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2003 J. Phys. A: Math. Gen. 36 L237

(http://iopscience.iop.org/0305-4470/36/15/104)

J. Phys. A: Math. Gen. **36** (2003) L237–L242 PII: S0305-4470(03)59969-3

LETTER TO THE EDITOR

Entanglement cost of three-level antisymmetric states

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Received 24 February 2003, in final form 4 March 2003
Published 3 April 2003
Online at stacks.iop.org/JPhysA/36/L237

Abstract
We show that the entanglement cost of the three-dimensional antisymmetric states is one ebit.

PACS numbers: 03.65.Ud, 03.67.−a

The concept of entanglement is the key to quantum information processing. To quantify the resource of entanglement, its measures should be additive, such as bits for classical information. One candidate for such additive measures is entanglement of formation. In [1], it is shown that the entanglement cost E_c of creating some state can be asymptotically calculated from the entanglement of formation. In this sense, the entanglement cost has an important physical meaning. Since the known results are, nevertheless, not so much [6, 7], we pay attention to antisymmetric states that are easy to deal with.

As is already shown [2], the entanglement of formation for two states in $\mathcal{S}(\mathcal{H}_-)$ is additive. Furthermore, the lower bound for the entanglement cost of density matrices in d-level antisymmetric space, obtained in [3], is $\log_2 \frac{d}{d-1}$ ebit. In this paper, we show that the entanglement cost of three-level antisymmetric states ($d = 3$) in $\mathcal{S}(\mathcal{H}_-)$ is exactly one ebit.

We first define the three-level antisymmetric states. Let us consider a bipartite qutrit system, $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^3$. The antisymmetric subspace $\mathcal{H}_-$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ is defined as follows:

$$\mathcal{H}_- := \mathrm{span}_{\mathbb{C}}\{|01\rangle - |10\rangle, |12\rangle - |21\rangle, |20\rangle - |02\rangle\} \subset \mathcal{H}_A \otimes \mathcal{H}_B.$$

Then, the antisymmetric state on $\mathcal{H}_-^{\otimes n}$ shared with Alice and Bob is, in general,

$$|\psi\rangle = \sum_{\substack{j_1,j_2,\ldots,j_n=0\\ k_1,k_2,\ldots,k_n=0}}^{2} \alpha_{j_1,j_2,\ldots,j_n;k_1,k_2,\ldots,k_n} |j_1, j_2, \ldots, j_n; k_1, k_2, \ldots, k_n\rangle$$
$$\in \mathcal{H}_-^{\otimes n} \subset \mathcal{H}_A^{(1)} \otimes \mathcal{H}_A^{(2)} \otimes \cdots \otimes \mathcal{H}_A^{(n)} \otimes \mathcal{H}_B^{(1)} \otimes \mathcal{H}_B^{(2)} \otimes \cdots \otimes \mathcal{H}_B^{(n)} \tag{1}$$

$$\alpha_{j_1,j_2,\ldots,j_n;k_1,k_2,\ldots,k_n} := \left(\frac{1}{\sqrt{2}}\right)^n \sum_{i_1,i_2,\ldots,i_n=0}^{2} a_{i_1,i_2,\ldots,i_n} \prod_{m=1}^{n} \epsilon_{i_m j_m k_m} \tag{2}$$

where $\mathcal{H}_{A(B)}^{(i)}$ means the ith space of Alice (resp. Bob) and ϵ is the Levi-Civita symbol, i.e.,

$\epsilon_{ijk} = 1$ for $(ijk) = (123)$ and its even permutations, -1 for odd permutations and 0 otherwise. Henceforth, we identify the above coefficient $\alpha_{j_1,\ldots,j_n;k_1,\ldots,k_n}$ with the entries of a matrix $\alpha \in M(3^n; \mathbb{C})$ with respect to the rows $\{j_1, \ldots, j_n\}$ and the columns $\{k_1, \ldots, k_n\}$ with lexicographical order.

The entanglement of formation E_f is defined as follows:

$$E_f(\rho) = \inf \sum_j p_j E(|\psi_j\rangle) \tag{3}$$

where p_j and $|\psi_j\rangle$ are decompositions such that $\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$ and E is the entropy of entanglement

$$E(|\psi\rangle) = S(\mathrm{Tr}_B |\psi\rangle\langle\psi|).$$

The subadditivity of E_f is well known [6]:

Lemma 1 (Subadditivity)**.** *Let $\rho^{(i)}$ be density matrices on $\mathcal{H}_A \otimes \mathcal{H}_B$, i.e., bipartite states. Then,*

$$E_f\left(\otimes_{i=1}^n \rho^{(i)}\right) \leqslant \sum_{i=1}^n E_f(\rho^{(i)}).$$

In [6], it is also shown that $E_f(\rho) = 1$ for any $\rho \in \mathcal{S}(\mathcal{H}_-)$. Using their result, we obtain the following:

Corollary 1. *For any $\rho^{(i)} \in \mathcal{S}(\mathcal{H}_-)$,*

$$E_f\left(\otimes_{i=1}^n \rho^{(i)}\right) \leqslant n.$$

To prove $E_c = 1$, it is therefore sufficient that we show the superadditivity $E_f\left(\otimes_{i=1}^n \rho^{(i)}\right) \geqslant n$. For the states in $\mathcal{H}_-^{\otimes n}$, we can prove the following lemma:

Lemma 2. *For any $|\psi\rangle \in \mathcal{H}_-^{\otimes n}$,*

$$E(|\psi\rangle) \geqslant n. \tag{4}$$

We give a proof of this lemma in the appendix. The following corollary immediately follows from this lemma because the definition of the entanglement of formation (3) is a linear combination of (4).

Corollary 2. *For any $\rho \in \mathcal{S}\left(\mathcal{H}_-^{\otimes n}\right)$,*

$$E_f(\rho) \geqslant n.$$

Theorem 1. *For any $\rho^{(i)} \in \mathcal{S}(\mathcal{H}_-)$,*

$$E_f\left(\otimes_{i=1}^n \rho^{(i)}\right) = n.$$

Proof. From corollaries 1 and 2, this theorem holds. □

Hence, as a corollary of this theorem, we obtain the main result:

Corollary 3 (Main result)**.** *For any $\rho \in \mathcal{S}(\mathcal{H}_-)$,*

$$E_f(\rho^{\otimes n}) = n.$$

Therefore,

$$E_c(\rho) := \lim_{n\to\infty} \frac{1}{n} E_f(\rho^{\otimes n}) = 1.$$

Acknowledgment

The author is grateful to T Shimono, K Matsumoto and H Fan for helpful discussions on related works and seminars.

Appendix. Proof of lemma 2

It is well known that the entanglement of pure states is defined by von Neumann entropy of the reduced density matrix $\rho_A = \mathrm{Tr}_B|\psi\rangle\langle\psi| = \alpha\alpha^\dagger$, where α is a $3^n \times 3^n$ matrix, which is defined in (1). Let λ_i be the eigenvalues of ρ_A and its elementary symmetric functions

$$
\begin{aligned}
s_1 &:= \sum_i \lambda_i = \mathrm{Tr}\,\rho_A = 1 \\
s_2 &:= \sum_{i<j} \lambda_i \lambda_j \\
&\vdots \\
s_{3^n} &:= \prod_i \lambda_i = \det \rho_A
\end{aligned}
$$

the power sum $I_k(\rho_A) = \sum_i \lambda_i^k = \mathrm{Tr}{\rho_A}^k$, respectively. Note that $\sqrt{s_2}$ is the generalized concurrence [10–12]. As we will see later, the value of this generalized concurrence is closely related to the entanglement of formation in our case.

Proposition 1. *Let α be the coefficient of $|\psi\rangle \in \mathcal{H}_-^{\otimes n}$ and $\rho_A = \alpha\alpha^\dagger$. Then,*

$$
I_2(\rho_A) \leqslant \frac{1}{2^n}. \tag{5}
$$

Proof. The calculation of $I_2(\rho_A)$ is lengthy but straightforward. First, let us choose two rows $J := (j_1, j_2, \ldots, j_n)$, $J' := (j_1', j_2', \ldots, j_n')$ and two columns $K := (k_1, k_2, \ldots, k_n)$, $K' := (k_1', k_2', \ldots, k_n')$ for a 2×2 minor of matrix α. Since $s_k(\rho_A)$ is equal to the square sum of all $k \times k$ minors of α, i.e., due to the Cauchy–Binet theorem [4], we therefore obtain (see also [5])

$$
\begin{aligned}
s_2(\rho_A) &= \frac{1}{4} \sum_{\substack{j_1, j_2, \ldots, j_n = 0 \\ j_1', j_2', \ldots, j_n' = 0 \\ k_1, k_2, \ldots, k_n = 0 \\ k_1', k_2', \ldots, k_n' = 0}}^{2} \left| \alpha_{j_1,\ldots,j_n;k_1,\ldots,k_n} \alpha_{j_1',\ldots,j_n';k_1',\ldots,k_n'} - \alpha_{j_1,\ldots,j_n;k_1',\ldots,k_n'} \alpha_{j_1',\ldots,j_n';k_1,\ldots,k_n} \right|^2 \\
&= \frac{1}{4}\left(\frac{1}{2^n}\right)^2 \sum_{JJ'KK'} \\
&\quad \times \left| \left(\sum_{p_1,\ldots,p_n=0}^{2} a_{p_1,\ldots,p_n} \prod_{m=1}^{n} \epsilon_{p_m j_m k_m} \right) \left(\sum_{p_1',\ldots,p_n'=0}^{2} a_{p_1',\ldots,p_n'} \prod_{m'=1}^{n} \epsilon_{p_{m'}' j_{m'}' k_{m'}'} \right) \right. \\
&\quad \left. - \left(\sum_{p_1,\ldots,p_n=0}^{2} a_{p_1,\ldots,p_n} \prod_{m=1}^{n} \epsilon_{p_m j_m k_m'} \right) \left(\sum_{p_1',\ldots,p_n'=0}^{2} a_{p_1',\ldots,p_n'} \prod_{m'=1}^{n} \epsilon_{p_{m'}' j_{m'}' k_{m'}} \right) \right|^2 \\
&= \frac{1}{2^{2n+2}} \sum_{JJ'KK'} \left| \sum_{PP'} a_{p_1,\ldots,p_n} a_{p_1',\ldots,p_n'} \left(\prod_{m=1}^{n} \epsilon_{p_m j_m k_m} \prod_{m'=1}^{n} \epsilon_{p_{m'}' j_{m'}' k_{m'}'} \right. \right.
\end{aligned}
$$

$$
\begin{aligned}
&\left.\left.-\prod_{m=1}^{n}\epsilon_{p_m j_m k'_m}\prod_{m'=1}^{n}\epsilon_{p'_{m'} j'_{m'} k_{m'}}\right)\right|^2\\
&=\frac{1}{2^{2n+1}}\sum_{JJ'KK'}\sum_{PP'QQ'}a_{p_1,\ldots,p_n}a_{p'_1,\ldots,p'_n}a^*_{q_1,\ldots,q_n}a^*_{q'_1,\ldots,q'_n}\\
&\quad\times\left(\prod_{m_1}\epsilon_{p_{m_1} j_{m_1} k_{m_1}}\prod_{m_2}\epsilon_{p'_{m_2} j'_{m_2} k'_{m_2}}\prod_{m_3}\epsilon_{q_{m_3} j_{m_3} k_{m_3}}\prod_{m_4}\epsilon_{q'_{m_4} j'_{m_4} k'_{m_4}}\right.\\
&\quad\left.-\prod_{m_1}\epsilon_{p_{m_1} j_{m_1} k_{m_1}}\prod_{m_2}\epsilon_{p'_{m_2} j'_{m_2} k'_{m_2}}\prod_{m_3}\epsilon_{q_{m_3} j_{m_3} k'_{m_3}}\prod_{m_4}\epsilon_{q'_{m_4} j'_{m_4} k_{m_4}}\right)
\end{aligned}\tag{6}
$$

where we denote $\sum_P \equiv \sum_{p_1,p_2,\ldots,p_n=0}^{2}$ and $a_P \equiv a_{p_1,p_2,\ldots,p_n}$, etc, for simplicity. Let us divide (6) into two parts.

1. First term

$$
\begin{aligned}
&\sum_{JJ'KK'}\left(\prod_{m_1=1}^{n}\epsilon_{p_{m_1} j_{m_1} k_{m_1}}\prod_{m_2=1}^{n}\epsilon_{p'_{m_2} j'_{m_2} k'_{m_2}}\prod_{m_3=1}^{n}\epsilon_{q_{m_3} j_{m_3} k_{m_3}}\prod_{m_4=1}^{n}\epsilon_{q'_{m_4} j'_{m_4} k'_{m_4}}\right)\\
&\qquad=\sum_{j_2,\ldots,j_n=0}^{2}\sum_{J'KK'}\left(\sum_{j_1=0}^{2}\epsilon_{p_1 j_1 k_1}\epsilon_{q_1 j_1 k_1}\right)\\
&\qquad\quad\times\left(\prod_{m_1=2}^{n}\epsilon_{p_{m_1} j_{m_1} k_{m_1}}\prod_{m_2=1}^{n}\epsilon_{p'_{m_2} j'_{m_2} k'_{m_2}}\prod_{m_3=2}^{n}\epsilon_{q_{m_3} j_{m_3} k_{m_3}}\prod_{m_4=1}^{n}\epsilon_{q'_{m_4} j'_{m_4} k'_{m_4}}\right)\\
&\qquad=\sum_{K}\left[\prod_{m=1}^{n}\left(\delta_{k_m k_m}\delta_{p_m q_m}-\delta_{k_m p_m}\delta_{k_m q_m}\right)\right]\\
&\qquad\quad\times\sum_{K'}\left[\prod_{m=1}^{n}\left(\delta_{k'_m k'_m}\delta_{p'_m q'_m}-\delta_{k'_m p'_m}\delta_{k'_m q'_m}\right)\right]\\
&\qquad=2^{2n}\prod_{m=1}^{n}\delta_{p_m q_m}\delta_{p'_m q'_m}
\end{aligned}
$$

where we use the relation $\sum_{j_1=0}^{2}\epsilon_{p_1 j_1 k_1}\epsilon_{q_1 j_1 k_1}=\delta_{k_1 k_1}\delta_{p_1 q_1}-\delta_{k_1 p_1}\delta_{k_1 q_1}$.

2. Second term

$$
\begin{aligned}
&\sum_{JJ'KK'}\left(\prod_{m_1=1}^{n}\epsilon_{p_{m_1} j_{m_1} k_{m_1}}\prod_{m_2=1}^{n}\epsilon_{p'_{m_2} j'_{m_2} k'_{m_2}}\prod_{m_3=1}^{n}\epsilon_{q_{m_3} j_{m_3} k'_{m_3}}\prod_{m_4=1}^{n}\epsilon_{q'_{m_4} j'_{m_4} k_{m_4}}\right)\\
&\qquad=\sum_{j_2,\ldots,j_n=0}^{2}\sum_{J'KK'}\left(\sum_{j_1=0}^{2}\epsilon_{p_1 j_1 k_1}\epsilon_{q_1 j_1 k'_1}\right)\\
&\qquad\quad\times\left(\prod_{m_1=2}^{n}\epsilon_{p_{m_1} j_{m_1} k_{m_1}}\prod_{m_2=1}^{n}\epsilon_{p'_{m_2} j'_{m_2} k'_{m_2}}\prod_{m_3=2}^{n}\epsilon_{q_{m_3} j_{m_3} k'_{m_3}}\prod_{m_4=1}^{n}\epsilon_{q'_{m_4} j'_{m_4} k_{m_4}}\right)
\end{aligned}
$$

$$= \sum_{KK'} \prod_{m=1}^{n} \left(\delta_{k_m k'_m}\delta_{p_m q_m} - \delta_{k_m p_m}\delta_{k'_m q_m}\right)\left(\delta_{k'_m k_m}\delta_{p'_m q'_m} - \delta_{k'_m p'_m}\delta_{k_m q'_m}\right)$$

$$= \prod_{m=1}^{n} \left(\delta_{p_m q_m}\delta_{p'_m q'_m} + \delta_{p'_m q_m}\delta_{p_m q'_m}\right).$$

We summarize these terms and obtain the following:

$$s_2(\rho_A) = \frac{1}{2^{2n+1}} \sum_{PP'QQ'} a_P a_{P'} a_Q^* a_{Q'}^* \left[2^{2n} \prod_{m=1}^{n} \delta_{p_m q_m}\delta_{p'_m q'_m} - \prod_{m=1}^{n} \left(\delta_{p_m q_m}\delta_{p'_m q'_m} + \delta_{p'_m q_m}\delta_{p_m q'_m}\right)\right]$$

$$= \frac{1}{2} - \frac{1}{2^{2n+1}} \sum_{PP'QQ'} a_P a_{P'} a_Q^* a_{Q'}^* \prod_{m=1}^{n} \left(\delta_{p_m q_m}\delta_{p'_m q'_m} + \delta_{p'_m q_m}\delta_{p_m q'_m}\right)$$

and

$$I_2(\rho_A) = s_1(\rho_A)^2 - 2s_2(\rho_A)$$

$$= \frac{1}{2^{2n}} \sum_{PP'QQ'} a_P a_{P'} a_Q^* a_{Q'}^* \prod_{m=1}^{n} \left(\delta_{p_m q_m}\delta_{p'_m q'_m} + \delta_{p'_m q_m}\delta_{p_m q'_m}\right)$$

$$= \frac{1}{2^{2n}} \sum_{PP'QQ'} \prod_{m=1}^{n} \left(\delta_{p_m q_m}\delta_{p'_m q'_m} + \delta_{p'_m q_m}\delta_{p_m q'_m}\right)$$

$$\times \frac{1}{2}[-|a_P a_{P'} - a_Q a_{Q'}|^2 + |a_P a_{P'}|^2 + |a_Q a_{Q'}|^2]$$

$$= \frac{1}{2^n} - \frac{1}{2^{2n+1}} \sum_{PP'QQ'} \prod_{m=1}^{n} \left(\delta_{p_m q_m}\delta_{p'_m q'_m} + \delta_{p'_m q_m}\delta_{p_m q'_m}\right) |a_P a_{P'} - a_Q a_{Q'}|^2$$

$$\leqslant \frac{1}{2^n}.$$

We have thus proved proposition 1. □

The following theorem is important:

Theorem 2 (Furuta; special case of [8, 9])**.** *Let A be an invertible positive operator. Then for any positive $x \in \mathbb{R}$*

$$-A \log A \geqslant (1 - \log x)A - \frac{1}{x}A^2.$$

This inequality holds even for singular A under the convention $0 \log 0 = 0$. By diagonalizing A and applying $-x \log x \geqslant (1 - \log x_0)x - x^2/x_0$ for positive x and x_0, we can obtain this inequality.

Corollary 4. *Let $S(A) = -\mathrm{Tr}(A \log_2 A)$ and ρ_A a normalized density matrix (i.e. $\mathrm{Tr}\rho_A = 1$). Then for positive x,*

$$S(\rho_A) \geqslant [(1 - \log x) - I_2(\rho_A)/x]/\log 2 \geqslant -\log_2 I_2(\rho_A)$$

where the lower bound holds when $x = I_2(\rho_A)$.

Hence, $S(\rho_A) \geqslant n$ and this ends the proof of lemma 2.

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