
[7] P. G. Ferreira and M. Joyce, Phys. Rev. Lett. **79**, 4740 (1997); Phys. Rev. D **58**, 023503 (1998).
[8] E. J. Copeland, A. R. Liddle and D. Wands, Phys. Rev. D **57**, 4686 (1998).
[9] B. Ratra and P. J. E. Peebles, Phys. Rev. D **37**, 3406 (1988).
[10] A. G. Muslimov, Class. Quantum Grav. **7**, 231 (1990); J. D. Barrow, Phys. Lett. B **235**, 40 (1990).
[11] L. Kofman, A. Linde and A. Starobinsky, Phys. Rev. Lett. **73**, 3195 (1994); Phys. Rev. D **56**, 3258 (1997); D. H. J. Chung, E. W. Kolb and A. Riotto, *ibid.* **59**, 023501 (1999).
[12] M. S. Turner, Phys. Rev. D **28**, 1243 (1983).
[13] B. Spokoiny, Phys. Lett. B **315**, 40 (1993); M. Joyce and T. Prokopeć, Phys. Rev. D **57**, 6022 (1998).
[14] S. Chandrasekhar, *An Introduction to the Study of Stellar Structure* (University of Chicago Press, Chicago, 1939).
[15] L. Wang (private communication).