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Electron bound by a potential well in the presence of a constant uniform magnetic field

V R Khalilov[1] **and F Kh Chibirova**[2]

[1] Faculty of Physics, Moscow State University, 119899, Moscow, Russia
[2] Karpov Institute of Physical Chemistry, 103064, Moscow, Russia

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Abstract
We study the effect of a constant uniform magnetic field on an electrically charged massive particle (an electron) bound by a potential well, which is described by means of a single attractive $\lambda\delta(\mathbf{r})$ potential. A transcendental equation that determines the electron energy spectrum is derived and solved. The electron wavefunction in the ground (bound) state is approximately constructed in a remarkably simple form. It is shown that there arises the probability current in the bound state in the presence of a uniform constant magnetic field. This (electric) current, being by the gauge invariant quantity, must be observable and involve (and exercise influence on) the electron scattering. The probability current density resembles a stack of 'pancake vortices' whose circulating 'currents' around the magnetic field direction (z-axes) are mostly confined within the plane $z = 0$. We also compute the tunnelling probability of an electron from the bound to free state under a weak constant homogeneous electric field, which is parallel to the magnetic field. The model under consideration is briefly discussed in two spatial dimensions.

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1. Introduction

The behaviour of quantum nonrelativistic systems in external electromagnetic fields has attracted permanent interest in view of possible applications of the corresponding models in many phenomena of quantum mechanics. Bound electron states play an important role in quantum systems in condensed matter. When the external field configuration has the cylindrical symmetry a natural assumption is that the relevant quantum mechanical system is invariant along the symmetry (for example, z) axis and the quantum mechanical problem then becomes essentially two dimensional in the xy plane. Such is the case when the external field configuration is a superposition of a constant uniform magnetic field and a cylindrically symmetric potential. Nonrelativistic electrons in such external field backgrounds are good quantum mechanical models for studying remarkable macroscopic quantum phenomena such as the fractional quantum Hall effect [1] and high-temperature superconductivity [2]. A related problem is the behaviour of electrons in trapping potentials in the presence of a constant uniform magnetic field in an effective mass approximation. Magnetic fields seem likely to have an effect on weakly bound electrons into singular potentials of defects in the defect films [3] and solids [4].

Pure two-dimensional models are also of significant interest. The effect of magnetic fields on a weakly bound electron in two spatial dimensions was studied by us in [5]. This model is of interest because it gives a good example of a nonrelativistic analogue of the so-called dimensional transmutation phenomenon first discovered by Coleman and Weinberg in [6] in the massless scalar electrodynamics. Whereas above nonrelativistic systems can be described by the Schrödinger equation, relativistic systems, related to the Dirac Hamiltonian in 2 + 1 dimensions in a constant uniform magnetic field, show up in a certain type of doped two-dimensional semimetals [7]. Similar problems are also related to a number of problems in quantum theory, for example, the parity violation, the theory of anyons (particles satisfying a fractional statistics), the Aharonov–Bohm effect [8] and other. At last recently, a new type of spectral problem has been found [9, 10] in quantum mechanics of planar electrons in a superposition of constant uniform magnetic and cylindrically symmetric potential fields. For this new class of spectral problems, the so-called quasi-exactly solvable (QES) models the energy spectra exist if a certain relation between the parameters characterizing the intensity of the interaction of an electron with external fields holds and then solutions of the corresponding equations of quantum mechanics can be expressed as the product of a weight function and a finite polynomial. Some physical examples of QES models, which include the two-dimensional Schrödinger or Dirac equation for an electron in a superposition of a constant uniform magnetic field and an attractive Coulomb field, were studied in [11].

The purpose of this paper is to study the effect of a constant uniform magnetic field on an electron bound by a single attractive $\lambda\delta(\mathbf{r})$ potential. We derive a simple transcendental equation determining the electron energy spectrum and construct the approximate wavefunction for a bound electron state in the presence of a constant uniform magnetic field. We show that the sizes of the electron localization region change and the probability current arises even when the electron is in the bound state in a superposition of a constant uniform magnetic field and a single attractive $\lambda\delta(\mathbf{r})$ potential. The probability current in three-dimensional space resembles a stack of 'pancake vortices' whose circulating (around the z-axes) 'currents' are mostly confined in the weak magnetic field within the plane $z = 0$.

The equation for determining the energy levels of the electron states is also obtained for the model under study in two spatial dimensions. We show that in difference from the three-dimensional case the binding energy is not analytical in λ.

The tunnelling of an electron from the bound to free state under a weak electric field is of importance. In the present paper, we compute the tunnelling probability of an electron from the bound to free state under a weak constant homogeneous electric field, which is parallel to the magnetic field.

2. Electron in a potential well in the presence of a constant uniform magnetic field

Let us consider an electron with the charge $e < 0$ in an attractive singular potential of the form

$$U(r) = -\frac{\hbar^2\lambda}{2m}\delta(\mathbf{r}) \qquad (1)$$

and a constant uniform magnetic field $\mathbf{B}$, which is specified in Cartesian coordinates as

$$\mathbf{B} = (0, 0, B) = \nabla \times \mathbf{A}, \qquad \mathbf{A} = (-yB, 0, 0). \tag{2}$$

In (1) λ is a positive coupling constant of the length dimension, $\delta(\mathbf{r})$ is the three-dimensional Dirac delta function and $m > 0$ is the effective mass of an electron. It is well to note that the attractive $\lambda\delta(\mathbf{r})$ potential can be considered as the limit of a sequence of appropriate narrow rectangular potential wells

$$U(r) = -U_0, \quad r < R, \qquad U(r) = 0, \quad r > R. \tag{3}$$

Then the parameter λ is expressed via U_0 and R as

$$\lambda = \frac{2mU_0R^3}{\hbar^2}.$$

The Schrödinger equation is

$$\frac{1}{2m}\left[\left(-\mathrm{i}\hbar\frac{\partial}{\partial x} + \frac{eB}{c}y\right)^2 - \hbar^2\frac{\partial^2}{\partial y^2} - \hbar^2\frac{\partial^2}{\partial z^2} - \hbar^2\lambda\delta(\mathbf{r})\right]\Psi_E(\mathbf{r}) = E\Psi_E(\mathbf{r}). \tag{4}$$

The electron wavefunction in the magnetic field (2) can be found in the form [12]

$$\psi_{np}(t, \mathbf{r}) = \frac{1}{2}\,\mathrm{e}^{-\mathrm{i}E_{ns}t/\hbar}\,\mathrm{e}^{\mathrm{i}(p_1x+p_3z)/\hbar}V_n(Y)\begin{pmatrix}1+s\\1-s\end{pmatrix}, \tag{5}$$

where

$$E_{ns} = \hbar\omega\left(n + \frac{1}{2}\right) + \frac{p_3^2}{2m} + s\hbar\omega\frac{m}{2m_e} \tag{6}$$

is the eigenenergy, $\omega = |eB|/mc$, m_e is the mass of the free electron, p_1, p_3 are the eigenvalues of a generalized momentum operator and $s = \pm 1$ is the spin quantum number. Note that for $m_e = m$ and for sign $eB < 0$, all the energy levels except one with $n = 0, s = -1$ are doubly degenerate: the coincident levels are those with $n, s = 1$ and $n + 1, s = -1$. In this paper, we consider the case $m_e = m$. Then, the energy eigenvalues depend only on the number n.

Note that p_1 is constrained by $|p_1| \leqslant eBL/c$ (see [12]). The functions

$$V_n(Y) = \frac{1}{(2^n n!\pi^{1/2}a)^{1/2}}\exp\left(-\frac{(y-y_0)^2}{2a^2}\right)H_n\left(\frac{y-y_0}{a}\right)$$

are expressed through the Hermite polynomials $H_n(z)$, the integer $n = 0, 1, 2, \ldots$ indicates the Landau level number,

$$a = \sqrt{\frac{\hbar c}{|eB|}} \equiv \sqrt{\frac{\hbar}{m\omega}}$$

is the so-called magnetic length and $y_0 = -cp_1/eB$. It should be recalled that the classical trajectory of an electron in the xy plane is a circle. The quantity y_0 corresponds to the classical y-coordinate of the circle centre. All the electron states (5) are not localized in the x-, z-directions.

Solutions of equation (4) are sought in the form

$$\Psi_E(\mathbf{r}) = \sum_{n=0}^{\infty}\int \mathrm{d}p_1\,\mathrm{d}p_3\,C_{Enp}\psi_{np}(\mathbf{r}) \equiv \sum_{n,p} C_{Enp}\psi_{np}(\mathbf{r}), \tag{7}$$

where $\psi_{np}(\mathbf{r})$ is the spatial part of wavefunctions (5).

Then, for coefficients C_{Enp}, one obtains

$$C_{Enp}\left(n + cp_3^2 + b\right) = \lambda_0\sum_{l,k} C_{Elk}V_l(0)V_n(0), \tag{8}$$

where

$$c = \frac{1}{2m\hbar\omega}, \qquad b = -\frac{E}{\hbar\omega}, \qquad \lambda_0 = \frac{\lambda}{8\pi^2 m\hbar\omega}, \qquad V_l(0) \equiv V_l(y=0).$$

Note that terms on the right-hand side of equation (8) are the matrix elements of the interaction operator of the electron with the singular potential (1). Correctly, these matrix elements must be obtained at first for the interaction potential (3) with finite U_0, R and then by proceeding to the limits $R \to 0, U_0 \to \infty$. But for finite U_0 and R the main contribution in the matrix element $U_{np_3,n'p_3'}$ gives the quantum number regions $|p_3 - p_3'|R < \hbar$, $\sqrt{n-n'}R < a$ because the matrix element falls off quickly due to the oscillations outside the above regions. Moreover, for the sharp potentials, one can obtain $U_{np_3,n'p_3'} \sim -\lambda$ for $n, n' < N$ and $p_3, p_3' < P$ and $U_{np_3,n'p_3'} = 0$ for $n, n' > N$ and $p_3, p_3' > P, \lambda \to 0\, N, P \to \infty$ as $R \to 0$. So, for such potentials the sum taken over n on the right-hand side of equation (8) is limited by the value N and the bound-state energy becomes dependent on N and λ, which, as well as U_0 and R, are considered finite. In the limits $R \to 0, U_0 \to \infty$, according to the above given estimations, we have $\lambda \to 0, N \to \infty$.

Thus, when transforming to the singular potential the bound-state energy remains finite (and, moreover, any given) value only if N will tend to infinity as

$$N^{3/2} = \frac{12\sqrt{2}a\pi}{\lambda} + \left(-\frac{E_0}{\hbar\omega}\right)^{3/2} \tag{9}$$

as $R \to 0$.

Let us write C_{Enp} as

$$C_{Enp} = C_E \frac{V_n(0)}{n + cp_3^2 + b}, \tag{10}$$

insert (10) into (8) and take account of the formulae

$$\sum_{n,p} |C_{Enp}|^2 = 1, \tag{11}$$

$$\int \mathrm{d}p_1 V_n(0) V_k(0) = \frac{\hbar}{a^2}\delta_{n,k}. \tag{12}$$

Then, one obtains equations

$$1 = \frac{\lambda}{8\pi^2 m\hbar\omega} \sum_{n=0}^{N} \int_{-\infty}^{\infty} \mathrm{d}p_3 \frac{1}{n + b + cp_3^2}, \tag{13}$$

$$(C_E)^{-2} = \sum_{n=0}^{N} \int_{-\infty}^{\infty} \mathrm{d}p_3 \frac{1}{\left(n + b + cp_3^2\right)^2}. \tag{14}$$

Integrating the right-hand side of equation (13) we obtain the following transcendental equation:

$$1 = \frac{\lambda\sqrt{m}}{8\sqrt{2}\pi} \sum_{n=0}^{N} \frac{\omega}{\sqrt{|\hbar\omega n - E|}}, \tag{15}$$

the roots of which determine the energy levels of the electron in the considered combination of fields. This equation can be solved numerically (graphically); its roots $x_n = E_n/\hbar\omega$ for

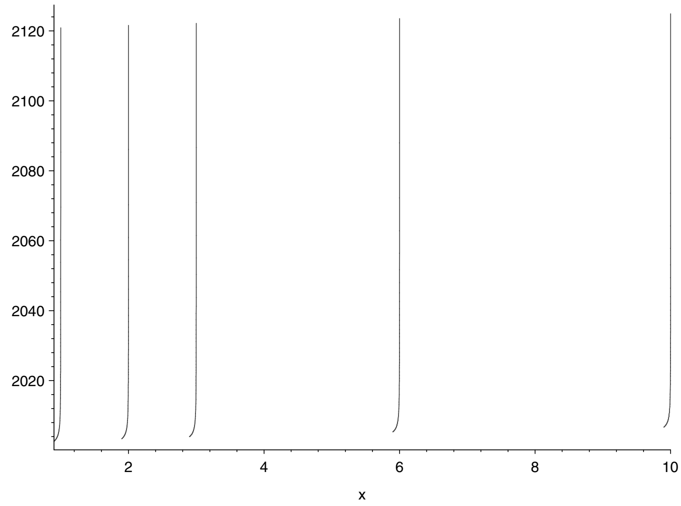

Figure 1. The plot of the function $f(x)$.

different λ are determined by the crossing of horizontal $8\sqrt{2}a\pi/\lambda$ with different values λ with the function

$$f(x) = \sum_{n=0}^{N} \frac{1}{\sqrt{|n-x|}}. \tag{16}$$

It is seen from equation (16) that for an attractive potential the main contributions in the sum gives small lengths Δx, which adjoins to the corresponding root with n to the left. The plot of $f(x)$ as a function of x near the roots with $n = 1, 2, 3, 6, 10$ is given in figure 1 for $N = 10^6$.

It follows from equation (15) that if the potential energy is small compared with $\hbar\omega$ then the energy level E_n is located between two energy levels $\hbar\omega n$ and $\hbar\omega(n+1)$ with the exception of the negative energy level E_0.

The energy levels can be approximately calculated if

$$\frac{\lambda}{8\pi a\sqrt{2}} \ll 1.$$

Then, representing

$$E_n = \hbar\omega n + \delta_n, \qquad \delta < 0 \tag{17}$$

for any number $n = 0, 1, \ldots,$ we obtain

$$\delta_n = -\frac{\lambda^2 m\omega^2}{32\pi^2}. \tag{18}$$

First, let the singular potential can be considered as a perturbation; therefore, the coupling constant $\lambda < a$. All the levels δ_n are shifted down with respect to the corresponding Landau levels $\hbar\omega n$ for any $n = 0, 1, \ldots$. In this case $|\delta_n| \ll \hbar\omega$. If $E_0 < 0$ is the electron energy in the pure singular $\lambda\delta(\mathbf{r})$ potential, then δ_0 is the magnetic field correction to the level E_0.

If the coupling constant λ obeys the inequality $8\pi\sqrt{2} \gg \lambda > a$, then $|E| > \hbar\omega$ and E is determined by equation (18). It should be emphasized that there exists the only energy level ($E_0 < 0$) in a single attractive $\lambda\delta(\mathbf{r})$ potential.

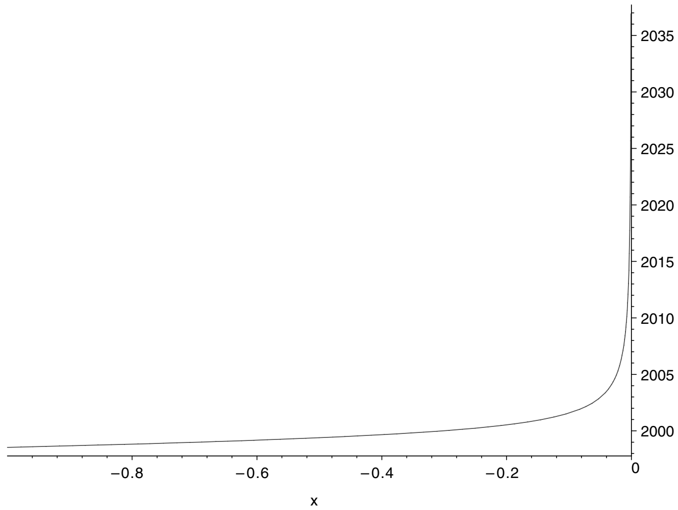

Figure 2. Function $f(x)$ near the root $n = 0$.

The plot of $f(x)$ as a function of x near the root with $n = 0$, which corresponds to the ground (bound) state, is given in figure 2 for $N = 10^6$.

Graphically the root $x_0 = E_0/\hbar\omega$ is determined by the crossing of horizontal $8\sqrt{2}a\pi/\lambda$ with the function plotted in figure 2. We also see that the energy of the bound electron state is analytical in λ.

The electron energy in the bound state is not analytical in λ for the model under discussion in the two-dimensional case. Indeed, equation (15) should be replaced by the equation

$$1 = \frac{\lambda}{4\pi}\sum_{n=0}^{N}\frac{1}{n+b}, \qquad b = -\frac{E}{\hbar\omega} \tag{19}$$

for the two-dimensional case (see [5]). Here we took into account all remarks concerning the number N, which were given when deriving of equation (15). The energy of the ground state (that is a negative root of equation (19) $E_0 < 0$) can be easily found for $N \gg 1$. Replacing the summation over n by the integration, we find as a result

$$1 = \frac{\lambda}{4\pi}\ln\left(\frac{N}{b}\right) \tag{20}$$

and for the electron energy of the bound state we obtain

$$E_0 = -\hbar\omega N\exp\left(-\frac{4\pi}{\lambda}\right). \tag{21}$$

It is important to emphasize that in the two-dimensional case the coupling constant λ is the dimensionless constant. Nevertheless, there exists a bound state in the attractive $\lambda\delta(\mathbf{r})$ potential. In the limits $N \to \infty, \lambda \to 0$, we must require that N should depend on the dimensionless constant λ so that the binding energy $-E_0$ would remain finite as $N \to \infty$. Thus, the cutoff dimensionless parameter N, which tends to infinity, transmutes in the arbitrary binding energy $|E_0|$. This is the nonrelativistic analogue of the dimensional transmutation phenomenon. For the model under discussion in the absence of a magnetic field this phenomenon was first considered in [13].

3. Electron wave function and probability current in the bound state

Now, we shall construct the wavefunction of the electron in the negative-energy state with $n = 0$. It can be found from equation (7) by putting $n = 0$. Then, using equations (10), (14) we integrate over p_1 by means of the following integral [14]:

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}xy} U_n(x+z) U_k(x+u) = \frac{1}{a} \exp\left[\frac{\mathrm{i}y(z+u)}{2} + \mathrm{i}(n-k)\arctan\frac{y}{z-u}\right] I_{nk}(\rho), \tag{22}$$

where the function $I_{nk}(\rho) = \exp(-\rho/2) L_n^{n-k}(\rho)$ is the Laguerre function of the argument

$$\rho = \frac{y^2 + (u-z)^2}{2}, \tag{23}$$

$L_n^{n-k}(\rho)$ is the Laguerre polynomial and in our problem $n = k = 0$.

Integration over p_3,

$$I = \int_{-\infty}^{\infty} \mathrm{d}p_3 \frac{\mathrm{e}^{-\mathrm{i}p_3 z/\hbar}}{p_3^2 + 2m|E_0|}, \tag{24}$$

can be carried out in the complex plane, closing the integration contour into the lower half-plane for $z > 0$ and in the upper half-plane for $z < 0$. The case $\lambda \ll a$ is of physical interest. For this case, one obtains

$$I = \frac{\mathrm{e}^{-\sqrt{2m|E_0|}|z|/\hbar}}{\sqrt{2m|E_0|)}}. \tag{25}$$

Then, simple calculations lead to the normalized electron wavefunction in the form (see also [5, 15, 16])

$$\Psi_0(\mathbf{r}) \sim \frac{\hbar}{\sqrt{2\pi l a}} \exp\left(-\frac{x^2+y^2-2\mathrm{i}xy}{4a^2}\right) \exp\left(-\frac{|z|}{l}\right), \qquad l = \frac{\hbar}{\sqrt{2m|E_0|}}. \tag{26}$$

It is of great interest that there is the probability current even when the electron is in the bound state in a superposition of a constant uniform magnetic field and a single attractive $\lambda\delta(\mathbf{r})$ potential. The probability current density is

$$\begin{aligned} J_x &= -\frac{\hbar}{2\pi l m a^4} y \exp\left(-\frac{x^2+y^2}{2a^2} - \frac{2\sqrt{2}|z|}{l}\right) \equiv -J_0 y \exp\left(-\frac{x^2+y^2}{2a^2} - \frac{2\sqrt{2}|z|}{l}\right), \\ J_y &= J_0 x \exp\left(-\frac{x^2+y^2}{2a^2} - \frac{2\sqrt{2}|z|}{l}\right), \qquad J_z = 0. \end{aligned} \tag{27}$$

We see from equation (27) that the divergence of the probability current density is equal to zero everywhere. Therefore, the probability density $|\Psi_0(\mathbf{r})|^2$ and the current density (27) satisfy the continuity equation

$$\frac{\partial|\Psi_0(\mathbf{r})|^2}{\partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} \equiv \frac{\partial|\Psi_0(\mathbf{r})|^2}{\partial t} + (\nabla \times \boldsymbol{J}) = 0 \tag{28}$$

everywhere and the function $|\Psi_0(\mathbf{r})|^2$ is conserved in time.

The electron wavefunctions in an external electromagnetic field are known to have the ambiguity, which is related to the ambiguity of the 4-potential $(\Phi, \mathbf{A})$ of the electromagnetic field. The latter is determined just with the exactness up to the gauge transformation

$$\Phi \to \Phi - \frac{1}{c}\frac{\partial f}{\partial t}, \qquad \mathbf{A} \to \mathbf{A} + \nabla \boldsymbol{A}, \tag{29}$$

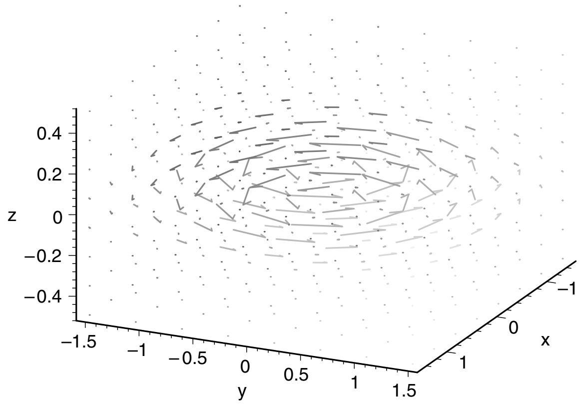

Figure 3. Probability current density in space.

where f is an arbitrary function of coordinates. The Schrödinger equation does not change provided the replacement of the vector potential in the Hamiltonian is carried out simultaneously with the replacement of wavefunction

$$\Psi \to \Psi \exp\left(\frac{\mathrm{i}e}{\hbar c} f\right). \tag{30}$$

In the considered case, we can cancel the phase factor in wavefunction (26) by means of the gauge transformation of the vector potential with the function $f = Bxy/2$. Under such a transformation the vector potential transforms from equation (2) to

$$\mathbf{A} = \frac{B}{2}(-y, x, 0), \tag{31}$$

the wavefunction (26) becomes real, but the (electric) current density

$$\mathbf{J} = \frac{\mathrm{i}e\hbar}{2m}[(\nabla\Psi^*)\Psi - \Psi^*\nabla\Psi] - \frac{e^2}{mc}\mathbf{A}\Psi^*\Psi, \tag{32}$$

by the gauge invariant quantity, does not change. Note that

$$\mathbf{J} = -\frac{e^2}{mc}\mathbf{A}\Psi^*\Psi \tag{33}$$

at the gauge (31).

The probability current density is shown in figure 3 in which the coordinates x, y, z are measured in units of a.

The vector field $\mathbf{J}$ can be treated semiclassically. Let the two-dimensional vector field $\mathbf{j}$ be a complex quantity $\mathbf{j} = j_x + i j_y$ in any plane $z =$ constant whose components j_x and j_y are functions of the complex variable $v = x + \mathrm{i}y$ and depend on z. The vector field $\mathbf{j}(x, y)$ is a vortex field. Computing $\mathbf{C} = [\nabla \times \boldsymbol{j}]$, we obtain

$$[\nabla \times \boldsymbol{j}] = \frac{\partial j_y}{\partial x} - \frac{\partial j_x}{\partial y} = \frac{A}{\pi}\left(1 - \frac{vv^*}{2a^2}\right)\exp\left(-\frac{vv^*}{2a^2}\right), \tag{34}$$

where $A = e\hbar F(z)/lma^4$, the function

$$F(z) = \exp\left(-\frac{2|z|}{l}\right)$$

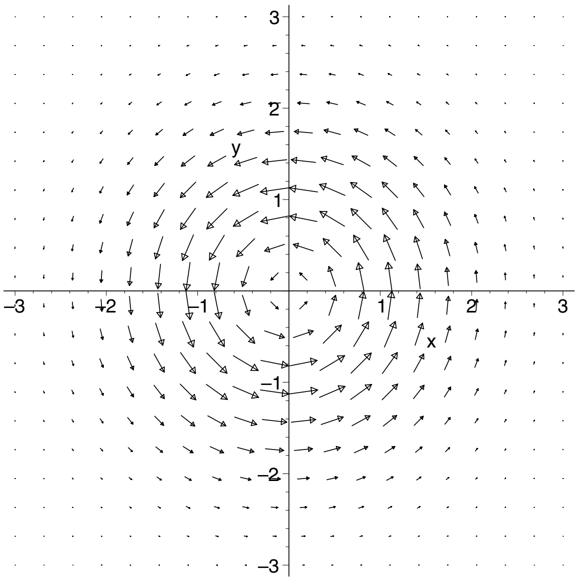

Figure 4. Probability current density in the $z = 0$ plane.

should be calculated on the plane $z = \text{constant}$ and v^* is the complex conjugate of v. In each plane $z = \text{constant}$ the vector $\mathbf{j}(x, y)$ determines the vector field of a point-like vortex located at the point $x, y = 0$.

One can write $\mathbf{j}$ in the form

$$\mathbf{j} = \frac{\mathrm{i}I(v)}{2\pi v^*}, \tag{35}$$

where

$$I(v) = Avv^* \exp\left(-\frac{vv^*}{2a^2}\right) \tag{36}$$

is the vortex intensity that is the vector-field circulation $\mathbf{j}(\mathbf{x}, \mathbf{y})$ on any closed contour encircling the vortex at the point $x, y = 0$.

The normalized two-dimensional vector field $\mathbf{j}(x, y)$ is shown in the plane $z = 0$ in figure 4 in which the coordinates x, y are measured in units of a.

One can see from equation (27) that the probability current density resembles a stack of 'pancake vortices' whose circulating 'currents' around the magnetic field direction (z-axes) are mostly concentrated within the plane $z = 0$.

4. Effect of a homogeneous electric field on the bound electron

Now we consider the effect of a weak constant homogeneous electric field on an electron bound by an attractive singular potential in the presence of a constant uniform magnetic field. Related problems are the ionization of negative charged ions in accelerators as well as the removal of electrons from trapping potentials by a constant electric field. It is worthwhile to note that the problem concerning the removal of a charged nonrelativistic particle from a spherically symmetric potential well by a constant homogeneous electric field was first partly solved in [17] and the more rigorous formulae were given in [18] for the total and in [19] for the differential probability. Here, we consider this problem for the model studied with the inclusion of a weak constant homogeneous electric field specified by the potential

$$U(z) = -|e|\epsilon z, \tag{37}$$

where $\epsilon > 0$ is the electric-field strength.

The inclusion of the electric field in the model under discussion leads to the quasistationary of the bound electron state and the appearance of the nonzero particle flux at large distances from the potential well. It is apparent that for the considered electromagnetic field combination we need to calculate only the particle flux in the z-direction far from the potential well.

Let us find the decay probability of the bound state per unit time, which is equal to the electron flux across the plane perpendicular to the z-axis. To find this flux we need the Schrödinger equation solutions for an electron far from the potential well. So far as the electron flux is of interest in the z-direction the magnetic field cannot be taken into account far from the well. Solutions of the Schrödinger equation for an electron in field (37) are expressed via the Airy functions (see, for example, [19]).

So far as the particle flux has to be nonzero in the z-direction we must choose the solution with the complex Airy function for the real z:

$$\psi_{p_1,p_2,E} = \mathrm{e}^{-\mathrm{i}Et/\hbar}\,\mathrm{e}^{\mathrm{i}(p_1x+p_2y)/\hbar}V(Z), \tag{38}$$

where

$$Z = \left(\frac{2m|e|\epsilon}{\hbar^2}\right)^{1/3}\left(\frac{p_1^2+p_2^2}{2m|e|\epsilon} - \frac{E}{|e|\epsilon} - z\right), \tag{39}$$

p_1, p_2 are the eigenvalues of the operator of generalized momentum and $V(Z)$ is the complex Airy function for the real Z. For $Z \ll 1$

$$V(z) = \frac{\sqrt{\pi}}{z^{1/4}}\left[\exp\left(\frac{2z^3/2}{3}\right) + \frac{\mathrm{i}}{2}\exp\left(-\frac{2z^3/2}{3}\right)\right]. \tag{40}$$

Note that only keeping the exponentially small term in solution (40) will give rise to the nonzero flux.

Outside the well, the electron wavefunction for large z can be written as the superposition of solutions in the electric field with $E = -E_0$,

$$\psi(\mathbf{r}) = \mathrm{e}^{-\mathrm{i}E_0t/\hbar}\int \mathrm{d}p_1\,\mathrm{d}p_2\,\mathrm{e}^{\mathrm{i}(p_1x+p_2y)/\hbar}V(Z)F(p_1p_2). \tag{41}$$

The appearance probability of an electron far from the well in unit time w is equal to the flux across the plane perpendicular to the z-axes

$$w = -\frac{\mathrm{i}\hbar}{2m}\int \mathrm{d}x\,\mathrm{d}y\left(\psi^*\frac{\partial\psi}{\partial z} - \psi\frac{\partial\psi^*}{\partial z}\right). \tag{42}$$

The function $F(p_1p_2)$ is found for large z from equation (41) by the Fourier transform

$$F(p_1p_2) = \frac{1}{4\pi^2V(Z)}\int \mathrm{d}x\,\mathrm{d}y\,\psi(\mathbf{r})\,\mathrm{e}^{\mathrm{i}(E_0t-p_1x-p_2y)/\hbar}. \tag{43}$$

In a weak electric field $\epsilon \ll m^{1/2}|E_0|^{3/2}/\hbar|e| \equiv \epsilon_0$ the electron wavefunction (41) differs insignificantly from the wavefunction of the bound state (26) (we just consider this case) for $|z| \ll \sqrt{\hbar^2\epsilon_0/2m|E_0|\epsilon}$, so for such z we can substitute the function (26) instead of $\psi(\mathbf{r})$ in equation (43).

Carrying out simple calculations, we obtain

$$F(p_1p_2) = \frac{C\exp(-i-|z|/l)}{4\pi^2 V(Z)}\int \mathrm{d}x\,\mathrm{d}y\exp\left(-\frac{x^2+y^2-2\mathrm{i}xy}{4a^2}+i\frac{p_1x+p_2y}{\hbar}\right)$$
$$\approx \frac{Ca^2\exp(-|z|/l)}{\pi V(Z)}\exp\left(-\frac{\left(p_1^2+p_2^2\right)a^2}{\hbar^2}\right), \tag{44}$$

where $C = 1/(a\sqrt{2\pi l})$ is the normalization constant of the wavefunction of the bound state. Finally, for the total probability one obtains

$$w = -\frac{2\sqrt{\pi}\hbar a^2|e|\epsilon}{l^2a^2m|e|\epsilon + l\hbar^2}\left(1+\frac{ml^3|e|\epsilon}{2(a^2ml|e|\epsilon+\hbar^2)}\right)\exp\left(-\frac{2\hbar^2}{3ml^3|e|\epsilon}\right). \tag{45}$$

5. Summary

It is of interest to compare wavefunction (26) with the electron wavefunction of the bound state in the only singular attractive $\hbar^2\lambda\delta(\mathbf{r})/2m$ potential. The latter can be easily obtained in the form

$$\Psi(\mathbf{r}) = \sqrt{\frac{1}{2\pi l_0}}\frac{\mathrm{e}^{-r/l_0}}{r}, \tag{46}$$

where $l_0 = \sqrt{\hbar^2/2m|E_0|}$ and $E_0 < 0$ is the electron energy and the wavefunction (46) is normalized as follows:

$$\int |\Psi(\mathbf{r})|^2\,\mathrm{d}V = 1. \tag{47}$$

The distribution of probabilities of different coordinates of the electron

$$|\Psi(\mathbf{r})|4\pi r^2\,\mathrm{d}r = 2\exp\left(-\frac{2r}{l_0}\right)\frac{\mathrm{d}r}{l_0}$$

is spherically symmetrical. Here $r = \sqrt{x^2+y^2+z^2}$. One sees that without a magnetic field the electron is localized in the region $\sim l_0$ and the probability current density for the state (46) is equal to zero everywhere.

The distribution of probabilities of different coordinates of the electron in the state (26)

$$|\Psi_0(\mathbf{r})|2\pi\rho\,\mathrm{d}\rho\,\mathrm{d}z = \frac{\rho}{a}\exp\left(-\frac{\rho^2}{2a^2}-\frac{2|z|}{l}\right)\frac{\mathrm{d}\rho}{a}\frac{\mathrm{d}z}{l}$$

has the cylindrical symmetry. Here $\rho = \sqrt{x^2+y^2}$, z are the cylindrical coordinates. The electron is located in the region $x, y \sim \sqrt{2}a$ in the x, y plane perpendicular to the magnetic field and $z \sim 2l$ along the magnetic field direction.

It is of importance that the probability current arises even when the electron is in the bound state in a superposition of a constant uniform magnetic field and a single attractive $\lambda\delta(\mathbf{r})$ potential. This (electric) current is the gauge invariant (physically observable) quantity and, therefore, it must involve (and exercise influence on) the electron scattering.

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