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SUBALGEBRAS OF THE ALGEBRA OF ALL COMPLEX-VALUED CONTINUOUS FUNCTIONS ON THE CIRCLE.*

By JOHN WERMER.[1]

1. **Introduction.** Let $\boldsymbol{R}$ be the algebra of all real valued continuous functions on the circle and let $\boldsymbol{C}$ be the algebra of all complex-valued continuous functions on the circle.

The subalgebras we are concerned with are assumed to contain the constant 1.

A fundamental theorem of Stone yields that any uniformly closed subalgebra $\boldsymbol{R}'$ of $\boldsymbol{R}$ which separates points (i. e which is such that if $\lambda_1 \neq \lambda_2$ then there exists an f in $\boldsymbol{R}'$ with $f(\lambda_1) \neq f(\lambda_2)$) coincides with $\boldsymbol{R}$.

For the algebra $\boldsymbol{C}$ the situation is quite different. There exists a large class of proper subalgebras of $\boldsymbol{C}$ which separate points. The problem of classifying these subalgebras leads to the following question: What are the *maximal* subalgebras of $\boldsymbol{C}$?

A closed proper subalgebra $\boldsymbol{M}$ *of* $\boldsymbol{C}$ separating points is called *maximal* if there exists no closed subalgebra $\boldsymbol{M}'$ with $\boldsymbol{M} \subseteq \boldsymbol{M}'$ and $\boldsymbol{M}' \neq \boldsymbol{M}$, $\boldsymbol{M}' \neq \boldsymbol{C}$.

In [1] and [2] the author has given examples of certain maximal subalgebras. Here we shall exhibit a large class of maximal subalgebras, associated with Riemann surfaces.

Let $\mathfrak{F}$ be a Riemann surface,[2] $\mathfrak{M}$ a region on $\mathfrak{F}$ bounded by a simple closed analytic curve γ, such that $\mathfrak{M} + \gamma$ is compact. $\mathfrak{M}$ then has finite genus p. Since γ is topologically a circle, we may regard $\boldsymbol{C}$ as the space of continuous complex-valued functions on γ. We shall use $\| f \|$ to denote $\max_{\lambda \varepsilon \gamma} | f(\lambda) |$.

Definition 1. $\mathfrak{A}$ is the subalgebra of $\boldsymbol{C}$ consisting of all f which may be continued into $\mathfrak{M}$ to be analytic on $\mathfrak{M}$ and continuous on $\mathfrak{M} + \gamma$.

By the maximum principle for $\mathfrak{M}$, $\mathfrak{A}$ is a closed subalgebra of $\boldsymbol{C}$. Also

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[1] I am indebted to Professor M. Heins for a number of valuable suggestions which I have used in some of the proofs.

[2] For definitions and basic facts of the theory of Riemann surfaces we refer the reader to R. Nevanlinna, "Uniformisierung," Springer Verlag, 1953.

$\mathfrak{A}$ separates points on γ, since, given p, q on γ, $p \neq q$, we can find some f in $\mathfrak{A}$ with $f(p) = 0$, $f(q) \neq 0$.

The main object of this paper is to prove:

THEOREM 2. *$\mathfrak{A}$ is a maximal subalgebra of $\boldsymbol{C}$.*

When $\mathfrak{F}$ is the plane and γ is the unit circle $|\lambda| = 1$, $\mathfrak{A}$ becomes the algebra generated by the functions 1 and λ. This is the case discussed in [1].

In Section 2 we prove Theorem 1 in which we give the form of the general linear functional on $\boldsymbol{C}$ which annihilates $\mathfrak{A}$. In Section 3 we use Theorem 1 to prove Theorem 2. In Section 4 we find when two of our maximal subalgebras are isomorphic.

2. Fix ζ in $\mathfrak{M}$; let G_ζ denote the Green's function for $\mathfrak{M}$ singular at ζ. Then G_ζ is harmonic in $\mathfrak{M}$ except at ζ; G_ζ vanishes on γ; for some fixed local parameter z at ζ, $G_\zeta(z) + \log|z - \zeta|$ is regular neighborhood of ζ.

Let H_ζ be the (multiple-valued) conjugate function of G_ζ. Since γ is an analytic curve, $G_\zeta + iH_\zeta$ is analytic everywhere on γ.

Set $W_\zeta(z) = -d\{G_\zeta(z) + iH_\zeta(z)\}/dz$. Then W_ζ is a "covariant" on $\mathfrak{M}$, as defined in [3], p. 102. W_ζ is analytic on $\mathfrak{M}$ except for a simple pole at ζ, with residue 1. On γ we denote by ω_ζ the measure $\frac{1}{2\pi i} W_\zeta(\lambda)\, d\lambda$. Then ω_ζ is the harmonic measure for $\mathfrak{M}$ evaluated at ζ. In particular, for any set E on γ, $\omega_\zeta(E)$ is real and non-negative.

Fix some ζ_0 in $\mathfrak{M}$. From now on we shall omit the subscript ζ_0 when writing G_{ζ_0}, H_{ζ_0}, W_{ζ_0} or ω_{ζ_0}. We note that W has no zero on γ.

LEMMA 1. *W has $2p$ zeros in $\mathfrak{M}$, where each zero is counted with its multiplicity.*

Proof. We choose an analytic parametrization of $\gamma: \lambda = \lambda(t)$, $0 \leqq t \leqq 1$. In any coordinate neighborhood U of a point of γ with local parameter z, we define $z(t) = z(\lambda(t))$. We can then consider $W(z(t)) \cdot z'(t)$ in $U \cap \gamma$, where the prime indicates differentiation with respect to t. Direct computation shows that this expression is independent of the choice of local parameter. Then

$$\Delta = \int_0^1 d/dt\{\log W(z(t))z'(t)\}dt$$

is well-defined. By a formula given in [3], p. 133, we have

$$\Delta = 2\pi i(B - A - N),$$

where B is the number of zeros of W in $\mathfrak{M}$, A the number of poles, and N the Euler characteristic of $\mathfrak{M}$. On the other hand

$$\Delta = \int_0^1 d/dt \log d/dt - \{G(z(t)) + iH(z(t))\} dt$$

and this equals the variation of $\log d - \{G(z(t)) + iH(z(t))\}/dt$ over $0 \leqq t \leqq 1$. The properties of G and H on γ then yield directly that $\Delta = 0$. Thus $B = A + N$. But $A = 1$ and $N = 2p - 1$. Hence $B = 2p$, as asserted.

Let $\alpha_1, \cdots, \alpha_{2p}$ be a homology basis of closed curves on $\mathfrak{M}$.

Definition 2. Let $u(\lambda)$ be a real continuous function on γ and let $U(\zeta)$ be the harmonic function on $\mathfrak{M}$ with $U \equiv u$ on γ. Then $\Phi_\nu(u)$, $\nu = 1, 2, \cdots, 2p$, denotes the period of the conjugate function of U corresponding to α_ν.

Definition 3. For $\mu = 1, \cdots, 2p$, $\psi_\mu(\zeta)$ is a harmonic function on $\mathfrak{M}$, continuous on $\mathfrak{M} + \gamma$ and twice differentiable on γ, with $\Phi_\nu(\psi_\mu) = \delta_\mu{}^\nu$.

LEMMA 2. *There exist functions K_i, $i = 1, \cdots, 2p$, meromorphic on $\mathfrak{M}$ and continuous and real valued on γ such that for $i = 1, \cdots, 2p$, $\Phi_i = K_i d\omega$ as functionals, i. e. $\Phi_i(u) = \int_\gamma u(\lambda) K_i(\lambda) d\omega(\lambda)$, all u, and such that $W(\zeta) K_i(\zeta)$ is analytic on $\mathfrak{M}$ for each i.*

Proof. Let W have the zeros $z_1, \cdots, z_k$ in $\mathfrak{M}$ of orders $\nu_1, \cdots, \nu_k$. By Lemma 1, $\sum_{i=1}^{i=k} \nu_i = 2p$. At z_i we use local polar coördinates (r_i, θ_i) for $i = 1, \cdots, k$.

For each i, $i = 1, \cdots, k$ a well-known construction yields us functions $u_j{}^i$, $j = 1, \cdots, \nu_i$ and $v_j{}^i$, $j = 1, \cdots, \nu_i$ such that:

(i) $u_j{}^i$, $v_j{}^i$ are harmonic on $\mathfrak{M}$ except at z_i.

(ii) $u_j{}^i$ has at z_i a pole with principal part $(r_i)^{-j} \cos j\theta_i$ and $v_j{}^i$ has at z_i a pole with principal part $(r_i)^{-j} \sin j\theta_i{}^j$.

(iii) $u_j{}^i$ and $v_j{}^i$ vanish identically on γ.

The $4p$ functions $u_j{}^i$, $v_j{}^i$ together form a linearly independent set. For suppose $\sum_{i,j} c_j{}^i u_j{}^i + d_j{}^i v_j{}^i = 0$, $c_j{}^i$, $d_j{}^i$ being constants. Because of the poles of the $u_j{}^i$, $v_j{}^i$ this implies $c_j{}^i u_j{}^i + d_j{}^i v_j{}^i = 0$, whence $c_j{}^i \cos j\theta_i + d_j{}^i \sin j\theta_i = 0$. Since θ_i is arbitrary, we conclude $c_j{}^i = d_j{}^i = 0$, all i, j. Thus linear independence is established. We now seek constants $a_j{}^i, b_j{}^i$, $j = 1, \cdots, \nu_i$, $i = 1, \cdots, 2p$, such that the function $\sum_{i,j} a_j^i u_j^i + b_j^i v_j^i$ have a single-valued conjugate function. This gives $2p$ conditions on $4p$ unknowns, and so we obtain at least $2p$ linearly independent $4p$-tuples satisfying the conditions. We thus get the linearly independent functions

$$q^\nu(\zeta) = \sum_{i,j} a_{j\nu}{}^i u_j{}^i(\zeta) + b_{j\nu}{}^i v_j{}^i(\zeta), \qquad \nu = 1, \cdots, 2p.$$

where each q^ν has a single-valued conjugate function. Then $q^\nu(\zeta)$ is harmonic on $\mathfrak{M}$ except for possible poles at the points z_i of orders $\leqq \nu_i$ and $q^\nu(\lambda) = 0$ on γ for all ν.

Let $r^\nu(\zeta)$ be the conjugate function of q^ν with $r^\nu(\zeta_0) = 0$. Then $h_\nu(\zeta) = i(q^\nu(\zeta) + ir^\nu(\zeta))$ is, for $\nu = 1, \cdots, 2p$, a meromorphic function on $\mathfrak{M}$ such that $h_\nu(\lambda)$ is real and continuous on γ for each ν. Suppose $\sum_{\nu=1}^{2p} c_\nu h_\nu = 0$, where $c_\nu = a_\nu + ib_\nu$, $(a_\nu, b_\nu$ real$)$. Since $h_\nu(\lambda)$ is real for λ and γ, this gives $\sum_{\nu=1}^{2p} a_\nu h_\nu(\lambda) = \sum_{\nu=1}^{2p} b_\nu h_\nu(\lambda) = 0$ for λ in γ. But $\sum_{\nu=1}^{2p} a_\nu h_\nu(\zeta)$ is meromorphic and so $\sum_{\nu=1}^{2p} a_\nu h_\nu(\zeta) = 0$, $\zeta \in \mathfrak{M}$. Similarly $\sum_{\nu=1}^{2p} b_\nu h_\nu(\zeta) = 0$ on $\mathfrak{M}$. Hence $\sum_{\nu=1}^{2p} a_\nu q^\nu(\zeta)$ $= \sum_{\nu=1}^{2p} b_\nu q^\nu(\zeta) = 0$ on $\mathfrak{M}$. But the q^ν are linearly independent by construction, whence $a_\nu = b_\nu = 0$, all ν. Hence $c_\nu = 0$, all ν. Hence the h_ν are linearly independent.

Consider now the covariant $h_\nu(\zeta) W(\zeta)$. This has no poles except possibly at ζ_0 since the zeros of W cancel the poles of h_ν. Also $h_\nu(\zeta_0) = iq^\nu(\zeta_0)$ and so $\frac{1}{2\pi i} \int_\gamma h_\nu(\lambda) W(\lambda) d\lambda = iq^\nu(\zeta_0)$ by the residue theorem. Now $h_\nu(\lambda)$ is real on γ, and $\frac{1}{2\pi i} W(\lambda) d\lambda$ is a real-valued measure on γ. Hence the left hand side is real. Hence $q^\nu(\zeta_0) = 0$ and so $h_\nu(\zeta_0) = 0$. It follows that $h_\nu(\zeta) W(\zeta)$ is regular at ζ_0 and so everywhere on $\mathfrak{M}$.

Let now N be the space of all real continuous functions u on γ with $\Phi_\nu(u) = 0$ for $\nu = 1, \cdots, 2p$. Given u in N choose v twice differentiable on γ with $\| u - v \| < \epsilon$. Then there is a constant K so that

$$| \Phi_\nu(v) | = | \Phi_\nu(v - u) | \leqq K\epsilon, \qquad \nu = 1, \cdots, 2p.$$

Set $w(\lambda) = v(\lambda) - \sum_{\nu=1}^{2p} \Phi_\nu(v) \psi_\nu(\lambda)$. Then w is differentiable on γ, $\Phi_j(w) = 0$, $j = 1, \cdots, 2p$ and

$$\| u - w \| \leqq \| u - v \| + \sum_{\nu=1}^{2p} |\Phi_\nu(v)| \, \| \psi_\nu \| < \epsilon + K'\epsilon = K''\epsilon, K''$$

independent of ϵ.

Let $w(\zeta)$ be the harmonic function with boundary value $w(\lambda)$. Since $\Phi_j(w)=0$, $j=1,\cdots,2p$, w has a single-valued conjugate w_1 and since w is twice differentiable on γ, w_1 is continuous in $\mathfrak{M}+\gamma$. By the preceding $h_i(\zeta)W(\zeta)$ is analytic on $\mathfrak{M}$ for each i. The residue theorem yields:

$$0=\int_\gamma (w(\lambda)+iw_1(\lambda))h_i(\lambda)W(\lambda)d\lambda,$$

all i. Since $h_i(\lambda)$ is real, we get

$$0=\int_\gamma w(\lambda)h_i(\lambda)W(\lambda)d\lambda=\int_\gamma w(\lambda)h_i(\lambda)dw(\lambda)$$

for all i. Now

$$\left|\int_\gamma u(\lambda)h_i(\lambda)dw(\lambda)\right|=\left|\int_\gamma (u(\lambda)-w(\lambda))h_i(\lambda)dw(\lambda)\right|$$
$$\leqq K''\epsilon\int_\gamma |h_i(\lambda)|\,dw(\lambda).$$

Since ϵ is arbitrary, we get

$$\int_\gamma u(\lambda)h_i(\lambda)dw(\lambda)=0,$$

all i. Thus the functional $h_i dw$ annihilates N. Hence, by elementary vector-space reasoning, there exist constants b_ν^i, $i=1,\cdots,2p$, $\nu=1,\cdots,2p$, with

$$h_i(\lambda)dw(\lambda)=\sum_{\nu=1}^{2p} b_\nu\Phi_\nu, \qquad i=1,\cdots,2p.$$

Since the h_i are linearly independent, we can solve this system of equations to get

$$\Phi_\nu=\sum_{i=1}^{2p} c_\nu^i h_i(\lambda)dw(\lambda)=K_\nu(\lambda)dw(\lambda), \qquad \nu=1,\cdots,2p.$$

The properties of the h_i established above yield that the K_i satisfy the assertions of the Lemma.

LEMMA 3. *Let μ be any complex-valued Borel measure on γ such that $\int_\gamma f(\lambda)d\mu(\lambda)=0$ whenever $f\in\mathfrak{A}$. Then for closed sets E on γ, $\omega(E)=0$ implies $\mu(E)=0$.*

The analogous assertion was proved for the unit circle by F. and M. Riesz in [4]. A slight modification of their argument yields the following proof.

Proof of Lemma 3. Since E is closed, the complement of E on γ is the union of countably many disjoint arcs γ_n. Since

$$\int_\gamma d\omega(\lambda) < \infty, \qquad \sum_{n=1}^{\infty} \int_{\gamma_n} d\omega(\lambda) < \infty.$$

Hence we can find a sequence of positive numbers d_n with d_n increasing to infinity with n, such that $\sum_{n=1}^{\infty} (\int_{\gamma_n} d\omega(\lambda)) \cdot d_n < \infty$. For $n = 1, 2, \cdots$ we define a positive real twice differentiable function g_n on γ_n such that $I_n = \int_{\gamma_n} g_n(\lambda) d\omega(\lambda) < \infty$ and $g_n(\lambda)$ increases on γ_n to ∞ as λ approaches the endpoints of γ_n. Choose positive constants c_n with $\sum_{n=1}^{\infty} c_n I_n < \infty$.

Set $P(\lambda) = c_n g_n(\lambda) + d_n$ for $\lambda \varepsilon \gamma_n$, $n = 1, 2, \cdots$. Since $\omega(E) = 0$, P is defined almost everywhere on γ with respect to ω and so with respect to ω_ζ for every ζ in $\mathfrak{M}$. Let now $P(\zeta) = \int_\gamma P(\lambda) d\omega_\zeta(\lambda)$, $\zeta \varepsilon \mathfrak{M}$. Note that

$$\int_\gamma P(\lambda) d\omega(\lambda) = \sum_{n=1}^{\infty} (c_n I_n + d_n \int_{\gamma_n} d\omega(\lambda)) < \infty.$$

From the way $P(\lambda)$ was constructed, we see that $P(\lambda)$ is continuous and finite at each point $\lambda \not\varepsilon E$ and $P(\lambda)$ becomes continuously $+\infty$ at each point of E.

By elementary properties of the harmonic measures ω_ζ, we get then that $P(\zeta)$ is harmonic in $\mathfrak{M}$ and has $P(\lambda)$ as continuous boundary function on γ, if we make the obvious definition of continuous approach to ∞ for $P(\zeta)$ as $\zeta \to \lambda$, λ in E. Let $k_1, \cdots, k_{2p}$ be the periods of the conjugate function of $P(\zeta)$. Choose a constant d so that $P_1 = d + P - \sum_{i=1}^{2p} k_i \psi_i > 0$ on γ, and hence on $\mathfrak{M}$ and let Q_1 be the (single-valued) conjugate function of P_1. Set now $k(\zeta) = (1 + P_1 + iQ_1)^{-1} \cdot (P_1 + iQ_1)(\zeta)$. Then $k(\zeta)$ is analytic in $\mathfrak{M}$. For $\lambda \varepsilon \gamma$ and $\lambda \not\varepsilon E$, $k(\lambda) = (1 + x + iy)^{-1} \cdot (x + iy)$ with $0 < x < \infty$ and hence $|k(\lambda)| < 1$. Let $\lambda \varepsilon E$. As $\zeta \to \lambda$ $|P_1(\zeta) + iQ_1(\zeta)| \to \infty$, whence $k(\zeta) \to 1$. Thus $k(\lambda) = 1$ on E. In particular $k(\lambda)$ is continuous in $\mathfrak{M} + \gamma$, and so $k \varepsilon \mathfrak{A}$. Hence $k^n \varepsilon \mathfrak{A}$ for $n = 1, 2, \cdots$. Then

$$0 = \int_\gamma k^n(\lambda) d\mu(\lambda) = \int_E d\mu(\lambda) + \int_{\gamma - E} k^n(\lambda) d\mu(\lambda).$$

Letting $n \to \infty$ and recalling that $|k(\lambda)| < 1$ if $\lambda \varepsilon \gamma - E$, we conclude that $0 = \int_E d\mu(\lambda) = \mu(E)$, as asserted.

Definition 4. We denote by $L^p(\gamma)$ the class of functions $F(\lambda)$ on γ measurable with respect to ω and with $\int_\gamma |F(\lambda)|^p d\omega(\lambda) < \infty$.

COROLLARY. *If μ is a Borel-measure on γ such that $\int_\gamma f(\lambda)d\mu(\lambda) = 0$ for all $f \varepsilon \mathfrak{A}$, then there exists $F(\lambda) \varepsilon L^1(\gamma)$ such that $d\mu(\lambda) = F(\lambda)d\omega(\lambda)$ as measures on γ.*

Proof. By Lemma 3, $\omega(E) = 0$ implies $\mu(E) = 0$ for any closed set E. It follows that this implication holds for each Borel set E.

We can write $\mu = \mu^+ - \mu^- + i\nu^+ - i\nu^-$ where μ^+, μ^-, ν^+, ν^- are real non-negative measures. Let now E be any Borel set with $\omega(E) = 0$. For every Borel subset E' of E, $\omega(E') = 0$ and so $\mu(E') = 0$. Hence $\mu^+(E) = \mu^-(E) = \nu^+(E) = \nu^-(E) = 0$. Thus μ^+, etc. are all absolutely continuous with respect to ω. It follows by the Radon-Nikodym theorem, that $d\mu^+(\lambda) = F_1(\lambda)d\omega(\lambda)$, where $F_1 \varepsilon L^1(\gamma)$. Similiarly $d\mu^-(\lambda) = F_2(\lambda)d\omega(\lambda)$, $F_2 \varepsilon L^1(\gamma)$ and so on. Adding these equations we get the assertion.

Let now γ' be a simple closed analytic curve in $\mathfrak{M}$ such that γ and γ' together bound an annular subregion $\mathfrak{M}'$ of $\mathfrak{M}$. We choose γ' so that all zeros of W lie outside $\mathfrak{M}' + \gamma'$. We can then map $\mathfrak{M}'$ conformally onto the annulus $r' < |z| < 1$ in the plane, by a mapping $z = \chi(\zeta)$, $\zeta \varepsilon \mathfrak{M}'$. Since γ, γ' are analytic curves, χ is analytic on the boundary curves γ and γ'. It follows that for a fixed K, and each Borel set E on γ,

$$\frac{1}{K}\omega(E) \leqq m(\chi(E)) \leqq K\omega(E),$$

where m denotes Lebesgue measure on $|z| = 1$.

Let $F(\zeta)$ be analytic on $\mathfrak{M}'$. Then $F^o(z) = F(\chi^{-1}(z))$ is analytic in the annulus $r' < |z| < 1$. We shall omit the symbol "^{o}," since this omission introduces no ambiguity. Also for $g(\lambda)$ defined on γ, we write $g(e^{i\theta})$ for $g(\chi^{-1}(e^{i\theta}))$.

Definition 5. Let F be analytic in $\mathfrak{M}$. We say $F \varepsilon \mathfrak{H}'$, provided that (with the notations just given)

$$\int_0^{2\pi} |F(re^{i\theta})|\, d\theta = O(1) \quad as \quad r \to 1.$$

LEMMA 4. *Let $F(\zeta) \varepsilon \mathfrak{H}'$. Then there exists a function $F^*(\lambda)$ defined on γ* a. e.—$d\omega$ *such that*

(a) $F^* \varepsilon L^1(\gamma)$.

(b) $F(\zeta) = \frac{1}{2\pi i} \int_\gamma F^*(\lambda) W_\zeta(\lambda) d\lambda$, *all* $\zeta \varepsilon \mathfrak{M}$.

(c) $\lim_{\zeta \to \lambda} F(\zeta) = F^*(\lambda)$ a. e.—$d\omega$ *on* γ, *if* $\zeta \to \lambda$ *within some sector.*

(d) *Fix* $r_1 > r'$. *Then for some constant* K *independent of* F *and* r *we have for* $r_1 < r < 1$:

$$\int_0^{2\pi} | F(re^{i\theta}) | \, d\theta \leqq K \int_0^{2\pi} | F^*(e^{i\theta}) | \, d\theta.$$

Proof. By hypothesis $\int_0^{2\pi} | F(re^{i\theta}) | \, d\theta = O(1)$ as $r \to 1$, and $F(z)$ is analytic for $r' < | z | < 1$. We may write $F = F_1 + F_2$ where F_1 is analytic in $| z | < 1$ and F_2 is analytic in $| z | > r'$. We hence get

$$\int_0^{2\pi} | F_1(re^{i\theta}) | \, d\theta = O(1) \text{ as } r \to 1.$$

Classical results now give that $F_1^*(e^{i\theta}) = \lim_{z \to e^{i\theta}} F_1(z)$ exists for a. a. θ if the approach to the boundary lies within some sector, and that $F_1^*(e^{i\theta})$ is summable on $0 \leqq \theta < 2\pi$. Hence $\lim_{z \to e^{i\theta}} F(z)$ exists a. e. We denote it by $F^*(e^{i\theta})$; for λ on γ we write $F^*(\lambda)$ instead of $F^*(\chi(\lambda))$. Since sets of ω-measure 0 on γ correspond to sets of Lebesgue measure 0 on $| z | = 1$, we so get assertion (c).

Let now $r' < r < 1$ and let γ_r be the curve in $\mathfrak{M}'$ which χ maps into the circle $| z | = r$. The residue theorem gives for $\zeta \varepsilon \mathfrak{M}$, ζ outside the region bounded by γ_r and γ:

$$F(\zeta) = \frac{1}{2\pi i} \int_{\gamma_r} F(\lambda) W_\zeta(\lambda) d\lambda = \frac{1}{2\pi} \int_0^{2\pi} F(re^{i\theta}) W_\zeta(re^{i\theta}) re^{i\theta} d\theta.$$

By a classical theorem, $\lim_{r \to 1} \int_0^{2\pi} | F_1(re^{i\theta}) - F_1^*(e^{i\theta}) | \, d\theta = 0$; also $W_\zeta(re^{i\theta})$ is continuous for $r' < r \leqq 1$. It follows that

$$F(\zeta) = \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} F(re^{i\theta}) W_\zeta(re^{i\theta}) re^{i\theta} \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} F^*(e^{i\theta}) W_\zeta(e^{i\theta}) e^{i\theta} \, d\theta.$$

Hence $F(\zeta) = \frac{1}{2\pi i} \int_\gamma F^*(\lambda) W_\zeta(\lambda) d\lambda$, $\zeta \varepsilon \mathfrak{M}$. Thus (b) is proved.

Now

$$\int_\gamma | F^*(\lambda) | \, d\omega(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} | F^*(e^{i\theta}) | \, W(e^{i\theta}) e^{i\theta} \, d\theta < \infty,$$

since $\int_0^{2\pi} |F^*(e^{i\theta})|\, d\theta < \infty$ by construction of F^*, and W is continuous. Thus (a) holds. Finally, write $F(z) = F_1(z) + F_2(z)$, where $F_1(z)$ is analytic in $|z| < 1$ and $F_2(z) = -\frac{1}{2\pi i}\int_{|\tau|=r'} F(\tau)(\tau - z)^{-1} d\tau$; so that F_2 is analytic in $|z| > r'$. Let $\zeta = \chi^{-1}(\tau)$. Then $F(\tau) = \int_\gamma F^*(\lambda)(W(\lambda))^{-1} W_\zeta(\lambda) d\omega(\lambda)$ by (b), whence $|F(\tau)| \leqq M_\zeta \int_\gamma |F^*(\lambda)|\, d\omega(\lambda)$, where $M_\zeta = \max_{\lambda \varepsilon \gamma} |W^{-1}(\lambda) W_\zeta(\lambda)|$. Hence for $0 \leqq \theta < 2\pi$, $r' < r$,

$$|F_2(re^{i\theta})| \leqq 1/(r - r') \max_{|\tau|=r'} |F(\tau)| \leqq (M/2\pi) 1/(r - r') \int_\gamma |F^*(\lambda)|\, d\omega(\lambda),$$

where $M = \sup_{\zeta \varepsilon \gamma'} M_\zeta$. At last, $r' < r_1 \leqq r$,

$$\int_0^{2\pi} |F_2(re^{i\theta})|\, d\theta \leqq \int_\gamma |F^*(\lambda)|\, d\omega(\lambda) \cdot M_1 \leqq M_2 \cdot \int_0^{2\pi} |F^*(e^{i\theta})|\, d\theta,$$

where M_1 and M_2 are constants. On the other hand, since F_1 is analytic in $|z| < 1$,

$$\int_0^{2\pi} |F_1(re^{i\theta})|\, d\theta \leqq \int_0^{2\pi} |F_1^*(e^{i\theta})|\, d\theta \leqq \int_0^{2\pi} |F^*(e^{i\theta})|\, d\theta + \int_0^{2\pi} |F_2^*(e^{i\theta})|\, d\theta$$
$$\leqq (1 + M_2) \int_0^{2\pi} |F^*(e^{i\theta})|\, d\theta.$$

Hence

$$\int_0^{2\pi} |F(re^{i\theta})|\, d\theta = \int_0^{2\pi} |F_1(re^{i\theta}) + F_2(re^{i\theta})|\, d\theta$$
$$\leqq (1 + 2M_2) \int_0^{2\pi} |F^*(e^{i\theta})|\, d\theta.$$

This proves (d).

Definition 6. $\mathfrak{B}$ is the conjugate space of $\boldsymbol{C}$.

By the representation theorem of F. Riesz, $\mathfrak{B}$ may be identified with the space of all complex-valued Borel-measures on γ.

Definition 7. $\mathfrak{W}$ is the subspace of $\mathfrak{B}$ consisting of all measures μ of the form $d\mu(\lambda) = G^*(\lambda) d\omega(\lambda)$ where G^* is the boundary function of some G in $\mathfrak{H}'$ with $G(\zeta_0) = 0$.

LEMMA 5.[3] *$\mathfrak{W}$ is regularly closed as subspace of $\mathfrak{B}$.*

[3] The idea of using a lemma of this kind resulted from a conversation with Professor S. Kakutani.

Proof. By a theorem of Banach, [5], p. 124, it suffices to show that with each weakly convergent sequence of elements of $\mathfrak{W}$ the limit again is in $\mathfrak{W}$.

Let now $\mu_n \varepsilon \mathfrak{W}$, μ_n converge weakly to μ. By Definition 7, there exists $G_n \varepsilon \mathfrak{H}'$, $G_n(\zeta_0) = 0$ with $d\mu_n(\lambda) = G_n^*(\lambda) d\omega(\lambda)$. Then for each $f \varepsilon \mathfrak{A}$, $f \cdot G_n \varepsilon \mathfrak{H}'$, and so by Lemma 4,

$$0 = f(\zeta_0) G_n(\zeta_0) = \int_\gamma f(\lambda) G_n^*(\lambda) d\omega(\lambda) = \int_\gamma f(\lambda) d\mu_n(\lambda).$$

For $f \varepsilon \mathfrak{A}$, then, $0 = \int_\gamma f(\lambda) d\mu(\lambda)$. By the Corollary to Lemma 3, this implies that there exists $G_0 \varepsilon L^1(\gamma)$ with $G_0(\lambda) d\omega(\lambda) = d\mu(\lambda)$.

We shall show that G_n converges to a function G analytic on $\mathfrak{M}$ with $G \varepsilon \mathfrak{H}'$ and that $G^*(\lambda) = G_0(\lambda)$ a.e.—$d\omega$ on γ. From this it follows that $d\mu(\lambda) = G^*(\lambda) d\omega(\lambda)$ and so $\mu \varepsilon \mathfrak{W}$. Now

$$G_n(\zeta) = \frac{1}{2\pi i} \int_\gamma G_n^*(\lambda) W_\zeta(\lambda) d\lambda = \int_\gamma (W(\lambda))^{-1} W_\zeta(\lambda) d\mu_n(\lambda).$$

Hence $G(\zeta) = \operatorname{Lim}_{n\to\infty} G_n(\zeta)$ exists for $\zeta \varepsilon \mathfrak{M}$. Also

$$|G_n(\zeta)| \leqq M_\zeta \int_\gamma |G_n^*(\lambda)| d\omega(\lambda),$$

where $M_\zeta = \max_{\lambda \varepsilon \gamma} |(W(\lambda))^{-1} W_\zeta(\lambda)|$. Now since the sequence μ_n converges weakly, the total variation of μ_n has a bound K valid for all n. Hence

$$(1) \qquad \int_\gamma |G_n^*(\lambda)| d\omega(\lambda) < K.$$

Also M_ζ is bounded on each compact subset of $\mathfrak{M}$. Hence by Vitali's theorem, G is analytic on $\mathfrak{M}$ and $\operatorname{Lim}_{n\to\infty} G_n = G$ uniformly on each compact subset of $\mathfrak{M}$. Now

$$\int_\gamma |G_n^*(\lambda)| d\omega(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} |G_n^*(e^{i\theta})| W(e^{i\theta}) e^{i\theta} d\theta.$$

But $W(e^{i\theta}) e^{i\theta}$ has a positive lower bound on $(0, 2\pi)$. Hence $\int_0^{2\pi} |G_n^*(e^{i\theta})| d\theta < K'$, all n, by (1). Hence $\int_0^{2\pi} |G_n(re^{i\theta})| d\theta < K''$, all n, by (d) of Lemma 4, $r' < r_1 < r < 1$. It follows that $\int_0^{2\pi} |G(re^{i\theta})| d\theta < K''$, whence $G \varepsilon \mathfrak{H}'$. We claim that $G^*(\lambda) = G_0(\lambda)$ a.e.—$d\omega$.

Let now $r' < r < 1$. Set

$$U_n^1(z) = \frac{1}{2\pi i}\int_{|\tau|=r'} (\tau - z)^{-1} G_n(\tau)\, d\tau, \quad |z| > r'$$

$$U_n^2(z) = \frac{1}{2\pi i}\int_{|\tau|=r} (\tau - z)^{-1} G_n(\tau)\, d\tau, \quad |z| < r.$$

Then U_n^1 is analytic for $|z| > r'$, U_n^2 for $|z| < 1$, and

$$G_n(z) = U_n^2(z) - U_n^1(z), \quad r' < |z| < 1.$$

Clearly $U_n^{1*}(e^{i\theta}) = \lim_{z \to e^{i\theta}} U_n^1(z)$ exists for all θ. Also $G_n^*(e^{i\theta})$ exists a. e. Hence $U_n^{2*}(e^{i\theta})$ exists a. e. and $G_n^* = U_n^{2*} - U_n^{1*}$, a. e.

Now $G_n(\tau) \to G(\tau)$ uniformly on $|\tau| = r'$. Hence $U_n^1(z) \to U^1(z)$ uniformly for $r' < a \leqq |z| \leqq b < \infty$, and $U^{1*}(e^{i\theta})$ exists everywhere. Hence $U_n^2(z) \to U^2(z)$ uniformly in $r' < a \leqq |z| \leqq b < 1$, and $G(z) = U^2(z) - U^1(z)$. It follows that

$$\int_0^{2\pi} |U^2(re^{i\theta})|\, d\theta = O(1) \text{ as } r \to 1. \tag{2}$$

Also

$$G^*(e^{i\theta}) = U^{2*}(e^{i\theta}) - U^{1*}(e^{i\theta}) \text{ a. e.} \tag{3}$$

Fix $r, \phi; r < 1, 0 \leqq \phi < 2\pi$. Set $g(\theta) = (1 - r^2)(1 + r^2 - 2r\cos(\theta - \phi))^{-1}$. Now

$$\frac{1}{2\pi}\int_0^{2\pi} U_n^{2*}(e^{i\theta}) g(\theta)\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} (G_n^*(e^{i\theta}) + U_n^{1*}(e^{i\theta})) g(\theta)\, d\theta$$

and $U_n^{1*}(e^{i\theta}) \to U^{1*}(e^{i\theta})$ uniformly in $0 \leqq \theta < 2\pi$ and $G_n^*(\lambda)\, d\omega(\lambda)$ converges weakly to $G_0(\lambda)\, d\omega(\lambda)$. Hence

$$\lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} U_n^{2*}(e^{i\theta}) g(\theta)\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} (G_0(e^{i\theta}) + U^{1*}(e^{i\theta})) g(\theta)\, d\theta.$$

On the other hand,

$$U_n^2(re^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi} U_n^{2*}(e^{i\theta}) g(\theta)\, d\theta \to U^2(re^{i\phi}).$$

By (2) we get, since U^2 is analytic in $|z| < 1$,

$$U^2(re^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi} U^{2*}(e^{i\theta}) g(\theta)\, d\theta.$$

Hence

$$\int_0^{2\pi} U^{2*}(e^{i\theta}) g(\theta)\, d\theta = \int_0^{2\pi} (G_0(e^{i\theta}) + U^1(e^{i\theta})) g(\theta)\, d\theta,$$

or

$$0=\int_0^{2\pi}\{U^{2*}-G_0-U^{1*}\}(e^{i\theta})(1-r^2)(1+r^2-2r\cos(\theta-\phi))^{-1}d\theta.$$

This now holds for arbitrary r, ϕ. Since $U^{2*}-G_0-U^{1*}$ is summable on $(0,2\pi)$, we conclude

$$G_0(e^{i\theta})=U^{2*}(e^{i\theta})-U^{1*}(e^{i\theta})=G^*(e^{i\theta}), \text{ a.e.}$$

The conclusion now follows, as shown above.

THEOREM 1. *Let* $\mu_0\,\varepsilon\,\mathfrak{B}$ *and* $\int_\gamma f(\lambda)\,d\mu_0(\lambda)=0$ *for all* f *in* $\mathfrak{A}$. *Then there exists* $J\,\varepsilon\,\mathfrak{H}'$, $J(\zeta_0)=0$, *and constants* c_i *such that, setting*

$$L(\lambda)=J^*(\lambda)+\sum_{i=1}^{2p}c_iK_i(\lambda),$$

where K_i *are the functions of Lemma 2, we have* $d\mu_0(\lambda)=L(\lambda)\,d\omega(\lambda)$.

Proof. Let $\mathfrak{W}'$ be the vector-space obtained by adjoining to $\mathfrak{W}$ the measures $K_i(\lambda)\,d\omega(\lambda)$, $i=1,\cdots,2p$. Since $\mathfrak{W}$ is regularly closed, the same is true of $\mathfrak{W}'$. Our assertion amounts to the statement that $\mu_0\,\varepsilon\,\mathfrak{W}'$.

Suppose $\mu_0\not\varepsilon\,\mathfrak{W}'$. Since $\mathfrak{W}'$ is regularly closed, it follows by Banach's definition, that for some f_0 in $\boldsymbol{C}$

$$\int_\gamma f_0(\lambda)\,d\mu_0(\lambda)\neq 0 \tag{4}$$

$$\int_\gamma f_0(\lambda)\,d\mu(\lambda)=0 \text{ if } \mu\,\varepsilon\,\mathfrak{W}'. \tag{5}$$

Let now $\bar{\mathfrak{A}}$ be the closure in $L^2(\gamma)$ of $\mathfrak{A}$. Then we can decompose f_0 as follows: $f_0=H+G$, $H\,\varepsilon\,\bar{\mathfrak{A}}$, G orthogonal to $\bar{\mathfrak{A}}$. Let $f\,\varepsilon\,\mathfrak{A}$. Then

$$\int_\gamma \bar{f}(\lambda)G(\lambda)\,d\omega(\lambda)=0. \tag{6}$$

Hence

$$\int_\gamma G(\lambda)\,d\omega(\lambda)=0. \tag{6'}$$

Let now $H_n\,\varepsilon\,\mathfrak{A}$, $H_n\rightarrow H$ in the norm of $L^2(\gamma)$. Then by the residue theorem

$$\frac{1}{2\pi i}\int_\gamma H_n(\lambda)(f(\lambda)-f(\zeta_0))\,d\omega(\lambda)=0,$$

whence

$$\int_\gamma H(\lambda)(f(\lambda)-f(\zeta_0))\,d\omega(\lambda)=0. \tag{7}$$

Also $(f(\lambda)-f(\zeta_0))d\omega(\lambda)\,\varepsilon\,\mathfrak{W}'$, whence by (5)

$$\int_\gamma f_0(\lambda)(f(\lambda)-f(\zeta_0))d\omega(\lambda)=0.$$

Hence from $f_0=H+G$ and (7),

$$\int_\gamma G(\lambda)(f(\lambda)-f(\zeta_0))d\omega(\lambda)=0.$$

By (6′), then,

$$\int_\gamma G(\lambda)f(\lambda)d\omega(\lambda)=0. \tag{8}$$

It follows from (6) and (8) that $\int_\gamma u(\lambda)G(\lambda)d\omega(\lambda)=0$ for all real continuous functions u on γ with $u=\operatorname{Re}f$, for some $f\,\varepsilon\,\mathfrak{A}$. As in the proof of Lemma 2, we get from this that for some b_ν, $G(\lambda)d\omega(\lambda)=\sum_{\nu=1}^{2p}b_\nu\Phi_\nu$ as functionals, and hence that

$$G(\lambda)d\omega(\lambda)=\sum_{i=1}^{2p}c_iK_i(\lambda)d\omega(\lambda) \tag{9}$$

where K_i are the functions constructed in Lemma 2. Now

$$\int_\gamma H(\lambda)K_\nu(\lambda)d\omega(\lambda)=0, \qquad \nu=1,\cdots,2p;$$

also $K_\nu(\lambda)d\omega(\lambda)\,\varepsilon\,\mathfrak{W}'$. Hence by (5),

$$\int_\gamma f_0(\lambda)K_\nu(\lambda)d\omega(\lambda)=0, \qquad \nu=1,\cdots,2p.$$

Hence

$$\int_\gamma G(\lambda)K_\nu(\lambda)d\omega(\lambda)=0, \qquad \nu=1,\cdots,2p. \tag{10}$$

By (9) and (10),

$$\int_\gamma |G(\lambda)|^2d\omega(\lambda)=\int_\gamma G(\lambda)\sum_{i=1}^{2p}\bar{c}_iK_i(\lambda)d\omega(\lambda)$$
$$=\sum_{i=1}^{2p}\bar{c}_i\int_\gamma G(\lambda)K_i(\lambda)d\omega(\lambda)=0.$$

Hence $G(\lambda)=0$ a.e. and so $f_0=H$ a.e.

Now consider $H_n\,\varepsilon\,\mathfrak{A}$, $H_n\to H$ in the norm of $L^2(\gamma)$. Then for $\zeta\,\varepsilon\,\mathfrak{M}$,

$$H_n(\zeta)=\frac{1}{2\pi i}\int_\gamma H_n(\lambda)W_\zeta(\lambda)d\lambda.$$

It then follows that

$$\frac{1}{2\pi i}\int_\gamma f_0(\lambda)W_\zeta(\lambda)\,d\lambda=\lim_{n\to\infty}H_n(\zeta)$$

is analytic in $\mathfrak{M}$. Also $\frac{1}{2\pi i}\int_\gamma f_0(\lambda)W_\zeta(\lambda)\,d\lambda$ has f_0 as continuous boundary value on γ. Hence $f_0\,\varepsilon\,\mathfrak{A}$. Then $\int_\gamma f_0(\lambda)\,d\mu_0(\lambda)=0$. This contradicts (4). Hence the assertion must be true.

3. Proof of Theorem 2. Let $\mathfrak{A}'$ be a closed subalgebra of $\boldsymbol{C}$ with $\mathfrak{A}'\neq\boldsymbol{C}$ and $\mathfrak{A}\subseteq\mathfrak{A}'$. Since $\mathfrak{A}'$ is a proper closed subspace of $\boldsymbol{C}$, a well-known theorem on Banach spaces guarantees the existence of a non-zero functional on $\boldsymbol{C}$ which annihilates $\mathfrak{A}'$. Thus there exists $\mu\,\varepsilon\,\mathfrak{B}$, $\mu\neq 0$ with $\int_\gamma g(\lambda)\,d\mu(\lambda)=0$ if $g\,\varepsilon\,\mathfrak{A}'$. This holds in particular if $g\,\varepsilon\,\mathfrak{A}$. Hence by Theorem 1, $d\mu(\lambda)=L_0(\lambda)\,d\omega(\lambda)$ where L_0 is meromorphic on $\mathfrak{M}$ and analytic on $\mathfrak{M}$ except at the poles of the K_i. Hence L_0 is analytic except at the points $z_1,\cdots,z_k$ where W vanishes. Also, $\lim_{\zeta\to\lambda}L_0(\zeta)$ exists for a. a. λ on γ, if $\zeta\to\lambda$ within some sector, and this limit $\neq 0$ a. e. on γ.

Fix now $\phi\,\varepsilon\,\mathfrak{A}'$. We shall show $\phi\,\varepsilon\,\mathfrak{A}$. For if $f\,\varepsilon\,\mathfrak{A}$, $f(\lambda)\phi^m(\lambda)\,\varepsilon\,\mathfrak{A}'$ for $m=1,2,\cdots$ whence $\int_\gamma f(\lambda)\phi^m(\lambda)\,d\mu(\lambda)=0$. Applying Theorem 1 to the measures $\phi^m(\lambda)\,d\mu(\lambda)$, we get $\phi^m(\lambda)\,d\mu(\lambda)=L_m(\lambda)\,d\omega(\lambda)$ where L_m has the same analyticity and boundary behavior as L_0. Hence $\phi^m(\lambda)L_0(\lambda)=L_m(\lambda)$ a. e. on γ. It follows that $(L_1(\lambda))^m=L_m(\lambda)(L_0(\lambda))^{m-1}$ a. e. on γ. On both sides we have non-tangential boundary values of functions analytic in the region $\mathfrak{M}_0$ obtained by deleting from $\mathfrak{M}$ the points $z_1,\cdots,z_k$. By a result of Lusin and Privaloff, [6], an analytic function possessing non-tangential boundary values on a set of positive measure is determined by these values. Hence $(L_1(\zeta))^m=L_m(\zeta)(L_0(\zeta))^{m-1}$ for ζ in $\mathfrak{M}_0$. Since this is true for all $m\geqq 1$, L_0 cannot have a zero at any point ζ' in $\mathfrak{M}_0$ of order α unless L_1 has at ζ' a zero of order $\geqq\alpha$. Hence $L_0^{-1}L_1$ is analytic in $\mathfrak{M}_0$. Also, since $\phi(\lambda)L_0(\lambda)=L_1(\lambda)$ a. e. on γ, ϕ is the non-tangential limit of $L_0^{-1}L_1$ a. e. on γ.

Set $T\phi(\zeta)=L_0^{-1}(\zeta)L_1(\zeta)$. The map $\phi\to T\phi$ then assigns to each ϕ in $\mathfrak{A}'$ an analytic function $T\phi$ on $\mathfrak{M}_0$ having boundary values $\phi(\lambda)$. By the theorem in [6] mentioned above, ϕ determines $T\phi$. Let now ϕ_1, ϕ_2 belong to $\mathfrak{A}'$. Then

$$\operatorname*{Lim}_{\zeta\to\lambda}T\phi_1(\zeta)\cdot T\phi_2(\zeta)=\phi_1(\lambda)\phi_2(\lambda)$$

and so $T(\phi_1 \cdot \phi_2) = T\phi_1 \cdot T\phi_2$. Similarly $T(\phi_1 + \phi_2) = T\phi_1 + T\phi_2$. Fix now z_0 in $\mathfrak{M}_0$. Then the map $\phi \to T\phi(z_0)$ is a multiplicative functional defined on $\mathfrak{A}'$. But a multiplicative functional on a Banach algebra is always bounded and has bound 1. Hence $|T\phi(z_0)| \leqq \|\phi\|$. Since z_0 is an arbitrary point in $\mathfrak{M}_0$, $T\phi$ is then bounded on $\mathfrak{M}_0$; hence $T\phi$ is analytic and bounded on $\mathfrak{M}$. Lemma 4 gives now that for ζ in $\mathfrak{M}$,

$$T\phi(\zeta) = \int_\gamma (T\phi)^*(\lambda)\, d\omega_\zeta(\lambda) = \int_\gamma \phi(\lambda)\, d\omega_\zeta(\lambda).$$

On the other hand the last integral represents a continuous function on $\mathfrak{M} + \gamma$ agreeing with $\phi(\lambda)$ on γ. Hence ϕ is in $\mathfrak{A}$, as asserted.

Hence $\mathfrak{A}' = \mathfrak{A}$, and so Theorem 2 is established.

4. (Added November 27, 1954.) Let now $\mathfrak{F}_1$, $\mathfrak{F}_2$ be Riemann surfaces, $\mathfrak{M}_1$, $\mathfrak{M}_2$ regions on them bounded by simple closed analytic curves γ_1, γ_2 with $\mathfrak{M}_i \cup \gamma_i$ compact, $i = 1, 2$. Let $\mathfrak{A}_i$ be the algebra of functions continuous on γ_i and extendable to be analytic on $\mathfrak{M}_i$, $i = 1, 2$. We assert:

THEOREM 3. *$\mathfrak{A}_1$ is isomorphic to $\mathfrak{A}_2$ as algebra if and only if $\mathfrak{M}_1$ is conformally equivalent to $\mathfrak{M}_2$.*

We need the following:

LEMMA. *If χ is a multiplicative functional on $\mathfrak{A}_i$, then there exists a point $p \varepsilon \mathfrak{M}_i \cup \gamma_i$ with $\chi(f) = f(p)$, all $f \varepsilon \mathfrak{A}_i$.*

Proof.[4] (We omit the subscript i from $\mathfrak{A}_i$, etc.) By the general representation theorem for bounded linear functions on spaces of continuous functions, there is a measure μ_0 on γ with

$$\chi(f) = \int_\gamma f(\lambda)\, d\mu_0(\lambda), \qquad f \varepsilon \mathfrak{A}.$$

Suppose now that the assertion of the Lemma is false. Then for each $p \varepsilon \mathfrak{M} \cup \gamma$ there exists $f_p \varepsilon \mathfrak{A}$ with $\chi(f_p) = 0$ and $f_p(p) \neq 0$.

Let $d\omega(\lambda)$ and $W(\zeta)$ have the same meaning as in the preceding sections. Let $\mathfrak{M}_0$ be the region obtained by deleting from $\mathfrak{M}$ the zeros of W.

Now for all $f \varepsilon \mathfrak{A}$ and p in $\mathfrak{M} \cup \gamma$

$$0 = \chi(f \cdot f_p) = \int_\gamma f(\lambda) f_p(\lambda)\, d\mu_0(\lambda).$$

Hence the measure $f_p(\lambda)\, d\mu_0(\lambda)$ annihilates $\mathfrak{A}$. By Theorem 1, then we can

[4] Cf. L. Carleson [7], Theorem 4, for a similar method of proof.

find a function L_p analytic on $\mathfrak{M}_0$ and with $L_p(\zeta)W(\zeta)$ regular on $\mathfrak{M}$, such that L_p has nontangential boundary-values $L_p(\lambda)$ for all λ in γ except for a set of ω-measure 0, and with $f_p(\lambda)\,d\mu_0(\lambda) = L_p(\lambda)\,d\omega(\lambda)$ as measures. Choose now p_1, p_2 distinct in $\mathfrak{M} \cup \gamma$. Then

$$f_{p_2}(\lambda)f_{p_1}(\lambda)\,d\mu_0(\lambda) = f_{p_2}(\lambda)L_{p_1}(\lambda)\,d\omega(\lambda)$$

and

$$f_{p_1}(\lambda)f_{p_2}(\lambda)\,d\mu_0(\lambda) = f_{p_1}(\lambda)L_{p_2}(\lambda)\,d\omega(\lambda)$$

whence $f_{p_2}\cdot L_{p_1} = f_{p_1}\cdot L_{p_2}$ a. e.—$d\omega$ on γ, whence by the result in [6] which we have quoted earlier, $f_{p_2}(\zeta)L_{p_1}(\zeta) = f_{p_1}(\zeta)L_{p_2}(\zeta)$ for all ζ in $\mathfrak{M}$.

Fix now p_0 in $\mathfrak{M}$ and set $F(\zeta) = f_{p_0}^{-1}(\zeta)L_{p_0}(\zeta)$. Since, for $q\,\varepsilon\,\mathfrak{M}_0$, L_q *and* f_q^{-1} are regular at q, we obtain that F is regular at q. Thus F is analytic on all $\mathfrak{M}_0$ and similarly we see that the covariant $F(\zeta)W(\zeta)$ is analytic on all of $\mathfrak{M}$.

Next, for each $q\,\varepsilon\,\gamma$, we choose an arc γ_q on γ with $|f_q(\lambda)| \geqq \delta_q$ for λ in γ_q, δ_q being a positive number. By the Heine-Borel theorem, some finite set of these arcs covers γ. We can hence get $\delta > 0$ and a decomposition $\gamma = \bigcup_{i=1}^{n} \gamma_i$ where the γ_i are disjoint half-open arcs and for each i there is some q_i with $|f_{q_i}(\lambda)| \geqq \delta$ on γ_i.

Now, $F(\zeta) = f_{q_i}^{-1}(\zeta)L_{q_i}(\zeta)$ for $\zeta\,\varepsilon\,\mathfrak{M}$, whence

$$F(\lambda) = \lim_{\zeta\to\lambda} F(\zeta) = f_{q_i}^{-1}(\lambda)L_{q_i}(\lambda) \text{ a. e. on } \gamma_i.$$

We now use annular coordinates r, θ: $r_0 \leqq r \leqq 1$, $0 \leqq \theta \leqq 2\pi$ in an annular subregion of $\mathfrak{M}$ bounded on one side by γ, with $r = 1$ being the equation of γ.

Let $g\,\varepsilon\,\mathfrak{A}$. Then for each i

$$\int_{\gamma_i} g(\lambda)\,d\mu_0(\lambda) = \int_{\gamma_i} g(\lambda)f_{q_i}^{-1}(\lambda)L_{q_i}(\lambda)\,d\omega(\lambda) = \int_{\gamma_i} F(\lambda)g(\lambda)\,d\omega(\lambda).$$

Hence

$$\int_{\gamma} g(\lambda)\,d\mu_0(\lambda) = \int_{\gamma} g(\lambda)F(\lambda)\,(2\pi i)^{-1}W(\lambda)\,d\lambda.$$

Now if γ_ρ is the curve with equation: $r = \rho$, $\rho < 1$,

$$\int_{\gamma_\rho} F(\zeta)g(\zeta)\,W(\zeta)\,d\zeta = 0$$

by the residue theorem. Also

$$\lim_{\rho\to 1}\int_{\gamma_\rho} F(\zeta)g(\zeta)W(\zeta)\,d\zeta=\int_\gamma g(\lambda)F(\lambda)W(\lambda)\,d\lambda$$

due to the boundary behavior of the functions L_p and f_p. Hence

$$\chi(g)=\int_\gamma g(\lambda)\,d\mu_0(\lambda)=\tfrac{1}{2\pi i}\int_\gamma g(\lambda)F(\lambda)W(\lambda)\,d\lambda=0.$$

This must hold for all $g\,\varepsilon\,\mathfrak{A}$, which is impossible. Hence the assertion of the Lemma must be true.

COROLLARY. *The space $\mathfrak{S}$ of multiplicative functionals on $\mathfrak{A}$ is homeomorphic to the set $\mathfrak{M}\cup\gamma$.*

Proof. By the Lemma, if $\chi\,\varepsilon\,\mathfrak{S}$, then there exists $p\,\varepsilon\,\mathfrak{M}\cup\gamma$ with $\chi(f)=f(p)$ for all $f\,\varepsilon\,\mathfrak{A}$. There cannot exist two distinct points p_1, p_2 with this property, for if $p_1\neq p_2$ then for some f in $\mathfrak{A}$, $f(p_1)\neq f(p_2)$. Hence the map $\chi\to p$ takes $\mathfrak{S}$ into $\mathfrak{M}\cup\gamma$. It is obviously one-one and it is onto $\mathfrak{M}\cup\gamma$ since each p in $\mathfrak{M}\cup\gamma$ defines some multiplicative functional on $\mathfrak{A}$. Finally, the map is easily seen to be bicontinuous.

Proof of Theorem 3. Let τ be an algebraic isomorphism of $\mathfrak{A}_1$ onto $\mathfrak{A}_2$. Fix p in $\mathfrak{M}_2\cup\gamma_2$. Map each f in $\mathfrak{A}_1$ into $\tau(f)(p)$. This map is a multiplicative functional on $\mathfrak{A}_1$, whence by the lemma there exists $\phi(p)$ in $\mathfrak{M}_1\cup\gamma_1$ with $\tau(f)(p)=f(\phi(p))$ if $f\,\varepsilon\,\mathfrak{A}_1$. The function ϕ then maps $\mathfrak{M}_2\cup\gamma_2$ onto $\mathfrak{M}_1\cup\gamma_1$ in a one-one and bicontinuous fashion. It follows that ϕ maps $\mathfrak{M}_2$ homeomorphically onto $\mathfrak{M}_1$.

Fix p_0 in $\mathfrak{M}_2$ and f_0 in $\mathfrak{A}_1$ with f_0 locally simple at $\phi(p_0)$. Then for p in some neighborhood of p_0, $f_0(\phi(p))=\tau(f_0)(p)$. Since f_0 and $\tau(f_0)$ are analytic functions and moreover f_0 is one-one in a neighborhood of $\phi(p_0)$, ϕ is analytic at p_0 as mapping from $\mathfrak{M}_2$ to $\mathfrak{M}_1$. This holds for each p_0 in $\mathfrak{M}_2$ and further ϕ is globally one-one. Hence ϕ provides a conformal map of $\mathfrak{M}_2$ onto $\mathfrak{M}_1$.

Conversely, suppose we are given a conformal map ϕ of $\mathfrak{M}_2$ on $\mathfrak{M}_1$. Classical results then give that ϕ is extendable to a homeomorphism of $\mathfrak{M}_2\cup\gamma_2$ onto $\mathfrak{M}_1\cup\gamma_1$. For each f in $\mathfrak{A}_1$ we can then define τf on $\mathfrak{M}_2\cup\gamma_2$ as follows: $\tau f(p)=f(\phi(p))$, $p\,\varepsilon\,\mathfrak{M}_2\cup\gamma_2$. Then $\tau f\,\varepsilon\,\mathfrak{A}_2$ and τ is an isomorphism from $\mathfrak{A}_1$ to $\mathfrak{A}_2$. This proves Theorem 3.

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