

Neutrino transport in accretion disks

R. F. Sawyer*

Department of Physics, University of California at Santa Barbara, Santa Barbara, California 93106, USA

(Received 7 April 2003; published 3 September 2003)

We test approximate approaches to solving a neutrino transport problem that presents itself in the analysis of some accretion-disk models. Approximation No. 1 consists of replacing the full, angular-dependent, distribution function by a two-stream simulation, where the streams are respectively outwardly and inwardly directed, with angles $\cos \theta = \pm 1/\sqrt{3}$ to the vertical. In this approximation the full energy dependence of the distribution function is retained, as are the energy and temperature dependences of the scattering rates. Approximation No. 2, used in recent works on the subject, replaces the distribution function by an intensity function and the scattering rates by temperature-energy-averaged quantities. We compare the approximations to the results of solving the full Boltzmann equation. Under some interesting conditions, approximation No. 1 passes the test; approximation No. 2 does not. We utilize the results of our analysis to construct a toy model of a disk at a temperature and density such that relativistic particles are more abundant than nucleons, and dominate both the opacity and pressure. The nucleons will still provide most of the energy density. In the toy model we take the rate of heat generation (which drives the radiative transfer problem) to be proportional to the nucleon density. The model allows the simultaneous solution of the neutrino transport and hydrostatic equilibrium problems in a disk in which the nucleon density decreases approximately linearly as one moves from the median plane of the disk upwards, reaching zero on the upper boundary.

DOI: 10.1103/PhysRevD.68.063001

PACS number(s): 04.40.Dg, 98.62.Mw

I. INTRODUCTION

Hyperaccreting black holes, which have been suggested as an energy source for powering gamma ray bursts, may provide conditions in which the close-in part of an accretion disk reaches a region of density and temperature such that neutrino radiation enters the dynamics in an important way [1–4]. In some extreme limits, the neutrino opacity of the disk may become important, necessitating a solution to a neutrino transport problem. A recent paper [5] specifically addresses the issue of the effects of neutrino opacity in the innermost portion of such an accretion disk. Despite the transitory nature of the phenomena, the various time scales are such that steady flow solutions of the neutrino transport problem are of interest. Reference [5] uses a simplified treatment of neutrino transport in this domain, a treatment that suppresses the energy dependence of the cross-sections in favor of a single parameter characterizing the thermal averaged opacity at an average temperature of the disk, and which further eliminates the angular degree of freedom in the Boltzmann equation in favor of a two-stream approximation of the angle dependence of the neutrino distribution.

The present paper investigates these two approximations in domains characteristic of the accretion disks of Ref. [5], by solving the Boltzmann equation directly, where we retain a neutrino energy spectrum and full angular dependence of the distribution function. The two-stream approximation to this equation, where we retain the energy dependence and the local temperature dependence of the opacities, is found to work quite adequately in most of the domains that we explore. However the further simplification of Ref. [5], in which the equation for a distribution function is replaced by

an equation for intensities, the details of which are explicated in Ref. [6], is not adequate in some domains that are of possible interest. We do not apply our results to a detailed physical system, but we do sketch a technique that seems useful in solving the kind of neutrino transport problem that arises in these systems. We also use the technique to solve a toy problem of a disk in which the relativistic particles dominate the neutrino opacities and the pressure, but in which nucleons still provide the energy density, and in which it is assumed that the local heat generation is proportional to the nucleon density. Hydrostatic equilibrium can be established in this case if the nucleon density decreases roughly linearly with height (i.e., the coordinate transverse to the disk) and vanishes on the disk's surface.

We consider a disk that is infinite and homogeneous in the x and y directions, with z taken as the normal coordinate. In appropriately scaled coordinates the surfaces of the disk are at $z = \pm 1$. The density and temperature will in general depend on z , with $\rho(-z) = \rho(z)$ and $T(-z) = T(z)$. The Boltzmann equation for the neutrino distribution function, $f(\mathbf{p}, z)$, is now to be solved in the region $0 < z < 1$, with boundary conditions $f(\mathbf{p}, 0) = f(-\mathbf{p}, 0)$, and with $f(\mathbf{p}, 1) = 0$ for $\mathbf{p} \cdot \hat{z} < 0$. There will be a prescribed amount of heat generated per unit volume at each height in the disk, $\dot{Q}(z)$, and therefore a prescribed profile for the energy flux upward. The temperature function must thus be adjusted so that the solution to the Boltzmann equation produces the input flux profile.

We include both absorption and isotropic elastic scattering in the collision terms, with reaction rates, $\lambda_{a,s}$. The methods of solution could be extended easily enough to include the simple angular dependences of standard model scattering rates, but since we are trying to answer generic questions that do not depend on neutrino flavor or the neutron-proton mix-

*Electronic address: sawyer@vulcan.physics.ucsb.edu

ture in the medium, we shall not include them. Experience with photon radiative transfer has shown that the angular dependence of Compton scattering matters little (as long as the correct weighting is included in the definition of the angular average of the scattering amplitude) in atmospheres that are (mathematically) similar to the ones that we consider [7]. Likewise, as long as there is significant absorption in the opacities, inelasticity in scattering should not be especially consequential.

Below we exhibit first the complete Boltzmann equation for this system, followed by two successive simplified forms.

A. Boltzmann equation

We define a variable $x = \cos(\theta)$ for the up-moving neutrinos, with $\cos(\theta) > 0$ and $x = -\cos(\theta)$ for the down-moving neutrinos. We introduce separate notations for the distribution function in the two hemispheres in momentum space, $f_+(z, p, x)$ for the up-moving hemisphere and $f_-(z, p, x)$ for the down-moving hemisphere, where p is the energy (or momentum) of the neutrino, and where we consider absorption and isotropic elastic scattering in the collision terms. Then we have

$$x \frac{\partial f_+(z, p, x)}{\partial z} = \lambda_a [f_{eq}(T(z), p) - f_+(z, p, x)] + \lambda_s \left[-f_+(z, p, x) + \frac{1}{2} \int_0^1 dx' [f_-(z, p, x') + f_+(z, p, x')] \right], \quad (1)$$

for the up-moving stream, and

$$x \frac{\partial f_-(z, p, x)}{\partial z} = -\lambda_a [f_{eq}(T(z), p) - f_-(z, p, x)] - \lambda_s \left[-f_-(z, p, x) + \frac{1}{2} \int_0^1 dx' [f_-(z, p, x') + f_+(z, p, x')] \right], \quad (2)$$

for the down-moving stream. Here f_{eq} denotes the Fermi function with chemical potential equal to zero, and a temperature that depends on z . The rate constants, λ_a and λ_s are functions of p and depend on z as well, the latter through variation of the density of scatterers or through temperature dependence of cross-sections. Of course, the temperature $T(z)$ must be adjusted so that the resulting distribution function produces the flux profile that is assumed, or calculated from other physics.¹

¹It would be of some interest to consider cases in which the “driving chemical potential,” $\hat{\mu} = \mu_e + \mu_p - \mu_n$, for electron neutrinos is not small compared to the temperature; indeed in the supernova atmosphere, where the temperatures are a little lower than in the accretion disk, at the same value of the density, it is essential to do so. In this case the Fermi function f_{eq} in Eqs. (1) and (2) would need to be evaluated with chemical potential $\hat{\mu}$ for neutrinos and $-\hat{\mu}$ for antineutrinos. In a steady state calculation, then, $\hat{\mu}(z)$ and $T(z)$ (and hence the electron-proton ratio), must be self-consistently determined in the solution of the transport problem, simultaneously fitting the requirements of a given lepton number flux (zero for example) and the externally prescribed flux profile. The only treatment in the literature that really solves this problem in the steady flow idealization is the work of Schinder and Shapiro [8] for the case of a hot newly formed neutron star atmosphere [8]. This work incorporates all important elastic scattering, inelastic scattering and absorption processes, as the present work does not. Its shortcomings are that it gives results for a rather specialized set of external parameters, most notably the neutrino fluxes arising from the core physics (which do not have an analogue in the disk problem), and that it provides little qualitative insight into which features of the microphysics drive which features of the results.

B. First simplification

We replace the continuum angular distribution by two streams at angle $\cos \theta = \pm 1/\sqrt{3}$ to the vertical. We then have,

$$\begin{aligned} \frac{\partial f_+(z, p)}{\partial z} &= \bar{\lambda}_a [f_{eq}(T(z), p) - f_+(z, p)] \\ &\quad + \bar{\lambda}_s [f_-(z, p) - f_+(z, p)]/2, \end{aligned} \quad (3)$$

and

$$\begin{aligned} \frac{\partial f_-(z, p)}{\partial z} &= -\bar{\lambda}_a [f_{eq}(T(z), p) - f_-(z, p)] \\ &\quad + \bar{\lambda}_s [f_-(z, p) - f_+(z, p)]/2, \end{aligned} \quad (4)$$

where $\bar{\lambda}_{a,s} = \sqrt{3} \lambda_{a,s}$, and where the local energy flux and energy densities are obtained from

$$\text{Flux}(z) = \frac{1}{4\pi^2\sqrt{3}} \int p^3 dp [f_+(z, p) - f_-(z, p)], \quad (5)$$

and

$$\rho_\nu(z) = \frac{1}{4\pi^2} \int p^3 dp [f_+(z, p) + f_-(z, p)]. \quad (6)$$

The inside boundary condition is $f_-(0, p) = f_+(0, p)$ and the outside $f_-(1, p) = 0$.

C. Second simplification

If in Eqs. (3) and (4), the opacities were energy-independent, then we could multiply by p^3 and integrate, to get equations for intensity,

$$\begin{aligned} \frac{dI_+(z)}{dz} &= \bar{\lambda}_a \{I_{eq}[T(z)] - I_+(z)\} \\ &+ \frac{1}{2} \bar{\lambda}_s [I_-(z) - I_+(z)], \end{aligned} \quad (7)$$

and

$$\begin{aligned} \frac{dI_-(z)}{dz} &= -\bar{\lambda}_a \{I_{eq}[T(z)] - I_-(z)\} \\ &+ \frac{1}{2} \bar{\lambda}_s [I_-(z) - I_+(z)], \end{aligned} \quad (8)$$

where $I_{eq}(T) = \int dp p^3 f_{eq}(T, p)$, with the inside boundary condition, $I_-(0) = I_+(0)$, and outside condition $I_-(1) = 0$. These are the equations discussed in Ref. [6] for the case of photon radiative transfer, a solution to which, adapted to the

neutrino case, is used in Ref. [5]. Of course, for the case of neutrino transport the rate functions are dependent on neutrino energy as well as the temperature of the medium. Thus the set (7) and (8), where we use some sort of thermal averaged rate functions, represents a further, and possibly drastic, approximation to the system.

II. INTEGRAL EQUATIONS

We define an (absorptive) optical thickness function $\zeta(z, p)$,

$$\zeta(z, p) = \int_0^z \lambda_a(y, p) dy. \quad (9)$$

The Boltzmann equations for the up-moving and down-moving distributions can now be written in integral forms,

$$\begin{aligned} f_+(z, p, x) &= e^{-\zeta(z, p)/x} \left\{ f_+(0, p, x) + \int_0^z dy x^{-1} e^{\zeta(y, p)/x} \left(\lambda_a(y, p) f_{eq}(y, p, x) \right. \right. \\ &\quad \left. \left. + \lambda_s(y, p) \left[-f_+(y, p, x) + \frac{1}{2} \int_0^1 dx' [f_+(y, p, x') + f_-(y, p, x')] \right] \right) \right\} \end{aligned} \quad (10)$$

and

$$\begin{aligned} f_-(z, p, x) &= e^{\zeta(z, p)/x} \left\{ \int_z^1 dy x^{-1} e^{-\zeta(y, p)/x} \left(\lambda_a(y, p) f_{eq}(y, p, x) \right. \right. \\ &\quad \left. \left. - \lambda_s(y, p) \left[-f_-(y, p, x) + \frac{1}{2} \int_0^1 dx' [f_+(y, p, x') + f_-(y, p, x')] \right] \right) \right\} \end{aligned} \quad (11)$$

with $f_+(0, p, x) = f_-(0, p, x)$ supplying the boundary condition in the interior. The external boundary condition is already incorporated into the above equations. The corresponding integral equations for the ‘‘first simplification’’ approximation are,

$$f_+(z, p) = e^{-\bar{\zeta}(z, p)} \left(f_+(0, p) + \int_0^z dy e^{\bar{\zeta}(y, p)} \left[\bar{\lambda}_a(y, p) f_{eq}(y, p) + \frac{1}{2} \bar{\lambda}_s(y, p) [-f_+(y, p) + f_-(y, p)] \right] \right) \quad (12)$$

and

$$f_-(z, p) = e^{\bar{\zeta}(z, p)} \left(\int_z^1 dy e^{-\bar{\zeta}(y, p)} \left[\bar{\lambda}_a(y, p) f_{eq}(y, p) - \frac{1}{2} \bar{\lambda}_s(y, p) [-f_+(y, p) + f_-(y, p)] \right] \right), \quad (13)$$

where $\bar{\zeta}(z, p) = \int_0^z \bar{\lambda}_a(y, p) dy$.

Beginning with an initial guess of the temperature profile and calculating the source term $f_{eq}(z, p)$, we perform the integral in the equation for f_- in Eq. (11), taking the scattering term to be zero. We then use $f_+(0, p, x) = f_-(0, p, x)$ as the initial condition to calculate f_+ from Eq. (10). This zero-order result for the distribution is used to calculate the scattering terms in Eqs. (1) and (2), the result to be used in the next iteration. In this way, if the iteration process is convergent, we arrive at a solution corresponding to a particular temperature distribution. Convergence problems, when scat-

tering is large compared to absorption, can be avoided by feeding back, in each iteration, a scattering term that is, say, 75% based on the distribution from the iteration one previous to the previous one.

The other part of our task is to adjust the temperature distribution so that the flux has the z dependence fixed (for now) externally. We can do this as a part of the same iteration procedure. After each iteration we calculate the flux from the distribution function which we then subtract (at each point z) from the desired flux function. We take these numbers multiplied by an appropriate constant and subtract

them from the (numerically calculated) $(d/dz)[T(z)]$; this difference is taken as the new $(d/dz)[T(z)]$, from which we construct the new function $T(z)$ by integration, keeping the central temperature fixed.

In some cases there are minor manual interventions necessary to converge to a result. In particular, for the domain in which the scattering is much larger than the absorption, it is necessary to ramp up the strength of the scattering while iterating, at a rate that keeps the errors in bounds at each stage, but that still gets the job done in a tolerable number of iterations. For the opacity functions that we have taken below, and for the “first simplification” form of the transport equation, it is surprisingly easy to get up to the range in which the scattering is six times the absorption, beginning in the zeroth approximation with a monotonically decreasing temperature function. For the Boltzmann equation itself, with its full angular dependence, this whole approach becomes much less efficient, owing to the x^{-1} factors in the exponentials, and we do not particularly recommend it as a technique, although we have used it for the purposes of making some of the comparisons to be summarized below.

III. NUMERICAL COMPARISONS

For the neutrino rate functions we have chosen two forms, which have the respective generic temperature, momentum, and baryon density dependence of the following cases:

(i) Opacity dominated by scattering and absorption on nucleons (under conditions of sufficiently high temperature so that electrons are nondegenerate and the number of neutrons is approximately equal to the number of protons),

$$\begin{aligned}\lambda_a(z,p) &= \beta_a n_N(z) p^2, \\ \lambda_s(z,p) &= \beta_s n_N(z) p^2\end{aligned}\quad (14)$$

where n_N is the nucleon number density.

(ii) Relativistic e^+ , e^- , ν , $\bar{\nu}$ atmosphere,

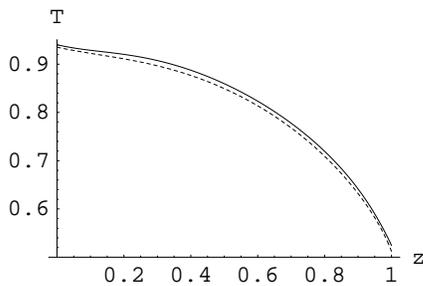


FIG. 1. Comparison of the temperature profiles in the first two-stream simplification with that obtained from solution of the full Boltzmann equation, for the case of the purely nucleonic opacity function, Eqs. (14), with scattering equal to absorption. The flux function, surface temperature, and optical thickness equal to 9.0 are taken to be identical in the two cases. The flux is taken to be proportional to height, z . The units of temperature, T , are arbitrary, but the same for the two cases. The solid curve comes from full Boltzmann equation and the dotted curve is from the two stream approximation.

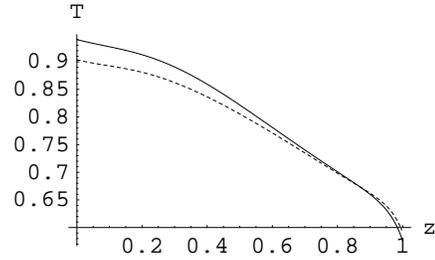


FIG. 2. The same comparison as in Fig. 1, but now for the case of purely relativistic opacity, and optical thickness of 11.0.

$$\begin{aligned}\lambda_a(T,p) &= \alpha_a p T^4, \\ \lambda_s(T,p) &= \alpha_s p T^4.\end{aligned}\quad (15)$$

First we examined a solution of the “two stream” approximation in which the temperature and momentum dependences are taken into account, that is to say, the “first simplification” (3) and (4). We then compared with the results of solution of the full Boltzmann equations (1) and (2). For purposes of making the comparisons we took the flux to increase linearly with z . Figure 1 compares the temperature profiles for a case with fixed temperature at the outer boundary, identical flux profiles, the “nucleonic” reaction rates given by Eq. (14), with equal scattering and absorption, and common optical thickness (of the one-half disk) of 9.0.² In this comparison and in all that follow the ordinates are plotted in arbitrary units; the ratios of results, once the total optical thicknesses of the disk are set, are independent of the absolute values of the input temperatures and densities.

Figure 2 shows the same comparison for the case of purely relativistic reaction rates given in Eqs. (15); the agreement is not quite as good as that shown in Fig. 1, but together the plots show a rather impressive agreement and provide a strong argument for using the first form of the two-stream simplification.

Next we address the comparison of the “first simplification,” or the two stream approximation with full temperature and energy dependence (3) and (4), and the “second simplification” (7) and (8), which subsumes all of this dependence in some average opacities. Popham and Narayan give an analytic solution for the latter equation for the case of z -independent opacities and flux that increases linearly with z , as in the two examples exhibited in Figs. 1 and 2,

$$(I_+ + I_-)/2 = \frac{1 + \hat{\tau}/2 - (z^2 \hat{\tau})/2}{1 + \hat{\tau}/2 + 1/\hat{\tau}_a} I_e(0),$$

²We define optical thickness of the half-layer in terms of the opacities (14) or (15) and the temperature profiles of the solutions to Eqs. (3),(4) themselves, determining a temperature-dependent mean opacity at each point, in the standard fashion, by taking the inverse of the appropriate thermal average of $\lambda_s^{-1} + \lambda_a^{-1}$, then integrating the mean opacity (evaluated at the local temperature) to obtain the optical thickness.

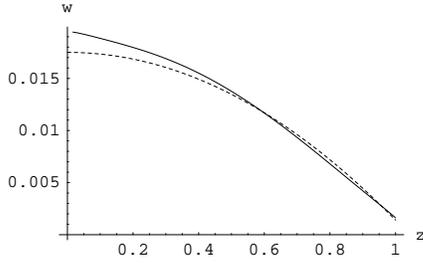


FIG. 3. The solid curve shows the neutrino energy density w as a function of height, as determined by the “first simplification” form of the equations, for the case of nucleonic opacity with equal elastic scattering and absorption rates, and optical thickness of 11.0, as in Fig. 1. The dotted curve shows the same plot for the solution (16) given by Popham and Narayan and specialized to the same parameters.

$$(I_+ - I_-)/2 = \frac{z}{1 + \hat{\tau}/2 + 1/\hat{\tau}_a} I_e(0) \quad (16)$$

where the z dependence of the source term I_e , given by,

$$I_e(z) = \frac{1 + \hat{\tau}/2 + 1/\hat{\tau}_a - (z^2)(\hat{\tau}/2)}{1 + \hat{\tau}/2 + 1/\hat{\tau}_a} I_e(0) \quad (17)$$

implicitly defines the temperature distribution. Here $\bar{\tau}$ and $\bar{\tau}_a$ are the respective total and absorption depths of the half-layer (times the factor $\sqrt{3}$ that enters the definition of the $\bar{\lambda}$ functions, above). In Fig. 3 we show a comparison of this solution with that of the “first simplified form” for the case of pure nucleonic opacity, with an optical thickness of 11.0 and the same flux profile, this time displaying the neutrino energy densities $w(z)$ as a function of height, in the two cases. The “second simplification” approximation is seen to work quite well, by the measure of the “first simplification” approximation, and therefore, in view of the previous comparisons, as a simulation of the full transport problem.

The case of purely relativistic opacity is more problematic. In Fig. 4. we show neutrino energy density plots for the case of equal scattering and absorption. The “second simplification” approximation here gives a greater than 30%

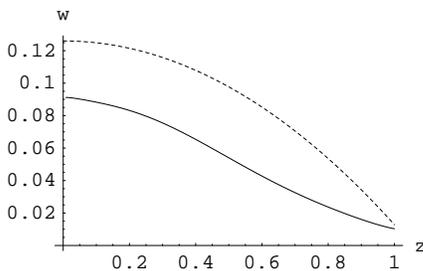


FIG. 4. The same as Fig. 3, but for purely relativistic opacity, with optical thickness 11.0 as in Fig. 1.

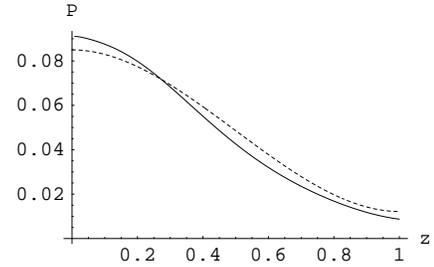


FIG. 5. The solid curve shows the total pressure (in arbitrary units) of relativistic particles as a function of height, as determined by the “first simplification” form of the equations, for the case of relativistic opacity with equal elastic scattering and absorption rates, where the flux is now taken as proportional to $z - z^2/2$. The optical thickness is 11. The dotted curve shows the function, $a + b(z^2/2 - z^3/3)$ that most nearly fits the pressure curve.

overestimate of the (central energy density)/(flux) ratio. But more significant, perhaps, are the shapes of the curves; with the same flux profile, the “first simplification” approximation has a quite different energy density profile from the “second simplification” result. In particular, note that the negative of the gradient of the energy density, in the outer region, is a decreasing function of z in the entire region $z > 0.5$ in our treatment that takes into account the energy and temperature dependences, while it is positive throughout in the case in which averaged opacities are used.

When we plot the $T(z)^4$ instead of the energy density of the neutrinos we find a shape that differs appreciably from that shown in Fig. 4 only in a region extremely near the surface (in fact, a region beginning above the average height for the last neutrino scattering). Thus the gradient of the pressure of all of the relativistic particles, taken together, declines steadily as we move upwards in the upper parts of the disk. Hydrostatic equilibrium requires that the total pressure gradient be proportional to $z\rho(z)$, where $\rho(z)$ is the total energy density, since the operative tidal gravitational field in the disk is proportional to z . Therefore if relativistic pressure were the whole story, the density would have to be a fairly rapidly decreasing function of z , in contrast to the uniform density assumption that appears to be used in [5]. In any of the regimes that we here consider, the total energy density is dominated by nucleons. But both pressure and neutrino opacities are dominated by relativistic particles when the temperature is such that the ratio of the number of relativistic particles to the number of nucleons is large.

One can therefore find regimes in which the relativistic particles predominate in both the pressure and opacity calculations while the mass density is still dominated by nucleons. We might achieve hydrostatic equilibrium in such a system by having a nucleon density that decreases significantly from the center to the top. For example, consider a nucleon density that is proportional to $(1 - |z|)$. Now assume as well that heat production is proportional to this density. Then we need to solve the neutrino transport problem, taking a flux profile proportional to $z - z^2/2$ (for $z > 0$). The resulting relativistic

pressure profile is shown in Fig. 5 for the case of (optical thickness=11.0).³

On the same curve we have drawn the profile of the pressure required for hydrostatic equilibrium for the case in which the total energy density is proportional to $(1 - |z|)$, giving $P = \text{const} \times (z^2/2 - |z|^3/3)$. It is clear that with a small redistribution of the nucleons, in this model we could simultaneously achieve our three goals: (a) Solution of the radiative transfer problem; (b) hydrostatic equilibrium; and (c) matching the energy flux to a local heating rate that is proportional to the nucleon density.

In the case of accretion disks that are more or less uni-

³The total relativistic pressure is the sum of the neutrino pressure and that of the e^\pm particles and the photons, the latter two computed from the local temperature; for the neutrino contribution we take the local energy density, divided by 3, from the solution of the radiative transfer problem. The shape of the energy density curve is the same as that of the total relativistic pressure except in the very outermost region.

form in density it appears that the temperature will never be quite high enough, at a given nucleon density, to lead to the complete dominance of pressure and opacity by the relativistic part of the medium [5]. But the above example does suggest the possibility of stable relativistic outlying regions, still well under the height of (average) last neutrino scattering, in which nucleon densities decrease markedly with height. The surface of the disk cannot be as simple as in this idealized model and one can easily imagine instabilities that lead to escape of some of the surface plasma. Thus it might be possible to achieve an electron-positron plasma in the region both near the surface and the axis that is more dense than that coming from neutrino-neutrino collisions in the vicinity of the axis, the latter a source that has been discussed in the literature [2].

ACKNOWLEDGMENTS

The author is indebted to Andrei Beloborodov and Chris Fryer for interesting conversations.

-
- [1] S.E. Woosley, *Astrophys. J.* **405**, 273 (1993).
 - [2] M. Ruffert, H.-Th. Janka, K. Takahashi, and G. Schafer, *Astron. Astrophys.* **319**, 122 (1997).
 - [3] R. Popham, S.E. Woosley, and C. Fryer, *Astrophys. J.* **518**, 356 (1999).
 - [4] A.I. MacFadyen and S.E. Woosley, *Astrophys. J.* **524**, 262 (1999).
 - [5] T. DiMatteo, R. Perna, and R. Narayan, *Astrophys. J.* **579**, 706 (2002).
 - [6] R. Popham and R. Narayan, *Astrophys. J.* **442**, 337 (1995); the relevant formulas in this paper were derived by simplifying a more exact result contained in a paper by I. Hubeny, *ibid.* **351**, 632 (1990).
 - [7] D. Mihalas, *Stellar Atmospheres* (Freeman, San Francisco, 1978).
 - [8] P.J. Schinder and S.L. Shapiro, *Astrophys. J.* **259**, 311 (1982).